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PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

NATURAL SELECTION AND MENDELIAN VARIATION

By H. T. J. Norton.

[Received and read 11 November, 1926.]

Part I.

The object of the present paper is to take the simplest known law of heredity, the Mendelian law, and to see how far it is possible to work out the effects of selection on a variety, inherited by this law, without making specializing assumptions. The conclusions, however, which are here reached are subject to the assumption that the ages of parents vary independently : that is to say, that in regard to the children born at any one moment, the age distribution in the fathers of those whose mothers are of age t is independent of t, and similarly with the age distribution in the mothers of those whose fathers are of age t. This, and the assumption that the birth-rates and death-rates in each type are functions merely of age and that the birth-rate is the same in all three types, are the main limitations. The equations involved are integral equations, and the generality of the results is a consequence of the well known fact that most properties of these equations are analogous to those of linear equations. The Mendelian equations are not linear, but the essential point is the analogue of the property of convergence possessed by the convergents to a continued fraction. A similar (but wider) generalization to difference equations of the third and higher order has been published by Perron*. The effect of selection, when generations do not overlap,

* *Math. Annalen*, 64 (1907), 1–76.

has been discussed by Haldane*, both in the case of the simple Mendelian law and of more complicated laws. So far as I can see, all the main results, which are true when generations do not interbreed, are also true when they do, and it is not necessary to assume any particular form for the function which connects the birth- and death-rates with age. In the following paper some preliminary questions are discussed in 1–10, and the Mendelian equations are stated and considered in 11–17.

It will be supposed that the population forms a community. By a community is meant an endogamous population, which consists at any time of the descendants of the original group which formed the population at the beginning of the period considered. The inhabitants of an oceanic island would form a community if there were no emigration nor immigration. The main result of §§ 1–9 is an asymptotic formula for the number of births in a community in terms of the instantaneous birth-rate and death-rate. These are the quantities that measure the chance that a member of a group, born at a certain time, should die at a certain age, or should become a parent at a certain age; the definitions are as follows: the instantaneous birth-rate and death-rate† at age t for births at time x are $\nu(t, x)$ and $\mu(t, x)$ when $\nu(t, x)dt$ is the ratio of the number of members of the group, that were born between x and $x+dx$ and become parents between the ages of t and $t+dt$, to the number that were born between x and $x+dx$ and are alive at age t; and when $\mu(t, x)dt$ is the ratio of the number, that were born between x and $x+dx$ and die at ages between t and $t+dt$, to the number that were born between x and $x+dx$ and survive to age t.

The only type of community considered here in detail is one in which births occur at all times in the year. But it will make the argument clearer to consider briefly the communities in which births occur only at breeding seasons which can be treated as momentary. In this case, u_n, the number of births in the n-th season, is given by

$$u_n = K(n, n-1)\,u_{n-1} + K(n, n-2)\,u_{n-2} + \ldots + K(n, n-r)\,u_{n-r}, \quad (1.1)$$

where $K(n, s)$ is the chance that an individual, born in the s-th season, should produce a child in the n-th, and r is the number of seasons in which he can produce children. $K(n, s)$ is easily expressible in terms of the instantaneous birth- and death-rates. (1.1) is a linear difference equa-

* J. B. S. Haldane, *Trans. Camb. Phil. Soc.*, 23 (1924), 19-41.

† *Encyclopédie des Sciences mathématiques*, tome 1, vol. 4, fasc. 3, p. 472. Only the definition of the instantaneous death-rate is given here, but that of the birth-rate is an obvious extension.

tion, and the general solution is expressible as a linear combination of at most r particular solutions. Suppose $r \geqslant s \geqslant 1$, and let $-2k(r, s)$ be the number of descendants, born in the n-th season, of a couple who were born in the s-th; the factor -2 is to keep the notation the same as that used in §§ 4–8. Then the general solution of (1 . 1) is of the form

$$u_n = -\sum_{s=1}^{r} C_s k(n, s), \qquad (1 . 2)$$

where C_1, C_2, ... are constants, depending on the initial age-distribution in the community. The actual value of C_s is

$$u_s - \sum_{t=1}^{s-1} K(s, t)\, u(t).$$

If, now, the eventual increase of the community is independent of its initial composition, u_n must be of the form $Q_n A_n$, where Q_n depends merely on the K's and A_n is a function depending upon the particular community, and tending to a constant positive limit as n tends to infinity. The necessary condition for this is that

$$\lim_{n\to\infty} \frac{k(n, s)}{k(n, 1)} \quad (s = 1, 2, \ldots, r)$$

should exist and be positive. Now $k(n, s)$, considered as a function of the K's, belongs to a well known type. In the particular case $r = 2$,

$$-k(2, 1) = K(2, 1); \quad -k(3, 1) = K(3, 2)\,K(2, 1) + K(3, 1),$$

$$-k(2, 2) = 1; \quad -k(3, 2) = K(3, 2), \text{ etc.}$$

Thus $-k(n, 1)$ is the denominator and $-k(n, 2)$ is the numerator of the n-th convergent to the continued fraction

$$\frac{1}{K(2, 1)+}\,\frac{K(3, 1)}{K(3, 2)+}\,\frac{K(4, 2)}{K(4, 3)+\ldots}.$$

For larger values of r, the functions $k(n, s)$ have similar properties, and have been considered by various writers. Thus it appears that, in the case $r = 2$, the necessary and sufficient condition that the eventual increase of the community should not depend on its initial composition is that the continued fraction should be convergent. Suppose births to occur continually. Then the main result of §§ 2–8 is to show that two communities with the same birth- and death-rates are asymptotically equivalent if condition (5 . 1) is satisfied. If we call the chance that an individual, born at x, should produce a child at the age t, "the survival

ratio at time x for age t'', this condition can be stated in the form that there is a certain age, the same for all values of x, at which the survival ratio is not zero and does not tend to zero as x increases. The general line of argument in §§ 4–7 is the same as that outlined above; an exact equation (4 . 4) is first found, and an asymptotic formula is deduced from it by an argument of which the first part is essentially the same as that used by Perron to show that, in the case of Jacobi's algorithm,

$$\frac{k(n, s)}{k(n, 0)}$$

tends to a limit.

2. In this section is given the equation which connects the number of births in a community with the number of births at previous moments.

Consider a group of organisms which forms a community during a certain period P. I take the origin of time at the beginning of P. Let $u(x)dx$ be the number of members of the community who are born in the short interval between x and $x+dx$; $u(x)$ is the *density* of births at x. It will be supposed that every member of the community is the child of two parents. Let the chance that an individual who was born between x and $x+dx$ should survive to y be $K_1(y, x)$, so that the number of survivors is $K_1(y, x)\, u(x)\, dx$. Some of the survivors may become parents between y and $y+dy$; if $\nu(t, x)$ is the instantaneous birth-rate at age t for births at time x, the number of parents is $\nu(y-x, x)\, dy\, K_1(y, x)\, u(x)\, dx$. The number of individuals in the whole community who become parents between y and $y+dy$ is obtained by integrating this expression over all values of x such that $y-x$ lies between the greatest and least ages at which members of the community breed. It is, therefore,

$$dy \int_{y-\lambda_2}^{y-\lambda_1} \nu(y-x, x)\, K_1(y, x)\, u(x)\, dx \quad (y \geqslant \lambda_2),$$

where λ_1 is the latter age and λ_2 the former. The values of λ_1 and λ_2 depend upon the peculiarities of the community; they will be supposed to be constants, and a period of λ_2 years will be called *a generation*. By definition, the children of members of the community belong to the community, and conversely, every member of the community, who is born during P, is the child of two members of the community. Therefore, the number of individuals who become parents in an interval is twice the number of births in that interval and, therefore,

$$2u(y) = \int_{y-\lambda_2}^{y-\lambda_1} \nu(y-x, x)\, K_1(y, x)\, u(x)\, dx \quad (y \geqslant \lambda_2). \quad (2.1)$$

This equation requires modification if y falls within the first generation of P. Let $\phi(x)\,dx$ be the number of individuals who are of ages between x and $x+dx$ at time 0, and let $K_2(y,\,x)\;\phi\;(x)\,dx$ be the number that survive to time y. The total number of individuals who were born before the beginning of P and who become parents between y and $y+dy$ is

$$dy\int_0^{\lambda_2-y}\nu(y+x,\,-x)\,K_2(y,\,x)\,\phi(x)\,dx,$$

provided that y is not greater than λ_2, and is otherwise zero. The number who were born after the beginning of P and become parents between y and $y+dy$ is

$$dy\int_0^{y-\lambda_1}\nu(y-x,\,x)\,K_1(y,\,x)\,u(x)\,dx,$$

provided that y is greater than λ_1, and is otherwise zero. Hence

$$\begin{aligned}2u(y) &= \int_0^{y-\lambda_1}\nu(y-x,\,x)\,K_1(y,\,x)\,u(x)\,dx\\ &\qquad+\int_0^{\lambda_2-y}\nu(y+x,\,-x)\,K_2(y,\,x)\,\phi(x)\,dx\quad(\lambda_2\geqslant y\geqslant\lambda_1)\\ &=\int_0^{\lambda_2-y}\nu(y+x,\,-x)\,K_2(y,\,x)\,\phi(x)\,dx\quad(\lambda_1\geqslant y\geqslant 0).\end{aligned}$$

The functions $K_1(y,\,x)$ and $K_2(y,\,x)$ are related to $\mu(t,\,x)$, the instantaneous death-rate at age t for births at time x, by the equations*

$$K_1(y,\,x)=\exp\Big[-\int_x^y\mu(y'-x,\,x)\,dy'\Big];\qquad(2\,.\,2)$$

$$K_2(y,\,x)=\exp\Big[-\int_0^y\mu(y'+x,\,-x)\,dy'\Big].\qquad(2\,.\,21)$$

It will be sufficient to prove the first. Of the $u(x)\,dx$ individuals, who were born between x and $x+dx$, $K_1(y,\,x)\,u(x)\,dx$ survive to time y, and $K_1(y+dy,\,x)\,u(x)\,dx$ survive to time $y+dy$; therefore the number that die between y and $y+dy$ is

$$[K_1(y,\,x)-K_1(y+dy,\,x)]\,u(x)\,dx.$$

But, as the individuals who were born between x and $x+dx$ are at time y of ages between $y-x-dx$ and $y-x$, the number that die before $y+dy$ is, by definition of μ, $\mu'\,dy\,K_1(y,\,x)\,u(x)\,dx$, where μ' lies between the greatest and the least values assumed by $\mu(y-x-a,\,x+a)$ as a varies from 0 to dx. These two expressions for the number of deaths must be

* Cf. *Enc. des Sci. Math.*, *loc. cit.*, 472-473.

equal, and, therefore, in the limit, when dx and dy tend to zero,

$$\frac{\partial K_1(y, x)}{\partial y} = -\mu(y-x, x) K_1(y, x).$$

(2 . 2) follows from this equation (by integration), since $K_1(x, x)$ is equal to one.

To sum up : if we put

$$\lambda = \lambda_2, \tag{2 . 31}$$

$$K(y, x) = 0 \qquad (y \geqslant 0,\ x \geqslant 0,\ y \geqslant x > y-\lambda_1) \tag{2 . 32}$$

$$= \tfrac{1}{2}\nu(y-x, x) \exp\left[-\int_x^y \mu(y'-x, x)\, dy'\right]$$

$$(y \geqslant \lambda_1,\ x \geqslant 0,\ y-\lambda_1 \geqslant x \geqslant y-\lambda_2), \tag{2 . 33}$$

$$f(y) = \tfrac{1}{2}\int_0^{\lambda_2-y} \nu(y+x, -x) \exp\left[-\int_x^y \mu(y'+x, -x)\, dy'\right] \phi(x)\, dx$$

$$(\lambda_2 \geqslant y \geqslant 0), \tag{2 . 34}$$

$u(y)$, the density of births in the community at time y, is given by the equations

$$u(y) = \int_{y-\lambda}^{y} K(y, x)\, u(x)\, dx \quad (y > \lambda) \tag{2 . 41}$$

$$= f(y) + \int_0^y K(y, x)\, u(x)\, dx \quad (\lambda \geqslant y \geqslant 0). \tag{2 . 42}$$

In this equation $K(y, x)$ is a never negative function, which is defined in the strip

$$y \geqslant 0, \quad x \geqslant 0, \quad y \geqslant x \geqslant y-\lambda\,;$$

$f(y)$ is never negative, and $f(\lambda)$ is equal to zero, and $u(y)$ is never negative.

3. In this and the following four sections, the solution of equations (2 . 41, 42) will be considered, the exact solution in § 4, and the asymptotic formula for $u(y)$, when y is large, in §§ 5–7.

It will be assumed that $K(y, x)$ is bounded and never negative in the infinite strip $y \geqslant 0$, $x \geqslant 0$, $y \geqslant x \geqslant y-\lambda$, and that its discontinuities, if it has any, are "regularly distributed"* in any finite portion of the strip. It will be assumed, further, that $f(y)$ is continuous in the interval $\lambda \geqslant y \geqslant 0$ and that $f(\lambda) = 0$. Finally it will be assumed that $u(y)$ is a continuous function.

* Cf. M. Bôcher, "Introduction to the theory of integral equations", *Cambridge Tracts*, No. 10, 3.

Equations of the type of (2 . 41, 42) are reducible to integral equations of the second kind. But the asymptotic formula, which is required, is not a consequence of the general theory, but depends essentially on the fact that $K(y, x)$ is zero if x is less than $y-\lambda$. (2 . 41, 42) may be considered as the generalization of a difference equation just as the general integral equation of the second kind is a generalization of a set of linear equations. An equation of the same form as (2 . 41, 42), but with $K(y, x)$ a function of $y-x$ only, has been studied in connexion with the theory of electrons by Paul Hertz and G. Herglotz*, and a beautiful analytical method of dealing with it is given by the latter. Some of his results are quoted below in § 9, but the method is not applicable to kernels of more general form. The essential part of the argument used in § 5 is the same as that used by Oskar Perron† in connexion with the theory of Jacobi's algorithm.

4. I shall begin by considering the existence of $u(y)$ and its expression as a function of $f(y)$ and $K(y, x)$. These follow from the theory of integral equations of the second kind, that is equations of the form

$$u'(y) = f'(y) + \int_a^y K'(y, x)\, u'(x)\, dx \quad (y \geqslant a). \tag{4 . 2}$$

For‡ if $K'(y, x)$ is finite and its discontinuities, if it has any, are regularly distributed, a necessary and sufficient condition that (4 . 2) should possess a continuous solution is that $f(x)$ be continuous, and if this condition is fulfilled, (4 . 2) has only one continuous solution, which is given by the equation

$$u'(y) = f'(y) - \int_a^y k'(y, x)\, f'(x)\, dx, \tag{4 . 3}$$

where $k'(y, x)$ is the function, reciprocal to $K'(y, x)$, defined by the absolutely and uniformly convergent series

$$\begin{aligned} -k'(y, x) = K'(y, x) &+ \int_x^y K'(y, \xi)\, K'(\xi, x)\, d\xi \\ &+ \int_x^y K'(y, \xi) \int_x^\xi K'(\xi, \xi')\, K'(\xi', x)\, d\xi\, d\xi' \\ &+ \ldots . \end{aligned} \tag{4 . 31}$$

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* *Math. Annalen*, 65 (1908), 1–86 and 87–106.

† *Math. Annalen*, 64 (1907), 1–76. The theorem which corresponds to (6 . 4) below is that given on p. 12 (Satz II).

‡ Max. Bôcher, *loc. cit.*, 19–24.

$k'(y, x)$ is a finite function, and it is discontinuous only at the points at which $K'(y, x)$ is.

If, in equation (2 . 41, 42), we put $f(y)$ equal to zero, when y is greater than or equal to λ, and $K(y, x)$ zero, if x is less than $y-\lambda$, the hypothesis of (4 . 3) is satisfied, so that, *if* $f(y)$ *and* $K(y, x)$ *satisfy the conditions stated in* § 3, (2 . 41, 42) *possesses one and only one continuous solution, which is given by the formulae*

$$u(y) = f(y) - \int_0^y k(y, x) f(x)\, dx \quad (\lambda \geqslant y \geqslant 0)$$

$$= -\int_0^\lambda k(y, x) f(x)\, dx \quad (y \geqslant \lambda), \tag{4 . 4}$$

where $k(y, x)$ *is the function reciprocal to* $K(y, x)$.

The reciprocal functions $K'(y, x)$ and $k'(y, x)$ satisfy the equations*

$$k'(y, x) + K'(y, x) = \int_x^y k'(y, \xi)\, K'(\xi, x)\, d\xi$$

$$= \int_x^y K'(y, \xi)\, k'(\xi, x)\, d\xi,$$

and these reduce in the present case to

$$k(y, x) + K(y, x) = \int_x^y k(y, \xi)\, K(\xi, x)\, d\xi \quad (x+\lambda \geqslant y \geqslant x), \tag{4 . 41}$$

$$= \int_x^y K(y, \xi)\, k(\xi, x)\, d\xi \quad (x+\lambda \geqslant y \geqslant x), \tag{4 . 42}$$

$$k(y, x) = \int_x^{x+\lambda} k(y, \xi)\, K(\xi, x)\, d\xi \quad (y > x+\lambda), \tag{4 . 43}$$

$$= \int_{y-\lambda}^y K(y, \xi)\, k(\xi, x)\, d\xi \quad (y > x+\lambda), \tag{4 . 44}$$

Further, in the present case, $-k(y, x)$ is a finite never negative function of its arguments in the area $y \geqslant 0$, $x \geqslant 0$, $y \geqslant x$, and it is discontinuous only at the points at which $K(y, x)$ is and possibly in addition on the line $y-x = \lambda$. In particular, $-k(y, x)$ is continuous if $y-x > \lambda$.

Before concluding this section, it will be convenient to prove two formulae which will be used in § 4 and § 5.

If $\xi > \xi'+\lambda$ *and if* $u(\theta)$ *and* $v(\theta)$ *are continuous* $(\xi \geqslant \theta \geqslant \xi')$ *and if*

$$u(\theta) = \int_{\theta-\lambda}^\theta K(\theta, \theta')\, u(\theta')\, d\theta' \quad (\xi'+\lambda \leqslant \theta \leqslant \xi),$$

---

* M. Bôcher, *loc. cit.*, 21, Eq. (9).

and
$$v(\theta) = \int_{\theta}^{\theta+\lambda} v(\theta')\, K(\theta', \theta)\, d\theta' \quad (\xi' \leqslant \theta \leqslant \xi - \lambda), \quad (4.501)$$

then
$$\int_{\xi-\lambda}^{\xi} v(\theta) \left\{ u(\theta) - \int_{\xi-\lambda}^{\theta} K(\theta, \theta')\, u(\theta')\, d\theta' \right\} d\theta$$
$$= \int_{\xi'}^{\xi'+\lambda} v(\theta) \left\{ u(\theta) - \int_{\xi'}^{\theta} K(\theta, \theta')\, u(\theta')\, d\theta' \right\} d\theta. \quad (4.51)$$

To prove this, let

$$F(t) = \int_{t-\lambda}^{t} v(\theta) \left\{ u(\theta) - \int_{t-\lambda}^{\theta} K(\theta, \theta')\, u(\theta')\, d\theta' \right\} d\theta.$$

Then $F(t)$ is continuous and differentiable, and if we differentiate with respect to t,

$$F'(t) = v(t) \left\{ u(t) - \int_{t-\lambda}^{t} K(t, \theta')\, u(\theta')\, d\theta' \right\} - v(t-\lambda)\, u(t-\lambda)$$
$$+ u(t-\lambda) \int_{t-\lambda}^{t} v(\theta)\, K(\theta, t-\lambda)\, d\theta$$
$$= 0 \quad (\xi' + \lambda \leqslant t \leqslant \xi).$$

Hence $F(\xi) = F(\xi' + \lambda)$, which proves the formula.

It is clear that this proposition will still hold if $u(\theta)$ has a finite number of abrupt finite changes of value in the interval $(\xi' \leqslant \theta \leqslant \xi' + \lambda)$ and if, similarly, $v(\theta)$ has a finite number of similar discontinuities in the interval $\xi - \lambda \leqslant \theta \leqslant \xi$. We may therefore substitute $k(\theta, x)$ for $u(\theta)$, provided that $x \leqslant \xi'$, and $k(y, \theta)$ for $v(\theta)$, provided that $y \geqslant \xi$.

Put, now, $u(\theta) = k(\theta, \xi')$, then

$$\int_{\xi-\lambda}^{\xi} v(\theta) \left\{ k(\theta, \xi') - \int_{\xi-\lambda}^{\theta} K(\theta, \theta')\, k(\theta', \xi')\, d\theta' \right\} d\theta$$
$$= \int_{\xi'}^{\xi'+\lambda} v(\theta) \left\{ k(\theta, \xi') - \int_{\xi'}^{\theta} K(\theta, \theta')\, k(\theta', \xi')\, d\theta' \right\} d\theta$$
$$= - \int_{\xi'}^{\xi'+\lambda} v(\theta)\, K(\theta, \xi')\, d\theta, \text{ by (4.42),}$$
$$= -v(\xi').$$

If $\xi - \lambda > \xi' + \lambda$, the left-hand side of this equation is equal to

$$\int_{\xi-\lambda}^{\xi} v(\theta) \int_{\theta-\lambda}^{\xi-\lambda} K(\theta, \theta')\, k(\theta', \xi')\, d\theta'\, d\theta, \text{ by (4.44);}$$

hence, if $v(\theta)$ satisfies (4.501),

$$v(\xi') = - \int_{\xi-\lambda}^{\xi} v(\theta) \int_{\theta-\lambda}^{\xi-\lambda} K(\theta, \theta')\, k(\theta', \xi')\, d\theta'\, d\theta \quad (\xi \geqslant \xi' + 2\lambda). \quad (4.52)$$

Put, in this equation,

$$\xi' = x, \quad v(\xi') = k(y, \xi');$$

$$k(y, x) = -\int_{\xi-\lambda}^{\xi} k(y, \theta) \int_{\theta-\lambda}^{\xi-\lambda} K(\theta, \theta')\, k(\theta', x)\, d\theta'\, d\theta$$

$$(y \geqslant \xi \geqslant x+2\lambda). \quad (4.53)$$

Finally, change the order of integration in (4.51), which is legitimate, and write $\xi = y$, $v(\theta) = k(y, \theta)$, $u(\theta) = k(\theta, x)$. We get

$$\int_{y-\lambda}^{y} k(\theta', x) \left\{ k(y, \theta') - \int_{\theta'}^{y} k(y, \theta)\, K(\theta, \theta')\, d\theta \right\} d\theta'$$

$$= \int_{\xi'}^{\xi'+\lambda} k(\theta', x) \left\{ k(y, \theta') - \int_{\theta'}^{\xi'+\lambda} k(y, \theta)\, K(\theta, \theta')\, d\theta \right\} d\theta'.$$

The left-hand side of this equation is equal to $-k(y, x)$ by (4.41, 43), and the right to

$$\int_{\xi'}^{\xi'+\lambda} k(\theta', x) \int_{\xi'+\lambda}^{\theta'+\lambda} k(y, \theta)\, K(\theta, \theta')\, d\theta\, d\theta' \quad (y > \xi'+2\lambda).$$

Hence

$$k(y, x) = -\int_{\xi'}^{\xi'+\lambda} k(\theta', x) \int_{\xi'+\lambda}^{\theta'+\lambda} k(y, \theta)\, K(\theta, \theta')\, d\theta\, d\theta'$$

$$(y > \xi'+2\lambda \geqslant x+2\lambda). \quad (4.54)$$

5. I now come to the asymptotic formula. Hitherto no limitations, except as regards continuity, have been placed upon $K(y, x)$; in future we make the following assumption:

Numbers α, β, ϵ exist such that

$$\alpha > \beta \geqslant 0, \quad \epsilon > 0,$$

and $$K(y, x) > \epsilon \quad \text{provided that} \quad \alpha \geqslant y - x \geqslant \beta. \quad (5.1)$$

The main proposition which will be proved is:

If $u(y)$ is a continuous solution of (2.41, 42), and if $K(y, x)$ and $f(x)$ satisfy the conditions stated in § 3, and if, in addition, $K(y, x)$ satisfies (5.1), then

$$u(y) = -k(y, 0) \int_0^{\lambda} f(x)\, v(x)\, dx + o\{k(y, 0)\, e^{-\eta y}\}, \quad (7.7)$$

where $v(x)$ is the function defined in § 7 below, and η is a positive constant which depends merely upon λ, α, β, ϵ, and M, the upper bound of $K(y, x)$.

In the next three sections it will be supposed that $K(y, x)$ satisfies (5.1) and that numbers α, β, ϵ are chosen once for all, such that (5.1)

is satisfied; and the symbol η, with or without a suffix*, will be used for a positive number which depends merely upon α, β, ϵ, λ, and M. The first step is to show that

$$\eta_1^{y-x} > -k(y, x) > \eta_2^{y-x} \quad (y-x \geqslant \eta_3). \tag{5 . 2}$$

The first part of this inequality follows at once from (4 . 45), for we have

$$\begin{aligned} -k(y, x) &= K(y, x)+\int_x^y K(y, \xi)\, K(\xi, x)\, d\xi+\ldots \\ &\leqslant M+M^2(y-x)+\frac{M^3}{2!}(y-x)^2+\ldots \\ &\leqslant Me^{M(y-x)}. \end{aligned}$$

In order to prove the second part we show first, by induction, that for any δ_0 less than $\frac{1}{2}(\alpha-\beta)$ and for any positive integer n, if

$$\begin{gathered} x+\alpha+n(\alpha-\delta_0) \geqslant y \geqslant x+\beta+n(\beta+\delta_0), \qquad (5 . 25) \\ -k(y, x) \geqslant \epsilon^{n+1}\delta_0^n. \end{gathered}$$

Now, by (4 . 45), if $x+\alpha \geqslant y \geqslant x+\beta$,

$$-k(y, x) \geqslant K(y, x) \geqslant \epsilon.$$

Thus the formula holds if n is put equal to zero. Assume that it holds for $n = n_0$. Now we have, by (4 . 22, 4 . 44),

$$-k(y, x) \geqslant -\int_{y-\alpha}^{y-\beta} K(y, \xi)\, k(\xi, x)\, d\xi \quad (y \geqslant x+\alpha).$$

Suppose that the values of ξ in this integral are represented by lengths measured from the same origin along a straight line. Then, since

$$K(y, \xi) \geqslant \epsilon \quad \text{provided that} \quad y-\beta \geqslant \xi \geqslant y-\alpha,$$

and

$$-k(\xi, x) \geqslant \epsilon^{n+1}\delta_0^n$$

provided that $\quad x+\alpha+n_0(\alpha-\delta_0) \geqslant \xi \geqslant x+\beta+n_0(\beta+\delta_0)$,

it follows that

$$-k(y, x) \geqslant \epsilon^{n_0+2}\delta_0^n\,\delta,$$

where δ is the common part of the two intervals

$$(y-\beta,\ y-\alpha) \quad \text{and} \quad \{(x+\alpha+n_0(\alpha-\delta_0),\ x+\beta+n_0(\beta+\delta_0)\};$$

and we shall have $\delta \geqslant \delta_0$, provided that

$$x+\alpha+n_0(\alpha-\delta_0)-(y-\alpha) \geqslant \delta_0$$

* η's with the same suffix represent the same number in all equations; those without a suffix do not necessarily do so.

and $$y-\beta-(x+\beta+n_0\overline{\beta+\delta_0}) \geqslant \delta_0,$$

that is, provided that

$$x+\alpha+(n_0+1)(\alpha-\delta_0) \geqslant y \geqslant x+\beta+(n_0+1)(\beta+\delta_0).$$

Hence $$-k(y, x) \geqslant \epsilon^{n_0+2}\,\delta_0^{n_0+1}$$

if $$x+\alpha+(n_0+1)(\alpha-\delta_0) \geqslant y \geqslant x+\beta+(n_0+1)(\beta+\delta_0).$$

This shows that the formula (5 . 25) holds for $n = n_0+1$ if it holds for $n = n_0$, and, since it has been shown to hold for $n_0 = 0$, it must, therefore, hold for all positive integral values of n.

Now the interval $[x+\alpha+n(\alpha-\delta_0),\ x+\beta+n(\beta+\delta_0)]$ overlaps the interval $[x+\alpha+(n+1)(\alpha-\delta_0),\ x+\beta+(n+1)(\beta+\delta_0)]$ provided that

$$n > \frac{2\beta+\delta_0-\alpha}{\alpha-\beta-2\delta_0}$$

and, therefore, if we take n to be the integer next to and not greater than $(y-x-\beta)/(\beta+\delta_0)$ it follows that, for all sufficiently large values of $y-x$,

$$-k(y, x) \geqslant \epsilon^{n+1}\,\delta_0^n = \eta^{y-x} \quad (y-x > \eta').$$

This proves (5 . 2).

Let, now, $y \geqslant y' \geqslant \eta_3+x+2\lambda$ and write $\xi = y'-\eta_3$.
We have, by (4 . 53),

$$-k(y, x) = \int_{\xi-\lambda}^{\xi} k(y, \theta) \int_{\theta-\lambda}^{\xi-\lambda} K(\theta, \theta')\, k(\theta', x)\, d\theta'\, d\theta,$$

and hence, substituting for $k(y, \theta)$ from (5 . 2),

$$(5\,.\,3) \quad -\eta_1^{y-\xi+\lambda} \int_{\xi-\lambda}^{\xi} \int_{\theta-\lambda}^{\xi-\lambda} K(\theta, \theta')\, k(\theta', x)\, d\theta'\, d\theta > -k(y, x) > -\eta_2^{y-\xi+\lambda} \int_{\xi-\lambda}^{\xi} \int_{\theta-\lambda}^{\xi-\lambda} K(\theta, \theta')\, k(\theta', x)\, d\theta'\, d\theta.$$

This assumes that $$\eta_1 \geqslant 1 \geqslant \eta_2,$$

which is obviously legitimate.

Writing $y = y'$, in (5 . 3),

$$-\eta_1^{\eta_3+\lambda} \int_{\xi-\lambda}^{\xi} \int_{\theta-\lambda}^{\xi-\lambda} K(\theta, \theta')\, k(\theta', x)\, d\theta'\, d\theta \geqslant -k(y', x) \geqslant -\eta_2^{\eta_3+\lambda} \int_{\xi-\lambda}^{\xi} \int_{\theta-\lambda}^{\xi-\lambda} K(\theta, \theta')\, k(\theta', x)\, d\theta'\, d\theta.$$

The double integral which occurs in these inequalities cannot be zero, for, if it were, $k(y, x)$ would be zero, contrary to (5 . 2). Hence, divid-

ing one by the other, we have

$$\frac{\eta_1^{y-\xi+\lambda}}{\eta_2^{\eta_3+\lambda}} \geqslant \frac{k(y, x)}{k(y', x)} \geqslant \frac{\eta_2^{y-\xi+\lambda}}{\eta_1^{\eta_3+\lambda}};$$

that is

$$\eta_4 \eta_1^{y-y'} \geqslant \frac{k(y, x)}{k(y', x)} \geqslant \eta_5 \eta_2^{y-y'} \quad (y \geqslant y' \geqslant x+\eta_6). \quad (5.4)$$

6. The next step is to show that $k(y, x)/k(y, 0)$ tends to a limit as $y \to \infty$.

To prove this, write

$$w(y) = \frac{k(y, x)}{k(y, 0)},$$

and we have, by (4.54),

$$\begin{aligned} w(y) &= \frac{k(y, x)}{k(y, 0)} \\ &= -\int_{\xi}^{\xi+\lambda} \frac{k(\theta', x)}{k(y, 0)} \int_{\xi+\lambda}^{\theta+\lambda} k(y, \theta) K(\theta, \theta') \, d\theta \, d\theta' \\ &\qquad\qquad (y > \xi+2\lambda \geqslant x+2\lambda) \\ &= -\int_{\xi}^{\xi+\lambda} w(\theta') \frac{k(\theta', 0)}{k(y, 0)} \int_{\xi+\lambda}^{\theta'+\lambda} k(y, \theta) K(\theta, \theta') \, d\theta \, d\theta' \\ &= \int_{\xi}^{\xi+\lambda} w(\theta') f(y, \xi, \theta') \, d\theta' \quad (y > \xi+2\lambda \geqslant x+2\lambda), \qquad (6.1) \end{aligned}$$

where $f(y, \xi, \theta')$ is never negative, and

$$\begin{aligned} \int_{\xi}^{\xi+\lambda} f(y, \xi, \theta') \, d\theta' &= -\int_{\xi}^{\xi+\lambda} \frac{k(\theta', 0)}{k(y, 0)} \int_{\xi+\lambda}^{\theta'+\lambda} k(y, \theta) K(\theta, \theta') \, d\theta \, d\theta' \\ &= 1, \text{ by } (4.54), \ (y > \xi+2\lambda). \qquad (6.11) \end{aligned}$$

Now $k(y, x)$ and $k(y, 0)$ are continuous if $y-x \geqslant \lambda$ and $-k(y, 0)$ is positive, by (5.2), provided that $y \geqslant \eta_3$. Therefore $w(y)$ is continuous, provided that $y-x \geqslant \eta_3+\lambda$; and it obviously is never negative. Thus it follows that $w(y)$ has a definite maximum and minimum value in any interval $\xi+\lambda \geqslant y \geqslant \xi$ provided that $\xi \geqslant x+\eta_3+\lambda$, and if M_ξ is the maximum value and N_ξ the minimum value, then M_ξ and N_ξ are finite and positive.

Now if $y > \xi+2\lambda$, and $\xi \geqslant x+\eta_3+\lambda$, then we have, by (6.11),

$$\begin{aligned} M_\xi &= \int_{\xi}^{\xi+\lambda} M_\xi f(y, \xi, \theta') \, d\theta' \\ &\geqslant \int_{\xi}^{\xi+\lambda} w(\theta') f(y, \xi, \theta') \, d\theta' \\ &= w(y), \text{ by } (6.1). \end{aligned}$$

Similarly $w(y) \geqslant N_\xi$, so that

$$M_\xi \geqslant w(y) \geqslant N_\xi \quad (y > \xi + 2\lambda \geqslant x + \eta_3 + 3\lambda). \quad (6.12)$$

In particular, suppose that $\xi_1, \xi_2, \ldots$ is a positive sequence tending to infinity and such that

$$\xi_n > \xi_{n-1} + 2\lambda, \quad \xi_1 > x + \eta_3 + \lambda,$$

and that M_n is the largest and N_n the smallest value of $w(y)$ in the interval $\xi_n + \lambda \geqslant y \geqslant \xi$; then we have, by (6.12),

$$M_1 \geqslant M_2 \geqslant M_3 \ldots \geqslant M_n \geqslant N_n \ldots \geqslant N_1.$$

Hence both M_n and N_n tend to a limit as n tends to infinity.

Finally, take $\xi_1 = x + \eta_6, \quad \xi_n = \xi_{n-1} + \eta_6 + 2\lambda,$

and let $M_n = w(y_n), \quad N_n = w(y'_n).$

It has already been shown that there are such values as y_n and y'_n in the interval $\xi_n + \lambda \geqslant y \geqslant \xi_n$.

Then we have, by (6.1),

$$M_n = w(y_n) = \int_{\xi_{n-1}}^{\xi_{n-1}+\lambda} w(\theta') f(y_n, \xi_{n-1}, \theta')\, d\theta'$$

$$= M_{n-1} - \int_{\xi_{n-1}}^{\xi_{n-1}+\lambda} \{M_{n-1} - w(\theta')\} f(y_n, \xi_{n-1}, \theta')\, d\theta' ;$$

that is $$M_n = M_{n-1} - \int_{\xi_{n-1}}^{\xi_{n-1}+\lambda} [M_{n-1} - w(\theta')] f(y_n, \xi_{n-1}, \theta')\, d\theta' ; \quad (6.21)$$

and similarly

$$N_n = N_{n-1} + \int_{\xi_{n-1}}^{\xi_{n-1}+\lambda} [w(\theta') - N_{n-1}] f(y'_n, \xi_{n-1}, \theta')\, d\theta'. \quad (6.22)$$

Since, now, if $\xi_{n-1} + 2\lambda \geqslant \theta \geqslant \xi_{n-1} + \lambda$, $y_n - \theta$, and $y'_n - \theta$ are not less than η_6, it follows, from (5.4), that

$$\frac{k(y_n, \theta)}{k(y'_n, \theta)} \geqslant \eta, \quad \frac{k(y'_n, 0)}{k(y_n, 0)} \geqslant \eta' ;$$

therefore $$\frac{k(y_n, \theta)}{k(y_n, 0)} \geqslant \eta_7 \frac{k(y'_n, \theta)}{k(y'_n, 0)}.$$

Hence $$f(y_n, \xi_{n-1}, \theta') = -\frac{k(\theta', 0)}{k(y_n, 0)} \int_{\xi_{n-1}+\lambda}^{\theta'+\lambda} k(y_n, \theta) K(\theta, \theta')\, d\theta$$

$$\geqslant -\eta_7 \frac{k(\theta', 0)}{k(y'_n, 0)} \int_{\xi_{n-1}+\lambda}^{\theta'+\lambda} k(y'_n, \theta) K(\theta, \theta')\, d\theta$$

$$= \eta_7 f(y'_n, \xi_{n-1}, \theta').$$

Substituting this in (6 . 21), we have

$$M_n \leqslant M_{n-1} - \eta_7 \int_{\xi_{n-1}}^{\xi_{n-1}+\lambda} \{M_{n-1} - w(\theta')\} f(y'_n, \xi_{n-1}, \theta')\, d\theta',$$

and, by (6 . 22),

$$N_n \geqslant N_{n-1} + \eta_7 \int_{\xi_{n-1}}^{\xi_{n-1}+\lambda} \{w(\theta') - N_{n-1}\} f(y'_n, \xi_{n-1}, \theta')\, d\theta' \quad (\eta_7 < 1);$$

and, subtracting,

$$M_n - N_n \leqslant (M_{n-1} - N_{n-1})(1-\eta_7) \leqslant \ldots \leqslant (M_1 - N_1)(1-\eta_7)^{n-1}.$$

Hence it follows that $M_n - N_n$ tends to zero as n tends to infinity, and, therefore, since M_n and N_n each tends to a limit, that the limit is the same in each case.

Let $v(x)$ be the limit. Then $v(x)$ is positive, for it is not less than N_1.

Suppose, now, that $y \geqslant x + \eta_6 + 2\lambda$, and take n to be the integer next to and not greater than $(y - x - \eta_6 - 2\lambda)/(\eta_6 + 2\lambda)$; then we have

$$\xi_{n+2} + 2\lambda > y \geqslant \xi_{n+1} + 2\lambda,$$

and hence, by (6 . 11),

$$|w(y) - v(x)| \leqslant (M_{n+1} - N_{n+1}) \leqslant M_1 (1-\eta_7)^n.$$

Since M_1 is the largest value of $k(y, x)/k(y, 0)$ in the interval

$$x + \eta_6 + \lambda \geqslant y \geqslant x + \eta_6,$$

$$M_1 = \frac{k(y_1, x)}{k(y_1, 0)} = \frac{k(y_1, x)}{k(x+\eta_6, 0)} \frac{k(x+\eta_6, 0)}{k(y_1, 0)} \leqslant - \frac{\eta_8}{k(x+\eta_6, 0)}$$

and

$$(1-\eta_1)^n \leqslant \eta_9 e^{-\eta(y-x)}.$$

Hence

$$\frac{k(y, x)}{k(y, 0)} = w(y) = v(x) + \frac{\eta(y, x)\, e^{-\eta(y-x)}}{k(x+\eta_6, 0)} \quad (y - x > \eta_{10}), \quad (6 . 4)$$

where

$$|\eta(y, x)| \leqslant \eta_{11}.$$

7. I shall give in this section two propositions concerning $v(x)$, and then prove the proposition stated at the beginning of § 5.

Consider first $v(x)$ as a function of x. It is positive for all values of x, as has already been shown, and $v(0) = 1$. It is continuous, since $k(y, x)/k(y, 0)$ is continuous, if $y \geqslant \eta_3$, and tends uniformly to $v(x)$ in any finite interval $b \geqslant x \geqslant 0$, and it satisfies the equation

$$v(x) = \int_x^{x+\lambda} v(\xi)\, K(\xi, x)\, d\xi \quad (x \geqslant 0), \qquad (7 . 1)$$

as follows from equation (4 . 43) and the uniformity of the convergence of $k(y, x)/k(y, 0)$.

These properties are sufficient to characterize $v(x)$ completely, that is to say :

If $K(y, x)$ satisfy the conditions stated in the hypothesis of (7 . 7), *there exists one, and only one, function $v(x)$ which is positive and continuous and equal to one when x is zero and which satisfies* (7 . 1), *and it is equal to* $\lim_{y\to\infty} k(y, x)/k(y, 0)$. That $\lim_{y\to\infty} k(y, x)/k(y, 0)$ exists and satisfies these conditions has just been shown. To prove that there is no other function which does, let $v'(x)$ be any one; then we have, by (4 . 52), writing x for ξ',

$$v'(x) = -\int_{\xi-\lambda}^{\xi} v'(\theta) \int_{\theta-\lambda}^{\xi-\lambda} K(\theta, \theta')\, k(\theta', x)\, d\theta'\, d\theta \quad (\xi > x+2\lambda).$$

Assume, now, that $\xi-2\lambda > x+\eta_{10}$, and substitute for $k(\theta', x)$ from (6 . 4). We have

$$v'(x) = -v(x)\int_{\xi-\lambda}^{\xi} v'(\theta) \int_{\theta-\lambda}^{\xi-\lambda} K(\theta, \theta')\, k(\theta', 0)\, d\theta'\, d\theta$$

$$-\frac{1}{k(x+\eta_6)} \int_{\xi-\lambda}^{\xi} v'(\theta) \int_{\theta-\lambda}^{\xi-\lambda} K(\theta, \theta')\, k(\theta', 0)\, \eta(\theta', x)\, e^{-\eta(\theta'-x)}\, d\theta'\, d\theta.$$

The first term is equal to $v(x)\ v(0) = v(x)$. The absolute value of the second is not greater than

$$\frac{-\eta_{11} e^{-\eta(\xi-x-2\lambda)} v'(0)}{k(x+\eta_6, 0)},$$

which tends to zero as ξ tends to infinity. Hence, since we can take ξ arbitrarily large,

$$v'(x) = v(x).$$

This holds for any positive value of x, so that $v'(x)$ and $v(x)$ are identical. This proves the proposition stated.

It follows from this that $v(x)$ may be *defined* as the positive continuous solution of (7 . 1), which is equal to one when x is zero.

We have, further,

$$\eta_{12} > -v(x)\, k(x, 0) > \eta_{13} \quad (x > \eta_{14}). \tag{7 . 3}$$

To prove this it will be sufficient to show that, for all sufficiently large values of N,

$$\eta > -\frac{k(N, x)\, k(x, 0)}{k(N, 0)} > \eta',$$

since the result stated follows on letting N tend to infinity. But, by (4 . 54),

$$k(N, 0) = -\int_x^{x+\lambda} k(\theta', 0) \int_{x+\lambda}^{\theta'+\lambda} k(N, \theta)\, \dot{K}(\theta, \theta')\, d\theta\, d\theta'. \qquad (7 . 31)$$

By (5 . 4)
$$\eta > \frac{k(\theta', 0)}{k(x, 0)} > \eta' \quad (x+\lambda \geqslant \theta' \geqslant x). \qquad (7 . 311)$$

And, again, by (4 . 54),

$$k(N, x) = -\int_\xi^{\xi+\lambda} k(\theta', x) \int_{\xi+\lambda}^{\theta'+\lambda} k(N, \phi)\, K(\phi, \theta')\, d\phi\, d\theta',$$

$$k(N, \theta) = -\int_\xi^{\xi+\lambda} k(\theta', \theta) \int_{\xi+\lambda}^{\theta'+\lambda} k(N, \phi)\, K(\phi, \theta')\, d\phi\, d\theta'.$$

If we choose ξ, so that $\xi - x$ is some constant A, independent of N, then, by (5 . 2),

$$\eta > \frac{k(\theta', x)}{k(\theta', \theta)} > \eta' \quad (x+A+\lambda \geqslant \theta' \geqslant x+A\,;\ x+2\lambda \geqslant \theta \geqslant x+\lambda),$$

and hence
$$\eta > \frac{k(N, x)}{k(N, \theta)} > \eta' \quad (x+2\lambda \geqslant \theta \geqslant x+\lambda). \qquad (7 . 321)$$

Substituting from (7 . 311), (7 . 321) in (7 . 31),

$$\eta > -\frac{k(N, 0)}{k(N, x)\, k(x, 0)} > \eta',$$

which is the required result*.

By the help of (7 . 3), the asymptotic expression for $k(y, x)$, which was found at the end of the last section, can be put into more convenient shape; we have

$$\frac{k(y, x)}{k(y, 0)} = v(x) \left\{1 + \frac{v(x+\eta_6)}{v(x)} \frac{\eta(y, x)}{v(x+\eta_6)\, k(x+\eta_6, 0)}\, e^{-\eta(y-x)}\right\},$$

where, by (7 . 3), $$\frac{1}{\eta_{13}} > \frac{1}{-v(x+\eta_0)\, k(x+\eta_6)} > \frac{1}{\eta_{12}},$$

* Formulae (7.3) and (7.9) are connected with the "logistic" curve, which has been used by Raymond Pearl to express the growth of a population. If in (7.9) we assume that $v(y)$ is of the form $v(y) = a + be^{-cy}$,

(7.9) takes the form $A/(a+be^{-cy}) > u(y) > A'/(a+be^{-cy})$,

where A, A', a, b, c are constants.

The logistic curve is $u(y) = A/a + be^{-cy}$.

[Cf. Pearl, *Studies in human biology* (Baltimore, 1924), 558–637.]

provided that x is large enough, and

$$\eta_{15} > \frac{v(x+\eta_6)}{v(x)} > \eta_{16}.$$

Hence $$k(y, x) = k(y, 0)\, v(x)\{1+\eta(y, x)\, e^{-\eta(y-x)}\} \quad (y-x > \eta'), \quad (7.4)$$

where $$|\eta(y, x)| < \eta''.$$

This now suffices to establish the proposition stated at the beginning of § 5. For if we substitute from (7 . 4) in the expression (4 . 4), which was found for solutions of (2 . 41, 2 . 42), we have

$$\begin{aligned} u(y) &= -\int_0^\lambda k(y, x) f(x)\, dx \\ &= -k(y, 0)\Big[\int_0^\lambda v(x) f(x)\, dx + e^{-\eta y}\int_0^\lambda \eta(y, x)\, v(x) f(x)\, e^{\eta x}\, dx\Big] \quad (y > \eta'+\lambda) \\ &= -k(y, 0)\Big[\int_0^\lambda v(x) f(x)\, dx + e^{-\eta y}\epsilon(y)\Big], \end{aligned}$$

where $$|\epsilon(y)| < \eta'' e^{\eta\lambda}\int_0^\lambda v(x)|f(x)|\, dx,$$

and, in particular, if $f(x)$ is never negative,

$$|\epsilon(y)| < \eta'''\int_0^\lambda v(x) f(x)\, dx.$$

Thus, finally :

If $K(y, x)$ and $f(x)$ satisfy the conditions stated in § 3, *and if, in addition, $K(y, x)$ satisfies* (5 . 1), *and if $u(y)$ be the continuous solution of* (2 . 41, 2 . 42), *then*

$$u(y) = -k(y, 0)\int_0^\lambda v(x) f(x)\, dx + o[e^{-\eta y} k(y, 0)], \quad (7.7)$$

and if $f(y)$ is never negative, then

$$u(y) = -k(y, 0)\int_0^\lambda v(x) f(x)\, dx\,[1+\eta''(y)\, e^{-\eta y}] \quad (y > \eta'), \quad (7.8)$$

where $|\eta'' y| \leqslant \eta''$, and $v(x)$ is the function defined in § 7.

(7 . 8) is the asymptotic formula required. The application of it to the case of a community is considered in the next section.

Before closing this section, there is one more formula to be noticed. If we substitute from (7 . 3) in (7 . 8), we have the result :

If $f(y)$ is never negative and is not identically zero, and if $u(y)$ is the continuous solution of (2 . 41, 2 . 42) *and the other conditions of the hypo-*

thesis of (7 . 7) *are satisfied, then*

$$\eta > \frac{u(y)\,v(y)}{\int_0^\lambda v(x)\,f(x)\,dx} > \eta' \quad (y > \eta''). \tag{7 . 9}$$

8. Consider the function $k(y, x)$. If we take any two short intervals x to $x+dx$, and y to $y+dy$, of which the former precedes the latter, and call them A and B, then $K(y, x)dy$ is the chance that an individual born in the former should produce a child in the latter. Similarly

$$dy \int_x^y K(y, \xi)\,K(\xi, x)\,d\xi$$

is the chance that a grandchild of his should be born in B. Thus, since

$$-k(y, x) = K(y, x) + \int_x^y K(y, \xi)\,K(\xi, x)\,d\xi + \ldots,$$

$-dyk(y, x)$ is the chance that some descendant of an individual, who was born in A, should be born in B, and $-2dyk(y, x)$ is the probable number of descendants, born in B, of a couple who were born in A.

I return now to the question asked at the beginning of the paper as to the connexion between the growth of a community during a certain period P and the instantaneous birth- and death-rates. If $u(y)$ is the density of births in the community at time y, we can write, by (7 . 8),

$$u(y) = -Ck(y, 0) + u_1(y), \tag{8 . 2}$$

where C is a constant, equal to

$$\int_0^\lambda v(x)\,f(x)\,dx,$$

and $u_1(y)$ is a variable term whose ratio to $k(y, 0)$ tends to zero more rapidly than $e^{-\eta y}$. Since both $u(y)$ and $k(y, 0)$ satisfy (2 . 41) if $y > \lambda$, $u_1(y)$ does so also. Further, if $\xi' > \lambda$,

$$\int_\xi^{\xi'+\lambda} v(\theta)\left[u_1(\theta) - \int_{\xi'}^{\theta} K(\theta, \theta')\,u_1(\theta')\,d\theta'\right] d\theta$$

$$= \int_{\xi'}^{\xi'+\lambda} v(\theta) \int_{\theta-\lambda}^{\xi'} K(\theta, \theta')\,u_1(\theta')\,d\theta'\,d\theta = 0.$$

This follows, at once, from (4 . 51) on substituting for u_1 from (8 . 2). Hence, since $v(\theta)$ and $K(\theta, \theta')$ are never negative, $u_1(\theta)$ must change sign at least once in any interval

$$\xi' \geqslant \theta \geqslant \xi' - \lambda.$$

The coefficient $$C = \int_0^\lambda v(x) f(x)\, dx$$

is necessarily positive or zero since v and f are; and it can only be zero if $f(x)$ is identically zero—that is, in the trivial case in which the members of the community at the beginning of P were too old to have any children. Therefore, apart from this exceptional case, *the curve, of which the ordinate is the density of births in the community and abscissa the time y, fluctuates about the curve*

$$x = -Ck(y, 0),$$

crossing it at least once in every generation; and the difference between the ordinates of the two curves tends to zero in proportion to either, though, in absolute magnitude, it may increase indefinitely.

Again, given two communities in which there is the same birth-rate and death-rate, the number of births at any moment in one tends, as the moment is taken later and later in P, to a constant proportion of the number of births at that moment in the other; and the difference between the proportion which actually exists at any time and the limit to which the proportion tends decreases at least in geometric ratio.

Finally, if the birth-rate and death-rate are given—and provided the conditions stated are fulfilled—the eventual age-distribution in the community is determinable; it will vary, of course, from moment to moment, but it is approximately the same at the same time in all communities in which the birth- and death-rates have their given values; in particular, it will not depend on the initial age-distribution at the beginning of P. This is so because the proportion of the population who, at time y, are between the ages of t and $t+dt$ is

$$\frac{dt\, u(y-t) \exp\left[-\int_0^t \mu(t', y-t)\, dt'\right]}{\int_{y-\lambda_4}^{y} u(x) \exp\left[-\int_0^{y-x} \mu(t', x)\, dt'\right] dx},$$

where λ_4 is the maximum age attained by members of the community, and this, by (7 . 8), tends to the limit

$$\frac{dt\, k(y-t, 0) \exp\left[-\int_0^t \mu(t', y-t)\, dt'\right]}{\int_{y-\lambda_4}^{y} k(x, 0) \exp\left[-\int_0^{y-x} \mu(t', x)\, dt'\right] dx},$$

which is independent of $f(x)$.

9. In conclusion, I shall discuss briefly a particular case*—that in which the instantaneous birth-rate and death-rate depend merely upon age. This may be called the case of a community in a constant environment.

If $\nu(t, x)$ and $\mu(t, x)$ are not functions of x, $K(y, x)$ is a function of $y-x$ alone. We have

$$K(y, x) = K(y-x),$$

$$k(y, x) = k(y-x).$$

Further, condition (5 . 1) is necessarily satisfied, provided that $K(t)$ is not zero for all values of t.

Equations (2 . 41, 2 . 42) reduce, in this case, to the form

$$\left.\begin{aligned} u(y) &= \int_0^\lambda u(y-t)\, K(t)\, dt \qquad (y > \lambda) \\ &= f(y) + \int_0^y u(y-t)\, K(t)\, dt \quad (\lambda \geqslant y \geqslant 0) \end{aligned}\right\} \quad (9 . 1)$$

Consider now the function of the complex variable z

$$F(z) = 1 - \int_0^\lambda e^{-zt} K(t)\, dt = 1 - \sum_0^\infty (-)^n \frac{z^n}{n!} \int_0^\lambda t^n K(t)\, dt. \qquad (9 . 15)$$

$F(z)$ is an integral function of z; it is continuous and is positive if z is real and positive and large, and negative if z is real, large, and negative, and it therefore possesses a real zero.

Let ρ_0 be a real zero; then $e^{-\rho_0 x}$ is continuous and positive for all real values of x, and equal to one when x is zero and

$$e^{-\rho_0 y} = \int_0^\lambda e^{-\rho_0 (y+t)} K(t)\, dt = \int_y^{y+\lambda} e^{-\rho_0 \xi} K(\xi - y)\, d\xi.$$

It follows, therefore, by the theorem given at the beginning of § 7 that

$$e^{-\rho_0 x} = \lim_{y \to \infty} \frac{k(y-x)}{k(y)} = v(x). \qquad (9 . 16)$$

It further follows from (9 . 16) that there can only be one real zero of $F(z)$, for, if ρ_1 were another we should, also, have

$$e^{-\rho_1 x} = \lim_{y \to \infty} \frac{k(y-x)}{k(y)} = e^{-\rho_0 x},$$

and, therefore, $\rho_1 = \rho_0$.

* This case was first discussed by Lotka: see a paper by Dublin and Lotka, *Journal of Amer. Statistical Assoc.* (Sept., 1925).

Finally, $F(z)$ cannot have a repeated zero at $z = \rho_0$, for, if it had,

$$F'(\rho_0) = \int_0^\lambda e^{-\rho_0 t} t\, K(t)\, dt$$

would be zero, which is impossible, since the subject of integration is never negative and is not identically zero.

Now we have

$$e^{\rho_0 y} = \int_0^\lambda e^{\rho_0(y-t)} K(t)\, dt \quad (y > \lambda)$$

$$= \int_y^\lambda e^{\rho_0(y-t)} K(t)\, dt + \int_0^y e^{\rho_0(y-t)} K(t)\, dt \quad (\lambda \geqslant y \geqslant 0).$$

Hence $e^{\rho_0 y}$ is that solution of (9 . 1) for which

$$f(y) = \int_y^\lambda e^{\rho_0(y-t)} K(t)\, dt \geqslant 0.$$

Writing, then, in (7 . 8), $u(y) = e^{\rho_0 y}$, $v(x) = e^{-\rho_0 x}$, we have

$$e^{\rho_0 y} = -k(y) \int_0^\lambda e^{-\rho_0 x} \int_x^\lambda e^{\rho_0(x-\xi)} K(\xi)\, d\xi\, dx \left[1 + \eta'(y)\, e^{-\eta y}\right]$$

$$= -k(y) \int_0^\lambda e^{-\rho_0 x} x\, K(x)\, dx \left[1 + \eta'(y)\, e^{-\eta y}\right];$$

and therefore

$$k(y) = -\frac{e^{\rho_0 y}}{\int_0^\lambda e^{-\rho_0 x} x\, K(x)\, dx} \left[1 + \eta'(y)\, e^{-\eta y}\right] \quad (\eta > 0), \quad (9 . 2)$$

where $|\, \eta'(y)\, | \leqslant \eta'$.

Finally, substituting from (9 . 2) in (7 . 8), we have the result *that any positive continuous solution of* (9 . 1) *has the asymptotic formula*

$$u(y) = e^{\rho_0 y} \frac{\int_0^\lambda e^{-\rho_0 x} f(x)\, dx}{\int_0^\lambda e^{-\rho_0 x} x\, K(x)\, dx} \left[1 + o(e^{-\eta y})\right] \quad (\eta > 0), \quad (9 . 3)$$

where ρ_0 *is the real zero of*

$$1 - \int_0^\lambda e^{-zt} K(t)\, dt.$$

This expression shows that, *if the instantaneous birth-rate and death-rate depend merely upon age, the number of births in the community at any time increases or decreases, on the whole, in fixed geometric ratio. The meaning of "on the whole" is that the curve, of which the ordinate is the number of births and the abscissa is the time* y, *fluctuates about the*

curve

$$x = \frac{\int_0^\lambda e^{-\rho_0 x} f(x)\,dx}{\int_0^\lambda e^{-\rho_0 x} x K(x)\,dx} \quad e^{\rho_0 y} = Ce^{\rho_0 y}, \tag{9.31}$$

crossing it at least once in every interval of length λ. *The constant of the geometric ratio,* e^{ρ_0}, *is fixed, given the instantaneous rates of mortality and birth, as being the only real zero of* (9.15). *As before, the eventual age-distribution in the community is the same, whatever the original age-distribution was, and, in the present case, it is the same at all sufficiently remote times, for the proportion of the population who, at time* y, *are between ages* t *and* $t+dt$ *tends to the value*

$$\frac{dt \exp\left[-\rho_0 t - \int_0^t \mu(\theta)\,d\theta\right]}{\int_0^{\lambda_4} \exp\left[-\rho_0 t' - \int_0^{t'} \mu(\theta)\,d\theta\right] dt'},$$

which is independent of y.

I shall now quote two propositions due to G. Herglotz; the proofs will be found in the article cited in § 4.

The first is that, provided that $K(t)$ is a function of limited total fluctuation, $F(z)$ possesses an infinity of zeros, of which only a finite number have their real parts greater algebraically than an arbitrarily assigned number*.

And the second†, that if ρ_h is the h-th zero of $F(z)$ in order of decreasing magnitude of real parts, and $\phi(y)$ is a real function of limited total fluctuation in the interval $0 \leqslant y \leqslant \lambda$, and $u(y)$ that solution of the integral equation

$$u(y) = \int_0^\lambda K(y-x)\,u(x)\,dx \quad (y \geqslant \lambda)$$

which is identical with $\phi(y)$ when $0 < y < \lambda$, then

$$\sum_{h=0}^{\infty} \frac{e^{\rho_h y}}{\int_0^\lambda K(t)\,e^{-\rho_h t}\,dt} \int_0^\lambda \phi(t)\,e^{-\rho_h t} \int_{\lambda-t}^\lambda K(\theta)\,e^{-\rho_h \theta}\,d\theta\,dt = \tfrac{1}{2}[u(y+0)+u(y-0)] \quad (0 < y < \lambda)$$

$$= u(y) \quad (y > \lambda). \tag{9.4}$$

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* *Loc. cit.*, 89. In quoting this and the next result I have altered the notation so as to correspond with my own.

† *Loc. cit.*, 99. These results are proved subject to the conditions that $K(t)$ and $\phi(t)$ are of limited total fluctuation and that $K(\lambda) \neq 0$. The first two conditions do not appear to be of importance from the point of view of the application to a community. The condition $K(\lambda) \neq 0$ is a limitation of rather more importance.

In this series*, the coefficient of $e^{\rho_h y}$ is a constant which can be easily shown to be equal to

$$\int_0^\lambda e^{-\rho_h t} f(t)\, dt \Big/ \int_0^\lambda e^{-\rho_h t}\, t K(t)\, dt.$$

Further, it follows from what has been already shown, that, in the present case, all the zeros of $F(z)$, except one, are complex or purely imaginary, and have their real parts less than the real zero. If that zero is positive, there may be some complex zeros whose real parts are positive, and there actually are some in certain cases, as it is easy to see, but, by the first proposition quoted above, the number whose real parts are positive is necessarily finite.

Hence, since† the series on the left of (9 . 4) is uniformly convergent in the infinite interval $y \geqslant y_0 > \lambda$, we have that for any arbitrarily assigned positive ϵ there exists an appropriate n_0 such that

$$u(y) = \sum_0^{n_0} e^{\rho_h y} \frac{\int_0^\lambda e^{-\rho_h t} f(t)\, dt}{\int_0^\lambda e^{-\rho_h t}\, tK(t)\, dt} + \epsilon(y) \quad (y \geqslant y_0 > \lambda), \tag{9 . 5}$$

where $|\epsilon(y)|$ is less than ϵ. Thus, to sum up: *In a community in which the birth-rate and death-rate are functions of age alone, the exact expression for the number of births at any time y in a given period P, consists of three parts: first, the dominant term which is a constant multiple of the function $e^{\rho_0 y}$ which varies exponentially with y; secondly, a finite number of terms, each of which is of the form*

$$Ce^{\sigma y} \begin{matrix} \sin \tau y \\ \cos \tau y \end{matrix};$$

and, thirdly, a term which remains negligibly small throughout P. (The length of the period of such periodic terms is less than twice the age of senility.) If ρ_0 is positive, so that the community as a whole increases, the amplitudes of some of the periodic terms may increase with y, though not so rapidly as $e^{\rho_0 y}$, and there may be purely periodic terms, with constant amplitudes. If ρ_0 is zero, so that the community is stationary on the whole, or negative, so that it decreases, the amplitudes of all the periodic terms diminish continually throughout P. It is

* It is assumed that $F(z)$ has no repeated zeros; but this limitation does not seem to be of importance for the present purpose.

† *Loc. cit.*, 97–98.

by setting up harmonic waves of this sort that a sudden disturbance in the number of births show itself in the birth-rate of subsequent years.

10. I propose in the next part to apply the preceding argument to the selection of varieties inherited by the Mendelian law, and I think it may be convenient to consider briefly a simpler case first. Suppose, then, that a community—for instance, one of annual flowers—has definite breeding seasons, which can be treated as momentary, and that no individual breeds in any season but that next after his birth; and let a simple pair of Mendelian factors be inherited in the community. If this is so, all the members of the community belong to one or another of three classes, the two pure types and the hybrid type, and there are rules assigning the probable types of the children, born of any couple. These are that all the children of two members of the same pure type belong to that type, and all the children of two members of different pure types are hybrids; that of the children of hybrids, half are hybrids and a-quarter are of each pure type, and of the children of a hybrid and a member of a pure type half are of that pure type, half hybrids. This being so, suppose that in the n-th season, u_n, w_n, $2v_n$ are the number born, the last being the number of hybrids. A proportion of each type will live and have children in the next breeding season; and if these proportions are supposed given, it is possible to calculate, on the assumption of random inter-breeding, the values of u_{n+1}, w_{n+1}, $2v_{n+1}$. The formulae to which this leads are

$$u_{n+1}+v_{n+1} = a_n u_n + b_n v_n,$$

$$v_{n+1}+w_{n+1} = b_n v_n + c_n w_n,$$

$$u_{n+1} w_{n+1} = v_{n+1}^2,$$

a_n, b_n, c_n being the proportion of each type that survive to have children. If we write $u_n : 2v_n : w_n = 1 : 2p_n : p_n^2$, this gives the formula

$$p_{n+1} = \frac{c_n p_n^2 + b_n p_n}{b_n p_n + a_n}.$$

If p_n tends to infinity, as n tends to infinity, the community will eventually consist of members of the w-type alone; if p_n tends to zero, of the u-type alone; and in all other cases it will contain all three indefinitely. There is one case of particular importance, in which the hybrids are indistinguishable from one of the pure types; when this is so, the factor is said to exhibit dominance, and the pure type, which the hybrids

resemble, is called the dominant type, and the other the recessive. Suppose, now, that in the difference equation just written c_n/a_n, b_n/a_n are constants independent of n. Then it is easy to show that

$$\text{if } \frac{c_n}{a_n} \geqslant \frac{b_n}{a_n} > 1, \quad \text{or } \frac{c_n}{a_n} > \frac{b_n}{a_n} = 1, \quad p_n \to \infty .$$

The case $c_n/a_n = b_n/a_n$ would include all cases of complete dominance in which the dominant is selected; the case $b_n/a_n = 1$ all cases in which the recessive is selected. Thus, whenever the hybrids are neither more advantageous nor less advantageous than both the pure types, the selected pure type eventually monopolizes the community. There are certain other results which follow as to the rate at which changes occur, but I need not mention them now. My main object is to generalize these results to the kind of community considered in the last part.

PART II.

11. I consider in the following sections the application of the preceding argument to the case of Mendelian varieties: it will be found that, in rough outline, the results follow easily from the equations of § 7. Suppose that, during a certain period P, the members of the community belong to the three zygotic types of a simple pair of Mendelian allelomorphic factors which are not correlated with sex or fertility: thus each type consists of those members of the community who possess a certain peculiarity of appearance or structure which is inherited according to the law stated in the next section; and every member of the community belongs to one type or other. If the death-rates are not the same in all three types, the distribution of the community between the types will vary with the time, and my object is to connect the changes in numbers of the types with the magnitudes of their advantages. It will be supposed that the death-rate and birth-rate in each type remain constant, for a given age, throughout the period P; that there is no difference in any type between the male and female death-rates, and that the ages of parents vary independently. By the last statement it is meant that the age distribution in the fathers of children whose mothers were aged t at the children's birth is independent of t, and so is the age distribution in the mothers of children whose fathers were of age t. The last two assumptions do not correspond to the facts, and are introduced to simplify the argument. Perhaps it may be said that in so far as the presence of a death-rate, varying with sex, or of a correlation between the ages of

parents affects the proportions of the types, selection is no longer ordinary selection, but is partly sex-limited or sexual.

12. Let x be any moment in P, and, in the short interval x to $x+dx$, let $u(x)dx$ and $w(x)dx$ be the numbers of the pure types born, and $2v(x)dx$ be the number of hybrids born. At a subsequent moment y, a certain proportion of the individuals born between x and $x+dx$ will still survive; let the proportion of the three types who still survive be respectively $K_1(y, x)$, $K_3(y, x)$, $K_2(y, x)$. Of those that do survive, a certain proportion will become parents in the short interval between y and $y+dy$. Suppose, now, that $\pi_y(t, \theta)\, dt\, d\theta\, dy$ is the total number of couples of individuals of respective ages between t and $t+dt$ and between θ and $\theta+d\theta$, who produce a child in the interval between y and $y+dy$. Then, it being supposed that there is no sexual selection of the typical characteristics, and no variation in fertility between the three types, the total number of couples, consisting of a member of the u-type, whose age is between t and $t+dt$, and a member of the v-type, whose age is between θ and $\theta+d\theta$, who produce a child between y and $y+dy$, is

$$\frac{2K_1(y, y-t)\, K_2(y, y-\theta)\, u(y-t)\, v(y-\theta)\, \pi_y(t, \theta)\, dt\, d\theta}{N(t)\, N(\theta)}\, dy,$$

where $N(t)dt$ has been written for the total number of persons between the ages t and $t+dt$, and is equal to

$$[u(y-t)\, K_1(y, y-t)+2v(y-t)\, K_2(y, y-t)+w(y-t)\, K_3(y, y-t)]\, dt.$$

The total number of children, born to members of the u-type and v-type in the interval y to $y+dy$ is, therefore,

$$dy \int_{\lambda_1}^{\lambda_2} \int_{\lambda_1}^{\lambda_2} \frac{2\pi_y(t, \theta)}{N(t)\, N(\theta)}\, K_1(y, y-t)\, K_2(y, y-\theta)\, u(y-t)\, v(y-\theta)\, dt\, d\theta.$$

Since it is obvious that

$$\pi_y(t, \theta) = \pi_y(\theta, t),$$

this can also be written

$$dy \int_{\lambda_1}^{\lambda_2} \int_{\lambda_1}^{\lambda_2} \frac{\pi_y(t, \theta)}{N(t)\, N(\theta)} [K_1(y, y-t)\, K_2(y, y-\theta)\, u(y-t)\, v(y-\theta) + K_1(y, y-\theta)\, K_2(y, y-t)\, u(y-\theta)\, v(y-t)]\, dt\, d\theta.$$

Similarly the number of children born to two u's is

$$\tfrac{1}{2} dy \int_{\lambda_1}^{\lambda_2} \int_{\lambda_1}^{\lambda_2} \frac{\pi_y(t, \theta)}{N(t)\, N(\theta)}\, K_1(y, y-t)\, K_1(y, y-\theta)\, u(y-t)\, u(y-\theta)\, dt\, d\theta.$$

By the definition of Mendelian allelomorphic factors, if u is a pure type, all the offspring of u-u matches, half those of u-v matches, and a quarter those of v-v matches are u's, and none of the offspring of other combinations. Adding the three numbers together, we get

$$\tfrac{1}{2}dy\int_{\lambda_1}^{\lambda_2}\int_{\lambda_1}^{\lambda_2}\frac{\pi_y(t,\theta)}{N(t)\,N(\theta)}\,[K_1(y,\,y-t)\,u(y-t)+K_2(y,\,y-t)\,v(y-t)]$$
$$\times\,[K_1(y,\,y-\theta)\,u(y-\theta)+K_2(y,\,y-\theta)\,v(y-\theta)]\,dt\,d\theta,$$

and this must be equal to $u(y)dy$. Similarly, if v is a hybrid type, half the offspring of u-v matches, v-v matches, and v-w matches, and all the offspring of u-w matches are hybrids. We thus get

$$2v(y)=\int_{\lambda_1}^{\lambda_2}\int_{\lambda_1}^{\lambda_2}\frac{\pi_y(t,\theta)}{N(t)\,N(\theta)}\,[K_1(y,\,y-t)\,u(y-t)+K_2(y,\,y-t)\,v(y-t)]$$
$$\times\,[K_2(y,\,y-\theta)\,v(y-\theta)+K_3(y,\,y-\theta)\,w(y-\theta)]\,dt\,d\theta.$$

And similarly,

$$w(y)=\tfrac{1}{2}\int_{\lambda_1}^{\lambda_2}\int_{\lambda_1}^{\lambda_2}\frac{\pi_y(t,\theta)}{N(t)\,N(\theta)}\,[K_3(y,\,y-t)\,w(y-t)+K_2(y,\,y-t)\,v(t)]$$
$$\times\,[K_2(y,\,y-\theta)\,v(y-\theta)+K_3(y,\,y-\theta)\,w(y-\theta)]\,dt\,d\theta.$$

If we add half the second equation to the first equation and then to the third,

$$u(y)+v(y)=\tfrac{1}{2}\int_{\lambda_1}^{\lambda_2}[K_1(y,\,y-t)\,u(y-t)+K_2(y,\,y-t)\,v(y-t)]\int_{\lambda_1}^{\lambda_2}\frac{\pi_y(t,\theta)}{N(t)}\,d\theta\,dt,$$

$$v(y)+w(y)=\tfrac{1}{2}\int_{\lambda_1}^{\lambda_2}[K_2(y,\,y-t)\,v(y-t)+K_3(y,\,y-t)\,w(y-t)]\int_{\lambda_1}^{\lambda_2}\frac{\pi_y(t,\theta)}{N(t)}\,d\theta\,dt.$$

The quantity $\displaystyle\int_{\lambda_1}^{\lambda_2}\frac{\pi_y(t,\theta)}{N(t)}\,d\theta$ is what was called, in the preceding part, the instantaneous birth-rate in the community, $\nu(t,\,y-t)$. If now we introduce the assumption that the ages of parents vary independently,

$$\pi_y(t,\,\theta)=\pi_y(t)\,\pi_y(\theta),$$

and we have, in the first place, the equation

$$v^2(y)=u(y)\,w(y), \qquad (12.1$$

and, secondly, we can consider $\nu(t,\,y-t)$ as independent of the age distribution in the community, and equal to $\nu(t)$. If the ages of parents do not vary independently, $\nu(t,\,y-t)$ is necessarily a function of the age distribution. Assuming, further, that the death-rates in the three

types are constant, and writing

$$\tfrac{1}{2}K_1(y,\, y-t)\,\nu(t) = K'(t), \quad \tfrac{1}{2}K_2(y,\, y-t)\,\nu(t) = K''(t),$$

$$\tfrac{1}{2}K_3(y,\, y-t)\,\nu(t) = K'''(t), \quad (12.11)$$

we have the equations

$$u(y)+v(y) = \int_{\lambda_1}^{\lambda_2} K'(t)\,u(y-t)\,dt + \int_{\lambda_1}^{\lambda_2} K''(t)\,v(y-t)\,dt, \quad (12.2)$$

$$(y \geqslant \lambda_2)$$

$$v(y)+w(y) = \int_{\lambda_1}^{\lambda_2} K'''(t)\,w(y-t)\,dt + \int_{\lambda_1}^{\lambda_2} K''(t)\,v(y-t)\,dt, \quad (12.3)$$

$$u(y)+v(y) = f_1(y) + \int_{\lambda_1}^{y} K'(t)\,u(y-t)\,dt + \int_{\lambda_1}^{y} K''(t)\,v(y-t)\,dt, \quad (12.21)$$

$$(\lambda_2 \geqslant y \geqslant 0)$$

$$v(y)+w(y) = f_2(y) + \int_{\lambda_1}^{y} K''(t)\,v(y-t)\,dt + \int_{\lambda_1}^{y} K'''(t)\,w(y-t)\,dt, \quad (12.31)$$

where, in the last two equations, the integrals vanish if $\lambda_1 \geqslant y$, and $f_1(y)$, $f_2(y)$ are given functions of their arguments.

13. With regard to the functions $K'(t)$, etc., it will be supposed that they are continuous and never negative if $\lambda_2 \geqslant t \geqslant \lambda_1$, and that numbers α, β, ϵ exist such that $\epsilon > 0$, $\lambda_2 \geqslant \alpha > \beta \geqslant \lambda_1$, and if $\alpha \geqslant t \geqslant \beta$, $K'(t)$, $K''(t)$, $K'''(t) > \epsilon$.

It will be supposed, as in the last part, that α, β, ϵ have been chosen once for all such that these conditions are satisfied; any function $K(y, x)$, which is continuous and never negative in the strip

$$0 \leqslant x_0 \leqslant x, \quad x \leqslant y \leqslant x+\lambda,$$

and is such that, if $\alpha \geqslant y-x \geqslant \beta$, $K(y, x) > \epsilon$, will be said to satisfy conditions A; and η will be used for a constant, which depends only on α, β, ϵ, λ_2, and M, the upper bound of $K'(t)$, $K''(t)$, $K'''(t)$. It will be convenient to define $K'(t)$, $K''(t)$, $K'''(t)$ as zero, if t is negative or is greater than λ_2. There is no objection to regarding the functions so defined as continuous.

The first question that arises is as to the existence of a solution (12.1.2); but, since $\lambda_1 > 0$, its existence is obvious. If $f_1(y)$, $f_2(y)$ are arbitrary continuous functions and are never negative, the right-hand sides of (12.21.31) are given functions for $0 \leqslant y \leqslant \lambda_1$, and hence, by (12.1), $u(y)$, $v(y)$, and $w(y)$ are given in that interval. If these are substituted in the right-hand sides of (12.21.31), the values of $u(y)$, $v(y)$, and $w(y)$ are found, with the help of (12.1), for

$2\lambda_1 \geqslant y \geqslant \lambda_1$. This process can be repeated indefinitely. The functions so obtained are finite and never negative, provided that $f_1(y)$, $f_2(y)$ are finite, and are continuous, except for a possible abrupt jump at $y = \lambda_2$.

Further, since the K's satisfy conditions A, (12 . 2) gives

$$u(y)+v(y) \geqslant \epsilon \int_\alpha^\beta [u(y-t)+v(y-t)\,dt.$$

It follows* from this that, from a certain point η on,

$$u(y)+v(y) > 0\,;$$

similarly with $v(y)+w(y)$, and hence with $u(y)$, $v(y)$, and $w(y)$. Since the initial point is arbitrary, it may be supposed that

$$u(y),\ \ v(y),\ \ w(y) > 0 \quad (y \geqslant 0).$$

On the conditions assumed, the three functions of z :

$$\int_0^\lambda e^{-zt} K'(t)\,dt-1, \quad \int_0^\lambda e^{-zt} K''(t)\,dt-1, \quad \int_0^\lambda e^{-zt} K'''(t)\,dt-1,$$

have each one, and only one, real zero. Let these real zeros be ρ, σ, τ ; it may be supposed, without loss of generality, that $\rho \leqslant \tau$. Consider, now, (12 . 2) ; it may be written in the form

$$z_1(y) = \int_{y-\lambda}^{y} Q_1(y,\, x)\, z_1(x)\,dx \quad (y > \lambda), \tag{14 . 1}$$

where $$Q_1(y,\, x) = \frac{K'(y-x)\,u(x)+K''(y-x)\,v(x)}{u(x)+v(x)}.$$

Thus $Q(y,\, x)$ satisfies conditions A, since $K'(y-x)$, $K''(y-x)$ do so, and $u(x)$ and $v(x)$ are positive and continuous. Hence, by § 7, if $q_1(y,\, x)$ is the function reciprocal to $Q_1(y,\, x)$, $\lim\limits_{y\to\infty} \dfrac{q_1(y,\, x)}{q_1(y,\, 0)}$ exists, and, if we write it equal to $L_1(x)$,

$$L_1(x) = \int_x^{x+\lambda} L_1(y)\, Q_1(y,\, x)\,dy,$$

and, by (7 . 9), (14 . 1),

$$\eta > \frac{z_1(x)\, L_1(x)}{\int_0^\lambda L_1(t) f_1(t)\,dt} > \eta'.$$

* Cf. the proof of (5 . 2).

Let p be a positive quantity and ω_p the real zero of

$$\int_0^\lambda e^{-zt}\left[\frac{K'(t)+pK''(t)}{1+p}\right]dt-1,$$

and suppose that $\sigma > \rho$. For any fixed value of p, the function of z just written decreases, as z increases from ρ to σ, and it is positive if $z=\rho$, negative if $z=\sigma$; therefore $\sigma > \omega_p > \rho$. Also, if $p' > p$,

$$\begin{aligned}
&\int_0^\lambda e^{-\omega_{p'}t}\left[\frac{K'(t)+p'K''(t)}{1+p'}\right]dt-1\\
&\quad=\int_0^\lambda e^{-\omega_{p'}t}\left[\frac{K'(t)+p'K''(t)}{1+p'}\right]dt-\int_0^\lambda e^{-\omega_{p'}t}\left[\frac{K'(t)+pK''(t)}{1+p}\right]dt\\
&\quad=\frac{(p-p')}{(1+p)(1+p')}\int_0^\lambda e^{-\omega_{p'}t}[K'(t)-K''(t)]\,dt\\
&\quad=\frac{(p-p')}{(1+p)(1+p')}\left[\int_0^\lambda\{e^{-\omega_{p'}t}-e^{-\rho t}\}K'(t)\,dt+\int_0^\lambda\{e^{-\sigma t}-e^{-\omega_{p'}t}\}K''(t)\,dt>0.\right.
\end{aligned}$$

Hence, if $p' > p$, $\omega_{p'} > \omega_p$.

Take any interval $x_1 \geqslant x \geqslant x_0$, and let p' be the maximum, p the minimum value of $v(x)/u(x)$ in it; p' and p are positive and finite. Then

$$L_1(x)\,e^{\omega_{p'}x}=\int_x^{x+\lambda}L_1(y)\,e^{\omega_{p'}y}\,Q_1(y,x)\,e^{-\omega_{p'}(y-x)}\,dy,$$

where

$$\begin{aligned}
&\int_x^{x+\lambda}e^{-\omega_{p'}(y-x)}Q_1(y,x)\,dy\\
&\qquad=\frac{u(x)}{u(x)+v(x)}\int_0^\lambda e^{-\omega_{p'}t}K'(t)\,dt+\frac{v(x)}{u(x)+v(x)}\int_0^\lambda e^{-\omega_{p'}t}K''(t)\,dt\leqslant 1.
\end{aligned}$$

Hence, if M_n be the greatest value of $L_1(x)e^{\omega_{p'}x}$ in the interval

$$x_0+(n-1)\lambda \leqslant x \leqslant x_0+n\lambda,$$

$$M_n \leqslant M_{n+1} \qquad (14.21)$$

Further, by (7.3), $\eta > -L_1(x)\,q_1(x,0) > \eta' \quad (x>\eta)$,

and, by (5.4), $$\eta > \frac{q_1(x',0)}{q_1(x,0)} > \eta' \quad (x+\lambda \geqslant x' \geqslant x > \eta).$$

Therefore if N_n is the least value of $L_1(x)e^{\omega_{p'}x}$ in

$$x_0+(n-1)\lambda \leqslant x \leqslant x_0+n\lambda$$

$$\eta > \frac{M_n}{N_n} > \eta'. \qquad (14.22)$$

It follows from (14 . 21 . 22) that

$$L_1(x) > \eta e^{-\omega_{p'}(x-x_0)} L_1(x_0) \quad (x_1 \geqslant x \geqslant x_0).$$

A similar argument holds for ω_p, so that

$$L_1(x_0)\,\eta' e^{-\omega_p(x-x_0)} > L_1(x) > \eta e^{-\omega_{p'}(x-x_0)} L_1(x_0)$$

$$(\sigma > \rho\,;\ x_1 \geqslant x \geqslant x_0). \quad (14 . 3)$$

If $\rho = \sigma$, then,* by § 7,

$$L_1(x) = e^{-\rho x} \quad (\sigma = \rho). \quad (14 . 31)$$

Again, if ω_p' is the real zero of

$$\int_0^\lambda e^{-zt}\left[\frac{K''(t)+pK'''(t)}{1+p}\right]dt - 1,$$

and $L_2(x)$ the function, analogous to $L_1(x)$, for the equation (12 . 3), then, since $w(x)/v(x) = v(x)/u(x)$,

$$L_2(x_0)\,\eta' e^{-\omega'(x-x_0)} > L_2(x) > \eta e^{-\omega_{p'}'(x-x_0)} L_2(x_0)$$

$$(\tau > \sigma\,;\ x_1 \geqslant x \geqslant x_0), \quad (14 . 32)$$

$$L_2(x) = e^{-\tau x} \quad (\tau = \sigma). \quad (14 . 33)$$

In addition to these inequalities we shall require the following formula :

If ω is any constant and

$$U(y) = \int_{y-\lambda}^{y} K(y, x)\, U(x)\, dx \quad (y > \lambda),$$

where $K(y, x)$ is finite and its discontinuities, if it has any, are regularly distributed, then

$$\int_t^{t+\lambda} e^{-\omega y} \int_{y-\lambda}^{t} K(y, x)\, U(x)\, dx\, dy$$

$$= \int_\lambda^{2\lambda} e^{-\omega y}\left[U(y) - \int_\lambda^y K(y, x)\, U(x)\, dx\right]dy$$

$$- \int_\lambda^t e^{-\omega x} U(x)\left[1 - \int_0^\lambda e^{-\omega\theta} K(x+\theta, x)\, d\theta\right]dx \quad (t \geqslant \lambda). \quad (14 . 4)$$

To prove this, write the left-hand side equal to $F(t)$, and differentiate

* For, $e^{-\rho x} = \int_x^{x+\lambda} e^{-\rho y} Q_1(y, x)\, dy$, and $e^{-\rho 0} = 1$; therefore, by § 7, $\lim \dfrac{q_1(y, x)}{q_1(y, 0)} = e^{-\rho x}$.

with respect to t,

$$\frac{\partial F}{\partial t} = -e^{-\omega t}\int_{t-\lambda}^{t} K(t, x)\, U(x)\, dx + U(t)\int_{t}^{t+\lambda} K(y, t)\, e^{-\omega y}\, dy$$

$$= -U(t)\, e^{-\omega t}\left[1-\int_0^{\lambda} e^{-\omega\theta} K(t+\theta, t)\, d\theta\right].$$

Hence, $$F(t) = F(\lambda) - \int_{\lambda}^{t} U(x)\, e^{-\omega x}\left[1-\int_0^{\lambda} e^{-\omega\theta} K(\theta+x, x)\, d\theta\right] dx.$$

Since $$F(\lambda) = \int_{\lambda}^{2\lambda} e^{-\omega y}\left[U(y) - \int_{\lambda}^{y} U(x)\, K(y, x)\, dx\right] dy,$$

this proves the result.

In particular, if we write $U = z_1$, $K = Q_1$, we have

$$\int_t^{t+\lambda} e^{-\omega y}\int_{y-\lambda}^{t} [K'(y-x)\, u(x) + K''(y-x)\, v(x)]\, dx\, dy$$

$$= \int_{\lambda}^{2\lambda} e^{-\omega y}\left[u(y)+v(y) - \int_{\lambda}^{y} [u(\theta)\, K'(y-\theta) + v(\theta)\, K''(y-\theta)]\, d\theta\right] dy$$

$$-\left[1-\int_0^{\lambda} e^{-\omega\theta} K'(\theta)\, d\theta\right]\int_{\lambda}^{t} e^{-\omega\theta}\, u(\theta)\, d\theta$$

$$-\left[1-\int_0^{\lambda} e^{-\omega\theta} K''(\theta)\, d\theta\right]\int_{\lambda}^{t} e^{-\omega\theta}\, v(\theta)\, d\theta. \qquad (14.5)$$

15. Consider, now, the cases $\rho \leqslant \sigma \leqslant \tau$.

(1) $\rho = \sigma = \tau$. This is the case of no selection. By (14.33.31) $L_1(x) = L_2(x) = e^{-\rho x}$. By (7.9),

$$\eta > \frac{z_1(x)\, L_1(x)}{\int_0^{\lambda} L_1(t)\, f_1(t)\, dt} > \eta' \quad (x > \eta''), \qquad (15.01)$$

with a similar inequality for $z_2(x)$. Hence

$$\eta > \frac{z_1(x)\, e^{-\rho x}}{\int_0^{\lambda} e^{-\rho t} f_1(t)\, dt} > \eta'; \quad \eta > \frac{z_2(x)\, e^{-\rho x}}{\int_1^{\lambda} e^{-\rho t} f_2(t)\, dt} > \eta \qquad \begin{matrix}(x > \eta''),\\ (\rho = \sigma = \tau).\end{matrix} \qquad (15.1)$$

Thus $z_1(x)e^{-\rho x}$ and $z_2(x)e^{-\rho x}$ oscillate between fixed limits.

(2) $\rho < \sigma = \tau$. This includes all cases of selected dominants. By (14.33), (7.9),

$$\eta > \frac{z_2(x)\, e^{-\tau x}}{\int_0^{\lambda} e^{-\tau t} f_2(t)\, dt} > \eta' \quad (x > \eta''). \qquad (15.21)$$

Put $\omega = \tau$ in (14 . 5) :

$$\int_t^{t+\lambda} e^{-\tau y} \int_{y-\lambda}^{t} [K'(y-x)\,u(x) + K''(y-x)\,v(x)]\,dx\,dy$$

$$= \int_\lambda^{2\lambda} e^{-\tau y} \Big[u(y) + v(y) - \int_\lambda^y \{u(x)\,K'(y-x) + v(x)\,K''(y-x)\}\,dx \Big] dy$$

$$- \Big[1 - \int_0^\lambda e^{-\tau\theta} K'(\theta)\,d\theta \Big] \int_\lambda^t e^{-\tau x} u(x)\,dx \quad (t > \lambda). \quad (15 . 22)$$

Since the term on the left is never negative, and the first term on the right is independent of t, and $1 > \int_0^\lambda e^{-\tau t} K'(t)\,dt$,

$$\int_\lambda^t e^{-\tau x} u(x)\,dx$$

must be less than a constant for all values of t ; and, therefore, since $u(x)$ is positive,

$$\int_\lambda^\infty e^{-\tau x} u(x)\,dx$$

must be convergent. Hence

$$\underline{\lim}\, e^{-\tau x} u(x) = 0.$$

Multiplying (15 . 21) by $u(x)e^{-\tau x}$, we have

$$\eta u(x) e^{-\tau x} > [e^{-2\tau x} u(x)\, w(x)] / C_2 = e^{-2\tau x} v^2(x) / C_2,$$

where, here and in future,

$$C_2 = \int_0^\lambda e^{-\tau t} f_2(t)\,dt, \quad C_1 = \int_0^\lambda e^{-\rho t} f_1(t)\,dt.$$

Hence

$$\underline{\lim}\, v(x)\, e^{-\tau x} = 0, \quad \underline{\lim}\, z_1(x)\, e^{-\tau x} = 0, \quad \overline{\lim}\, L_1(x)\, e^{\tau x} = \infty .$$

But

$$L_1(x)\, e^{\tau x} = \int_x^{x+\lambda} L_1(y)\, e^{\tau y}\, Q_1(y, x)\, e^{-\tau(y-x)}\,dy,$$

and

$$\int_x^{x+\lambda} e^{-\tau(y-x)}\, Q_1(y, x)\,dy \leqslant 1.$$

Thus, as before, if M_n, N_n are the greatest and least values of $L_1(x)e^{\tau x}$ in the interval $(n-1)\lambda \leqslant x \leqslant n\lambda$,

$$M_n \leqslant M_{n+1}, \quad \eta > \frac{M_n}{N_n} > \eta' ;$$

so that, since $\overline{\lim}\, M_n = \infty$,

$$M_n,\ N_n \to \infty\ ; \quad L_1(x)\, e^{\tau x} \to \infty\ ; \quad z_1(x)\, e^{-\tau x} \to 0.$$

It follows, from this and (15 . 21), that

$$\eta > \frac{w(x)\, e^{-\tau x}}{C_2} > \eta', \quad v(x)\, e^{-\tau x} \to 0, \quad u(x)\, e^{-\tau x} \to 0 \quad (\rho < \sigma = \tau). \quad (15 . 23)$$

Further, by (14 . 3), (15 . 01), (15 . 21), in any interval $x_1 \geqslant x \geqslant x_0$, in which the maximum and minimum values of $v(x)/u(x)$ are p' and p,

$$\frac{v(x_0)}{w(x_0)}\, \eta e^{-(\tau-\omega_{p'})(x-x_0)} > \frac{v(x)}{w(x)} > \eta' e^{-(\tau-\omega_p)(x-x_0)}\, \frac{v(x_0)}{w(x_0)}.$$

$\tau - \omega_p$ tends to zero as p tends to infinity, so that, for every positive δ,

$$e^{\delta x} v(x)/w(x) \to \infty\,. \quad (15 . 24)$$

Further, since the left-hand side of (15 . 22) tends to zero,

$$\left[1 - \int_0^\lambda e^{-\tau t} K'(t)\, dt\right] \int_\lambda^\infty e^{-\tau x} u(x)\, dx$$

$$= \int_\lambda^{2\lambda} e^{-\tau y} \left[u(y) + v(y) - \int_\lambda^y \{K'(y-x)\, u(x) + K''(y-x)\, v(x)\}\, dx\right] dy$$

$$(\rho < \sigma = \tau). \quad (15 . 25)$$

(3) $\rho = \sigma < \tau$. This includes all cases of selected recessives. By (14 . 31), (7 . 9),

$$\eta > \frac{[u(x) + v(x)]\, e^{-\rho x}}{C_1} > \eta' \quad (x > \eta''). \quad (15 . 3)$$

Further, $$L_2(x)\, e^{\rho x} = \int_x^{x+\lambda} L_2(y)\, e^{\rho y}\, Q_2(y, x)\, e^{-\rho(y-x)}\, dy,$$

and $$\int_x^{x+\lambda} e^{-\rho(y-x)}\, Q_2(y, x)\, dy \geqslant 1.$$

Hence, if M_n', N_n' are the greatest and the least values of $L_2(x)e^{\rho x}$ in the interval $(n-1)\lambda \leqslant x \leqslant n\lambda$,

$$N_n' \geqslant N_{n+1}', \quad \eta > \frac{M_n'}{N_n'} > \eta' \quad (n > \eta).$$

Thus $M_n' \leqslant \eta N_{n_0}'$, if $n > n_0$, and

$$\frac{\underline{\lim}\, z_2(x)\, e^{-\rho x}}{\int_0^\lambda L_2(t) f_2(t)\, dt} \geqslant \eta\, \underline{\lim} \left[\frac{1}{L_2(x)\, e^{\rho x}}\right] \geqslant \frac{\eta}{N_{n_0}'}.$$

Hence, by (15 . 3), $$\underline{\lim}\,\frac{v(x)}{u(x)} \geqslant \frac{C_4}{C_1}\frac{\eta}{N'_{n_0}} \quad (C_4 = \text{constant}),$$

and it follows, since $v(x)/u(x)$ is positive, that, in (14 . 32), we may put $p > p_0$, where p_0 is independent of x_1 so that

$$L_2(x)\, e^{+\rho(x-x_0)} \leqslant \eta e^{-(\omega'_{p_0}-\rho)(x-x_0)} L_2(x_0) \quad (x > x_0);$$

which tends to zero as x tends to infinity. Thus

$$w(x)\, e^{-\rho x} \to \infty\,; \quad u(x)\, e^{-\rho x} \to 0\,; \quad \eta > \frac{v(x)\, e^{-\rho x}}{C_1} > \eta' \quad (\rho = \sigma < \tau). \quad (15.31)$$

Again, in any interval $x_1 \geqslant x \geqslant x_0$, in which the greatest and least values of $v(x)/u(x)$ are p' and p,

$$\frac{v(x_0)}{w(x_0)}\eta e^{-(\omega'_p-\rho)(x-x_0)} > \frac{v(x)}{w(x)} > \eta' e^{-(\omega'_{p'}-\rho)(x-x_0)}\frac{v(x_0)}{w(x_0)}.$$

But $\omega'_{p'}-\rho$ tends to zero as p' tends to zero; and it follows that, given any interval $x_0+m\lambda \geqslant x \geqslant x_0+(m-1)\lambda$, there is no positive δ, such that

$$\frac{v(x)}{w(x)} < e^{-\delta(x-x_0)}\frac{v(x_0)}{w(x_0)} \quad (\rho = \sigma < \tau), \qquad (15.32)$$

for all values of $v(x_0)/w(x_0)$.

(4) $\rho < \sigma < \tau$. This includes those cases of imperfect dominance in which the advantages of the hybrids are intermediate between those of the pure forms. As before, if M_n, N_n are the greatest and least values of $L_1(x)e^{\sigma x}$ in $n\lambda \geqslant x \geqslant (n-1)\lambda$, and M'_n, N'_n are the greatest and least values of $L_2(x)\, e^{\sigma x}$ in the same interval,

$$M_n \leqslant M_{n+1}, \quad \eta > \frac{M_n}{N_n} > \eta',$$

$$N'_{n+1} \leqslant N_n, \quad \eta > \frac{M'_n}{N'_n} > \eta' \quad (x > \eta).$$

Thus $$\eta > \frac{M_n}{N'_n}\cdot\frac{M'_n}{N_n} > \eta',$$

or, since $$\frac{M_n}{N'_n} > \eta\,\frac{z_2(x)}{z_1(x)}\,\frac{C'_2}{C'_4},$$

$$\frac{M'_n}{N_n} > \eta\,\frac{z_1(x)}{z_2(x)}\,\frac{C_4}{C'_2},$$

where C_4', C_2' are constants, if p_n' is the greatest, and p_n the least value of $z_2(x)/z_1(x)$ in $n\lambda \geqslant x \geqslant (n-1)\lambda$,

$$\eta > \frac{p_n'}{p_n} > \eta';$$

hence, for any fixed m and x_0,

$$\eta^m > \frac{p'}{p} > \eta' \quad (x_0 + m\lambda \geqslant x \geqslant x_0),$$

where p', p are the maximum and minimum of $z_2(x)/z_1(x)$ in this interval. Further,

$$\int_0^\lambda e^{-\omega_{p'}t} K'(t)\,dt + p' \int_0^\lambda e^{-\omega_{p'}t} K''(t)\,dt = 1 + p' = 1 + p' \int_0^\lambda e^{-\sigma t} K''(t)\,dt;$$

so that $$\int_0^\lambda [e^{-\omega_{p'}t} - e^{-\sigma t}]\, K''(t)\,dt = +\frac{1}{p'}\left[1 - \int_0^\lambda e^{-\omega_{p'}t} K'(t)\,dt\right].$$

Similarly, $$\int_0^\lambda [e^{-\sigma t} - e^{-\omega_p' t}]\, K''(t) = +p\left[\int_0^\lambda e^{-\omega_p' t} K'''(t)\,dt - 1\right],$$

and therefore

$$\int_0^\lambda [e^{-\omega_{p'}t} - e^{-\omega_p' t}]\, K''(t)\,dt = \frac{1}{p'}\left[1 - \int_0^\lambda e^{-\omega_{p'}t} K'(t)\,dt\right] + p\left[\int_0^\lambda e^{-\omega_p' t} K'''(t)\,dt - 1\right].$$

Each term on the right-hand side is positive; as p tends to infinity, the second term is greater than $\epsilon_1 > 0$, and, as p' tends to zero, the first term is greater than $\epsilon_2 > 0$. Hence, if p', p vary in the triangle

$$\infty \geqslant p \geqslant 0, \quad p' \geqslant \eta p,$$

where η is a fixed positive number, the right-hand side must have a positive minimum, that is, if

$$\frac{p'}{p} > \eta, \quad \int_0^\lambda [e^{-\omega_{p'}t} - e^{-\omega_p' t}]\, K''(t)\,dt > \epsilon',$$

and, therefore, $\omega_p' - \omega_{p'} > \epsilon''$.

Now, by (14 . 3), (14 . 32),

$$L_2(x) < \eta e^{-\omega_p'(x-x_0)} L_2(x_0),$$

$$L_1(x) > \eta e^{-\omega_{p'}(x-x_0)} L_1(x_0),$$

so that $$\frac{L_2(x)}{L_1(x)} < \eta e^{-\epsilon(x-x_0)} \frac{L_2(x_0)}{L_1(x_0)}.$$

Since this holds for all values of x if $\delta > 0$,

$$e^{(\epsilon-\delta)x}\frac{L_2(x)}{L_1(x)} \to 0;$$

and, hence, for a positive δ,

$$e^{\delta x}\frac{v(x)}{w(x)} \to 0 \quad (\rho < \sigma < \tau). \tag{15.41}$$

This is to be contrasted with (15.24) and (15.32). The behaviour of $u(x)$, $v(x)$, $w(x)$ at infinity follows more easily from the lemma in the next section.

There are two cases remaining, $\sigma > \tau$, ρ and $\sigma < \tau$, ρ; I shall only consider them very briefly, but I am anxious not to omit them, for the sake of a general conclusion. If $\sigma > \tau$, ρ or $\sigma < \tau$, ρ there exist a real ω and a positive l such that

$$1+l = \int_0^\lambda e^{-\omega t} K'(t)\,dt + l\int_0^\lambda e^{-\omega t} K''(t)\,dt,$$

$$1+l = \int_0^\lambda e^{-\omega t} K''(t)\,dt + l\int_0^\lambda e^{-\omega t} K'''(t)\,dt.$$

This follows on eliminating l, when we get the equation

$$\left[1-\int_0^\lambda e^{-\omega t} K''(t)\,dt\right]^2 = \left[1-\int_0^\lambda e^{-\omega t} K'(t)\,dt\right]\left[1-\int_0^\lambda e^{-\omega t} K'''(t)\,dt\right].$$

This has a root between σ and τ, if $\sigma > \tau$, and one between σ and ρ, if $\sigma < \rho$; and the corresponding values of l are positive. If $\sigma > \tau$, (14.3), (14.32) lead to the inequality:

$$\eta p(x_0)\, e^{(\omega'_p-\omega_p)(x-x_0)} > p(x) > \eta p(x_0)\, e^{(\omega'_{p'}-\omega_{p'})(x-x_0)}$$
$$(x > x_0,\ \ \sigma > \tau \geqslant \rho),$$

where $\omega'_\pi - \omega_\pi \gtrless 0$ as $\pi \lessgtr l$. Hence there is a contradiction if either $\overline{\lim}\, p(x) < l$ or $\underline{\lim}\, p(x) > l$, so that

$$\overline{\lim}\, p(x) \geqslant l \geqslant \underline{\lim}\, p(x) \quad (\sigma > \tau \geqslant \rho).$$

(2) If $\sigma < \rho$, these inequalities become

$$\eta p(x_0)\, e^{(\omega'_{p'}-\omega_{p'})(x-x_0)} > p(x) > \eta p(x_0)\, e^{(\omega'_p-\omega_p)(x-x_0)},$$

where $\omega'_\pi - \omega_\pi \gtrless 0$ as $\pi \gtrless l$. Hence, if $p(x_0)$ is small enough, $p(x)$ tends to zero, and, if $p(x_0)$ is large enough, to ∞.

16. We can round off some of these results by means of the following lemma:

If $K'(t)$, $K''(t)$ satisfy conditions A; if $f(t)$ is continuous for

$0 \leqslant t \leqslant \lambda$, *and* $v(x)$, $u(x)$ *are never negative and continuous, and if*

$$u(x)+v(x) = \int_0^\lambda K'(t)\,u(x-t)\,dt + \int_0^\lambda K''(t)\,v(x-t)\,dt \quad (x > \lambda),$$

$$= f(x) + \int_0^x K'(t)\,u(x-t)\,dt + \int_0^x K''(t)\,v(x-t)\,dt \quad (\lambda \geqslant x \geqslant 0),$$

then (1), *if* $0 = \sigma > \rho$,

$$\int_0^\infty u(t)\,dt \leqslant \Big[\int_0^\lambda f(t)\,dt\Big] \Big/ \Big[1 - \int_0^\lambda K'(t)\,dt\Big],$$

$$u(x)+v(x) \to \Big[\int_0^\lambda f(t)\,dt - \Big(1 - \int_0^\lambda K'(t)\,dt\Big) \int_0^\infty u(t)\,dt\Big] \Big/ \int_0^\lambda tK''(t)\,dt,$$

and (2), *if* $0 = \sigma = \rho$, *and* $u(x) \to 0$,

$$v(x) \to \Big[\int_0^\lambda f(t)\,dt\Big] \Big/ \int_0^\lambda tK''(t)\,dt.$$

This can be proved by writing the equation in the form

$$u(x)+v(x) = \int_0^\lambda K''(t)\,[u(x-t)+v(x-t)]\,dt + \phi(x),$$

and applying (4 . 4), (9 . 2), (14 . 4).

If $k''(x)$ is the reciprocal function to $K''(x)$, we have, on multiplying (16 . 1) by $k(y-x)$ and integrating,

$$u(y)+v(y) = -\int_0^\lambda k''(y-t)\,f(t)\,dt + \int_0^y u(\theta)\,R(y-\theta)\,d\theta$$

$$- \int_0^y k''(y-\theta) \int_0^\theta u(x)\,R(\theta-x)\,dx\,d\theta, \quad (y > \lambda).$$

where $R(y-\theta) = K'(y-\theta) - K''(y-\theta)$, if $\lambda \geqslant y-\theta \geqslant 0$,

$= 0$, otherwise.

Take any positive number A, greater than λ. Then we can write the preceding equation

$$u(y)+v(y) = -\int_0^\lambda k''(y-t)\,f(t)\,dt$$

$$+ \int_{y-A}^y u(\theta) \Big[R(y-\theta) - \int_\theta^y k''(y-t)\,R(t-\theta)\,dt\Big]\,d\theta$$

$$- \int_0^{y-A} u(\theta) \int_\theta^{\theta+\lambda} k''(y-t)\,R(t-\theta)\,dt\,d\theta. \qquad (1)$$

If $\sigma = 0$, by (9 . 2),

$$-k''(y) = \left[\int_0^\lambda tK''(t)\,dt\right]^{-1}[1-\psi(y)\,e^{-\eta y}] \quad \text{where} \quad |\psi(y)| < \eta' \quad (y > \eta'').$$

Substitute this in the right-hand side of (1) :

$$-\int_0^\lambda k''(y-t)\,f(t)\,dt = \frac{\int_0^\lambda f(t)\,dt}{\int_0^\lambda tK''(t)\,dt} + \psi_2(y)\int_0^\lambda e^{-\eta(y-t)}f(t)\,dt,$$

where $$|\psi_2(y)| < \eta;$$

so that, as $y \to \infty$,

$$-\int_0^\lambda k''(y-t)\,f(t)\,dt \to \left(\int_0^\lambda f(t)\,dt\right)\Big/\int_0^\lambda tK''(t)\,dt,$$

$$\int_0^{y-A} u(\theta)\int_\theta^{\theta+\lambda} k''(y-t)\,R(t-\theta)\,dt\,d\theta = \frac{\left[\int_0^\lambda K'(t)\,dt - 1\right]}{\int_0^\lambda tK''(t)\,dt}\int_0^{y-A} u(\theta)\,d\theta + I_1,$$

where $$I_1 = \left[\int_0^{y-A} u(\theta)\int_\theta^{\theta+\lambda} e^{-\eta(y-t)}\psi(y-t)\,R(t-\theta)\,dt\,d\theta\right]\Big/\int_0^\lambda tK''(t)\,dt$$

$$\leqslant \eta e^{-\eta A}\int_0^{y-A} u(\theta)\,e^{-\eta(y-A-\theta)}\,d\theta;$$

finally

$$\left|\int_{y-A}^{y} u(\theta)\left[R(y-\theta) - \int_\theta^y k''(y-t)\,R(t-\theta)\,dt\right]d\theta\right| \leqslant \eta\int_{y-A}^{y} u(\theta)\,d\theta.$$

Hence

$$u(y)+v(y) = \frac{\int_0^\lambda f(t)\,dt}{\int_0^\lambda tK''(t)\,dt} + \frac{\int_0^\lambda K'(t)\,dt - 1}{\int_0^\lambda tK''(t)\,dt}\int_0^{y-A} u(\theta)\,d\theta + I_1 + \eta\int_{y-A}^{y} u(\theta)\,d\theta + \psi(y), \quad (16\,.\,1)$$

where $\psi(y) \to 0$, as $y \to \infty$.

By (14 . 5), if $0 = \sigma > \rho$,

$$\int_0^\infty u(\theta)\,d\theta$$

is convergent; hence, for every A,

$$\lim_{y\to\infty}\int_{y-A}^{y} u(\theta)\,d\theta = 0,$$

and, as y tends to infinity,

$$\lim I_1 \leqslant \eta e^{-\eta A}\int_0^\infty u(\theta)\,d\theta < Ce^{-\eta A} \quad (C = \text{constant}).$$

If, therefore, in equation (16 . 1), we first let y tend to infinity and then A,

$$\lim_{y\to\infty}[u(y)+v(y)] = \left[\int_0^\lambda f(t)\,dt - \left(1-\int_0^\lambda K'(t)\,dt\right)\int_0^\infty u(\theta)\,d\theta\right]\Big/\int_0^\lambda tK''(t)\,dt.$$

If $0 = \rho = \sigma$, and $u(x)\to 0$; then since, for every positive δ,

$$\int_0^y u(\theta)\,e^{-\delta(y-\theta)}\,d\theta \to 0 \quad \text{as} \quad y\to\infty,$$

provided that $$u(\theta)\to 0,$$

it follows that, for every positive A, as y tends to infinity,

$$I_1 e^{\eta A}\to 0.$$

And hence, letting y tend to infinity, in (16 . 1),

$$\lim_{y\to\infty}[u(y)+v(y)] = \int_0^\lambda f(t)\,dt\Big/\int_0^\lambda tK''(t)\,dt.$$

This proves the result.

If now we apply this result to cases (2), (3), (4), it leads to the conclusion that, in all cases,

$$e^{-\sigma y}u(y)\to 0,$$

$$e^{-\sigma y}v(y)\to \text{constant} \quad (\rho<\sigma\leqslant\tau,\ \rho=\sigma<\tau),$$

$$e^{-\tau y}w(y)\to \text{constant},$$

and there is no difficulty in proving that the only case in which the constants can be zero is $\rho<\sigma=\tau$, when

$$e^{-\sigma y}v(y)\to 0,$$

$$e^{-\tau y}w(y)\to \text{constant} > 0.$$

For instance, suppose that $\rho=\sigma<\tau$. In applying the lemma it is

arbitrary what is considered the initial point. Thus for any x_0, if

$$A_1 = \int_0^\lambda te^{-\sigma t} K''(t)\,dt, \qquad A_2 = \int_0^\lambda te^{-\tau t} K'''(t)\,dt,$$

$$1 - A_3 = \int_0^\lambda e^{-\tau t} K''(t)\,dt,$$

then $$e^{-\sigma x} v(x) \to \frac{1}{A_1}\int_\lambda^{2\lambda} e^{-\sigma y}\int_{y-\lambda}^{\lambda} [K'(y-\theta)\,u(\theta) + K''(y-\theta)\,v(\theta)]\,d\theta\,dy$$

$$= \frac{1}{A_1}\int_{x_0}^{x_0+\lambda} e^{-\sigma y}\int_{y-\lambda}^{x_0} [K'(y-\theta)\,u(\theta) + K''(y-\theta)\,v(\theta)]\,d\theta\,dy = C, \text{ say};$$

$$e^{-\tau x} w(x) \to \frac{1}{A_2}\Big[\int_{x_0}^{x_0+\lambda} e^{-\tau y}\int_{y-\lambda}^{x_0} [K''(y-\theta)\,v(\theta) + K'''(y-\theta)\,w(\theta)]\,d\theta\,dy$$

$$- A_3\int_{x_0}^{\infty} e^{-\tau y} v(y)\,dy\Big].$$

Given any ϵ, we can choose x_0 so large that

$$\left|\frac{e^{-\sigma x} v(x)}{C} - 1\right| < \epsilon \quad (x > x_0),$$

and, hence, so large that

$$\frac{C(1+\epsilon)\,e^{-(\tau-\sigma)x}}{\tau-\sigma} > \int_x^\infty e^{-\tau y} v(y)\,dy > \frac{C(1-\epsilon)\,e^{-(\tau-\sigma)x}}{\tau-\sigma} \quad (x > x_0).$$

If this is substituted in the expression for $e^{-\tau x}\,w(x)$,

$$\lim_{x\to\infty} e^{-\tau x} w(x) = \frac{1}{A_2}\int_{x_0}^{x_0+\lambda} e^{-\tau y}\int_{y-\lambda}^{x_0} \{[K'''(y-\theta)\,w(\theta) + K''(y-\theta)\,v(\theta)]$$

$$- \phi\,[K''(y-\theta)\,v(\theta) + K'(y-\theta)\,u(\theta)]\}\,d\theta\,dy,$$

where $$|\phi| < \text{a constant, independent of } x_0.$$

But the integral on the right is necessarily positive if x_0 is large enough, since

$$\frac{v(\theta)}{w(\theta)}, \quad \frac{u(\theta)}{w(\theta)} \to 0, \text{ as } \theta \to \infty.$$

Hence $$\lim e^{-\tau x} w(x) > 0.$$

The other case can be treated in the same way.

I think that equation (15 . 25) is worth noticing, because it has a simple interpretation in the present application. Since the initial point is

arbitrary, it can be written, if $\rho = \sigma < \tau$,

$$\int_x^\infty e^{-\tau t} u(t)\, dt$$

$$= \frac{\int_x^{x+\lambda} e^{-\tau t}\left[u(t)+v(t)-\int_x^t \{K'(t-\theta)\, u(\theta)+K''(t-\theta)\, v(\theta)\}\, d\theta\right] dt}{1-\int_0^\lambda e^{-\tau t} K'(t)\, dt}.$$

Suppose that $\tau = 0$.

The integral on the left-hand side is the total number of recessives which are born after a fixed date, when the survival ratio of the dominants corresponds to that in a stationary population; the integral on the right depends on the number of recessives and hybrids in the community in the first generation after that date; so that, in the case of the selected dominant, there is a simple relation connecting the two. Similarly, in the case of the selected recessive, we have

$$\int_{-\infty}^x e^{-\rho t} w(t)\, dt$$

$$= \frac{\int_x^{x+\lambda} e^{-\rho t}\left[v(t)+w(t)-\int_x^t [K''(t-\theta)\, v(\theta)+K'''(t-\theta)\, w(\theta)]\, d\theta\right] dt}{\int_0^\lambda e^{-\rho t} K'''(t)\, dt-1},$$

where the left-hand side is the total number of recessives born before a fixed date, on the assumption that the survival ratio of the dominant corresponds to that in a stationary population. Similarly, again, in the simple community mentioned in § 10, we have, if we make a slight change in the notation, and write p_n for the number of dominants born in the n-th breeding season, $2q_n$ for the number of hybrids, and r_n for the number of recessives, and suppose their survival ratios to be, respectively, a, b, c, then, if the dominant is selected,

$$\sum_{s=n}^{\infty} r_s a^{-s} = a^{-n}[q_n+r_n]\Big/\left(1-\frac{c}{a}\right) \quad (c < b = a),$$

and if the recessive is selected,

$$\sum_{s=-\infty}^{n-1} r_s a^{-s} = a^{-n}[q_n+r_n]\Big/\left(\frac{c}{a}-1\right) \quad (c > b = a).$$

Here, again, the left-hand sides are, respectively, the number of recessives born after a fixed date, and the number born before a fixed date, on the assumption that the survival ratio of the dominants corresponds to that in a stationary population.

In conclusion, it may be mentioned that (15 . 25) makes clear the sense in which selection is capable of establishing a dominant variety. This formula can be written

$$\int_\lambda^\infty e^{-\sigma t} u(t)\, dt = A \Big/ \Big[1 - \int_0^\lambda e^{-\sigma\tau} K'(t)\, dt\Big],$$

where A is a constant depending on the initial condition in the community. This is to be compared with the case in which the pure types do not interbreed, when

$$\int_\lambda^\infty e^{-\sigma t} u(t)\, dt = A' \Big/ \Big[1 - \int_0^\lambda e^{-\sigma t} K'(t)\, dt\Big],$$

where A' is another constant. If $\sigma = 0$, the dominants are stationary, the recessives decrease to zero, and the left-hand sides of (15 . 251 . 252) represent the total number of recessives born after a fixed date. The formulae are of the same type. The difference between the two cases lies in the time taken before the last recessive is born, which may be one generation in the second case, while it is indefinitely long in the other.

The main results of the previous part can be summarized as follows. If the ages of parents vary independently, and the instantaneous birth-rate is the same in all three types and the same for a given age at all moments, and if the death-rates are the same for a given type and age at all moments, the advantages possessed by the three types can be measured by the real zeros of the functions

$$\int_0^\lambda e^{-zt} K'(t)\, dt - 1, \quad \int_0^\lambda e^{-zt} K''(t)\, dt - 1, \quad \int_0^\lambda e^{-zt} K'''(t)\, dt - 1,$$

where $K'(t)$, $K''(t)$, $K'''(t)$ are the functions defined by (12 . 11). If ρ, σ, τ are the three respective zeros, and $u(x)$, $v(x)$, $w(x)$ the density of births in the types at moment x, then

$$\frac{w(x)}{v(x)}, \quad \frac{v(x)}{u(x)} \to \infty ;$$

and

$$\int_0^\infty e^{-\tau x} u(x)\, dx$$

is less than a constant depending on the initial state of the community; the integral last written represents the total number of the unselected pure type born after a fixed date, if the birth- and death-rates in the other pure type correspond to those in a stationary population. If $\rho < \sigma < \tau$, there exists a positive δ, depending only on the K's, such that

$$\frac{w(x)}{v(x)}\, e^{-\delta x} \to \infty ;$$

if $\rho < \sigma = \tau$, then in any interval $x_1 \geqslant x \geqslant x_0$ in which $w(x)/v(x) < p'$, there exists a positive δ, such that

$$e^{-\delta x}\frac{w(x)}{v(x)} > e^{-\delta x_0}\frac{w(x_0)}{v(x_0)},$$

but eventually, for every δ,

$$e^{-\delta x}\frac{w(x)}{v(x)} \to 0;$$

if $\rho = \sigma < \tau$, then in any interval in which $w(x)/v(x) \geq p > 0$ there is a positive δ such that

$$e^{-\delta x}\frac{w(x)}{v(x)} > e^{-\delta x_0}\frac{w(x_0)}{v(x_0)},$$

but there is no δ such that this holds for every p; if $\rho = \sigma = \tau$, $w(x)/v(x)$ oscillates between finite positive limits. These results can all be summarized in the statement that, subject to the conditions stated, the same results hold as in the case when different generations do not interbreed.

The preceding results are very incomplete. But I think that enough has been proved to make clear that, if the advantages possessed by a type are measured by the constant of the geometric ratio in which a community would increase, if it had the same birth-rate and death-rate as the type, and if the other conditions are fulfilled, then the changes in composition of the community depend essentially on the relative magnitude of the advantages of the types; and that, in particular, if the hybrids are neither the most nor the least advantageous type, then the pure type which has the greater advantages increases indefinitely in proportion to the others.

THE MODULAR EQUATIONS

By M. HANNA.

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1. During the last century much interest has been taken in the modular equations corresponding to prime-order transformations of elliptic functions, but, although five sets of such equations have been introduced, the labour involved in computing them is so great that only a small number of the equations has been calculated.

The purpose of this paper is to give a list of all the previous modular equations which I have been able to trace, with references to their original computers, and to give thirteen new ones, together with a few complex multiplication moduli, recently calculated by myself.

In presenting these results I am greatly indebted to Prof. W. E. H. Berwick, under whose supervision I have been working in the University of Leeds.

On page 47 I give a list of known modular equations, including those recorded here and believed to be new.

2. The method used in calculating the uv, $u'v'$ equations is a modification of that adopted by Russell.

(i) By extending Weber's work the irreducible form of the equation connecting uv, $u'v'$ is completely known.

(ii) The q-formulae, obtained from Weber's functions, are substituted and the coefficients of successive powers of q are equated to zero. Hence some (or all) of the coefficients are calculated.

(iii) The remaining coefficients (if any) are obtained by comparing known results with those arising from the substitutions $u = v'$, $v^s = \pm u'^s$ in the equations, s being 1, 2, or 4 according as $n \equiv 7 \pmod 8$, $3 \pmod 8$, or $1 \pmod 4$.

First, when $n \equiv 1 \pmod 4$, we put

$$L = kl+k'l'-1, \quad M = 64(klk'l'-kl-k'l'), \quad N = -32klk'l',$$

	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	67	71	79	83	103	107	127	167	191	239	
1: Jacobi, 1828	*	*																									
2: Sohncke, 1836			*	*	*	*	*																				u, v
Hanna, 1927								*		*				*													
3: Legendre, 1828	*																										
4: Guetzlaff, 1834			*																								
5: Schröter, 1861		*		*				*		*																	
6: Hurwitz, 1880														*													$uv, u'v'$
7: Fiedler, 1885							*											*	*								
8: Russell, 1887–89					*	*			*				*		*	*											
Hanna, 1927											*	*					*			*	*	*	*	*	*	*	
9: Schläfli, 1870	*	*	*	*	*	*	*																				
10: Weber, 1908								*																			uu', vv'
11: Berry, 1908									*	*	*																
12: Klein, 1879	*	*	*		*																						
13: Fricke, 1892				*			*																				J, J' (parametric)
14: Greenhill, 1910						*																					
15: Smith, 1878	*																										
16: Berwick, 1916		*																									J, J' (explicit)

1: *Fundamenta Nova* (1829), 23, 27. 2: *Journal für Math.*, 16 (1836), 97–130. 3: *Astr. Nachrichten*, 6 (1828), 205–8.
4: *Journal für Math.*, 12 (1834), 173–177. 5: *Journal für Math.*, 58 (1861), 378–379. 6: *Math. Annalen*, 17 (1880), 62–70 (69).
7: *Wolf's Vierteljahrsschrift, Zürich*, 30 (1885), 129–229. 8: *Proc. London Math. Soc.* (1), 19 (1887), 90–111; 21 (1889), 351–395.
9: *Journal für Math.*, 72 (1870), 360–369. 10: *Lehrbuch der Algebra*, 3 (1908), § 132. 11: *Amer. J. of Math.*, 30 (1908), 156–169.
12: *Math. Annalen*, 14 (1879), 111–172 (143). 13: *Math. Annalen*, 40 (1892), 469–502. 14: *Math. Annalen*, 68 (1910), 208–219.
15: *Proc. London Math. Soc.* (1), 10 (1878), 87–91. 16: *Quart. J. of Math.*, 47 (1916), 94–103.

Using these functions, I have calculated two new equations* :

$$37\quad \begin{cases} L^{\frac{1}{2}}-N^{\frac{1}{4}}(M^4-35L^2M^3+388L^4M^2-1387L^6M+737L^8) \\ \quad +L^{\frac{1}{4}}N(121M^3-4456L^2M^2+33792L^4M+105386L^6) \\ \quad -LN^{\frac{3}{4}}(4932M^2-136222L^2M+923722L^4) \\ \quad +L^{\frac{3}{4}}N^2(61848M-882163L^2) \\ \quad +N^{\frac{5}{4}}(3375M-73845L^2)-135000L^{\frac{1}{2}}N^3=0. \end{cases}$$

$$41\quad \begin{cases} L^7+N^{\frac{1}{3}}(M^3-26L^2M^2+171L^4M-150L^6) \\ \quad -LN^{\frac{2}{3}}(77M^2-892L^2M-5703L^4) \\ \quad +N(52M^2+1859L^2M-41782L^4)-LN^{\frac{4}{3}}(2152M-36785L^2) \\ \quad +N^{\frac{5}{3}}(880M-6716L^2)-18512LN^2+4800N^{\frac{5}{3}}=0. \end{cases}$$

My next three new equations are under $n \equiv 3 \pmod 8$, where

$$P=\sqrt{(kl)}+\sqrt{(k'l')}-1, \quad Q=16\{\sqrt{(klk'l')}-\sqrt{(kl)}-\sqrt{(k'l')}\},$$
$$R=-16\sqrt{(klk'l')}.$$

$$67\quad \begin{cases} P^{17}+R(Q^7-31P^2Q^6+389P^4Q^5-2530P^6Q^4+9044P^8Q^3 \\ \qquad\qquad -17233P^{10}Q^2+15254P^{12}Q-4159P^{14}) \\ \quad -PR^2(88Q^5-2359P^2Q^4+25325P^4Q^3-136278P^6Q^2 \\ \qquad\qquad +365067P^8Q-386963P^{10}) \\ \quad +R^3(172Q^4-2417P^2Q^3+11193P^4Q^2-27839P^6Q+42011P^8) \\ \quad -PR^4(6336Q^2-72400P^2Q+178384P^4) \\ \quad +R^5(6912Q-53568P^2)=0. \end{cases}$$

$$83\quad \begin{cases} P^7+R^{\frac{1}{3}}(Q^3-13P^2Q^2+46P^4Q-35P^6) \\ \quad +R^{\frac{2}{3}}P(8Q^2-84P^2Q+227P^4)+R(5Q^2-27P^2Q+72P^4) \\ \quad +R^{\frac{4}{3}}P(38Q-49P^2)+R^{\frac{5}{3}}(15Q+89P^2)+93PR^2+18R^{\frac{7}{3}}=0. \end{cases}$$

$$107\quad \begin{cases} P^9-R^{\frac{1}{3}}(Q^4-17Q^3P^2+94P^4Q^2-183P^6Q+74P^8) \\ \quad -PR^{\frac{2}{3}}(10Q^3-157P^2Q^2+755QP^4-1237P^6) \\ \quad +R(14Q^3-255P^2Q^2+1473P^4Q-3047P^6) \\ \quad +PR^{\frac{4}{3}}(116Q^2-1450P^2Q+4825P^4) \\ \quad -R^{\frac{5}{3}}(77Q^2-1230P^2Q+5291P^4)-PR^2(446Q-3680P^2) \\ \quad +R^{\frac{7}{3}}(192Q-2009P^2)+564PR^{\frac{8}{3}}-180R^3=0. \end{cases}$$

* The results are contrary to Russell's statement [*Proc. London Math. Soc.* (1), 21 (1889), 363] that the equation is of degree 19 and is not reducible to any simpler form.

Thirdly, when $n \equiv 7 \pmod{16}$, putting

$$S = \sqrt[3]{(kl)} + \sqrt[3]{(k'l')} - 1, \quad T = 4\{\sqrt[3]{(klk'l')} - \sqrt[3]{(kl)} - \sqrt[3]{(k'l')}\},$$
$$U = -4\sqrt[3]{(klk'l')},$$

$$103 \left\{ \begin{aligned} &S^{13} + U(T^5 - 18S^2T^4 + 117S^4T^3 - 335S^6T^2 + 418S^8T - 191S^{10}) \\ &\quad + SU^2(T^3 - 25S^2T^2 + 188S^4T - 201S^6) \\ &\quad + U^3(24T^2 - 246S^2T + 191S^4) + 200SU^4 = 0, \end{aligned} \right.$$

$$167 \left\{ \begin{aligned} &S^7 + U^3(T^3 - 10S^2T^2 + 25S^4T - 13S^6) + S^3U^3(5T + 6S^2) \\ &\quad + S^2U(12T + 5S^2) + U^2S(22T - 7S^2) \\ &\quad + U^3(8T + 8S^2) + 8SU^2 - 8U^3 = 0. \end{aligned} \right.$$

Fourthly, when $n \equiv 15 \pmod{16}$, we write

$$V = \sqrt[3]{(kl)} + \sqrt[3]{(k'l')} + 1, \quad Z = 4\sqrt[3]{(klk'l')}, \quad W = (V-2)^2 - Z.$$

$$127 \left\{ \begin{aligned} &W^8 - VZ(V^{12} + 16V^{10}W + 91V^8W^2 + 215V^6W^3 \\ &\qquad + 177V^4W^4 + 3V^2W^5 - 18W^6) \\ &\quad - Z^2(11V^{10} + 77V^8W + 102V^6W^2 + 88V^4W^3 - 65V^2W^4 + W^5) \\ &\quad + VZ^3(27V^6 + 451V^4W - 7V^2W^2 - 25W^3) \\ &\quad + Z^4(433V^4 - 154V^2W - 2W^2) - 25VZ^5 = 0. \end{aligned} \right.$$

$$191 \left\{ \begin{aligned} &W^4 - VZ^{\frac{1}{2}}(V^6 + 8V^4W + 13V^2W^2 - 3W^3) \\ &\quad + WZ^{\frac{1}{2}}(5V^4 + 40V^2W + 2W^2) \\ &\quad + VZ(7V^4 + 49V^2W - 23W^2) \\ &\quad - Z^{\frac{1}{2}}(13V^4 + 94V^2W - 4W^2) - VZ^{\frac{3}{2}}(13V^2 - 92W) \\ &\quad + Z^2(88V^2 - 40W) - 100VZ^{\frac{5}{2}} + 32Z^3 = 0. \end{aligned} \right.$$

$$239 \left\{ \begin{aligned} &W^5 - VZ^{\frac{1}{2}}(V^6 + 10V^4W + 26V^2W^2 + 7W^3 - 11W^4) \\ &\quad + Z^{\frac{1}{2}}(8V^8 + 62V^6W + 96V^4W^2 + 49V^2W^3 - W^4) \\ &\quad - VZ(26V^6 + 172V^4W + 132V^2W^2 + 5W^3) \\ &\quad + Z^{\frac{3}{2}}(29V^6 + 217V^4W + 162V^2W^2 + 6W^3) \\ &\quad + VZ^2(40V^4 - 53V^2W - 10W^2) - V^2Z^{\frac{5}{2}}(181V^2 + 62W) \\ &\quad + VZ^{\frac{5}{2}}(294V^2 + 132W) - Z^3(240V^2 + 24W) \\ &\quad + 120VZ^{\frac{7}{2}} - 16Z^4 = 0. \end{aligned} \right.$$

Three more equations of this type have been calculated, under $n = 27 = 3^3$, $n = 43$, and $n = 143 = 11 \cdot 13$.

$$27 \qquad P^9 - P^2R(Q^2 + 11P^2Q - 29P^4) + PR^2(3Q - 17P^2) - R^3 = 0;$$

whereas Fiedler gives the last term as

$$+12328\{\sqrt{(klk'l')}\}^3 \text{ instead of } +\{16\sqrt{(klk'l')}\}^3.$$

43
$$\begin{cases} P^{11}-R(Q^4-19P^2Q^3+125P^4Q^2-330P^6Q+286P^8) \\ \quad +PR^2(2Q^2-21P^2Q+97P^4)-R^3(8Q+8P^2)=0, \end{cases}$$

in this the last five coefficients complete Russell's equation.

143
$$\begin{cases} VW^3-V^2Z^{\frac{1}{3}}(V^4+6V^2W+4W^2)+VZ^{\frac{2}{3}}(V^4+10V^2W+4W^2) \\ \quad -Z(3V^4+7V^2W-W^2)-9Z^{\frac{1}{3}}V^3 \\ \quad +Z^{\frac{2}{3}}(13V^2+4W)-8Z^2V+4Z^{\frac{3}{3}}=0. \end{cases}$$

3. Various complex multiplication moduli, previously obtained by Russell, Weber, and others, have been confirmed from the above modular equations. Putting

$$y=\sqrt[4]{(16kk')}, \quad z=\sqrt[12]{(16kk')},$$

I have also obtained the following new results:

$\Delta=-87=-3\,.\,29$:

$$(2y^3+y^2+11y+5)^2=29(y^2+y+1)^2.$$

$\Delta=-111=-3\,.\,37$:

$$(2y^4-y^3+15y^2+15y+12)^2=37(y^3+y^2+3y+2)^2.$$

$\Delta=-127$: $z^5-z^4-2z^3+z^2+3z-1=0.$

$\Delta=-143=-11\,.\,13$:

$$(2z^5-3z^4+7z^3-2z^2+4z+3)^2=13(z^4-z^3+2z^2+1)^2.$$

$\Delta=-151$: $z^7+z^6+z^5+3z^3+z^2+3z-1=0.$

$\Delta=-167$:

$$z^{11}+z^{10}+5z^9+4z^8+10z^7+6z^6+11z^5+7z^4+9z^3+4z^2+2z-1=0.$$

$\Delta=-191$:

$$z^{13}-2z^{12}+4z^{10}-5z^9+z^8+5z^7-11z^6+19z^5-22z^4+16z^3-10z^2+6z-1=0.$$

$\Delta=-223$: $z^7+z^5+4z^4-z^3+5z-1=0.$

$\Delta=-239$:

$$z^{15}-4z^{14}+4z^{13}+4z^{12}-5z^{11}-13z^{10}+20z^9+4z^8-15z^7$$
$$-13z^6+27z^5-4z^4-8z^3-2z^2+6z-1=0.$$

4. In conclusion, I notice the simple form of the uv, $u'v'$ equation when $n=8p-1$.

(i) The equation is symmetric, homogeneous, and of degree p in uv, $u'v'$, $(-1)^p$.

(ii) Since

$$uv = 2q^p(1-q+\ldots), \quad u'v' = 1-2q+\ldots,$$

the term free from $u'v'$ must be $(uv-1)^p$. Many of the coefficients are deduced from considerations of symmetry.

(iii) To determine the remaining coefficients we divide the equation through by $u^p v^p$, substitute the q-series for $(uv)^{-1}$, $u'v'u^{-1}v^{-1}$, and equate to zero the coefficients of the successive powers of q. The simplicity of the process is due to the fact that

$$u^{-1} = 2^{-\frac{1}{2}} q^{-\frac{1}{8}}(1+q-q^2+q^4-q^6-q^7+2q^8+q^9-2q^{10}-q^{11}+2q^{12}+q^{13}\ldots),$$

a series with very small coefficients. Had Sohncke been aware of this he would certainly have economized labour and would probably have calculated more equations.

A further simplification arises from the relations

$$\frac{1}{uv} = \frac{1}{2q^{\frac{1}{2}}}\phi(q), \quad \frac{u'v'}{uv} = \frac{1}{2q^{\frac{1}{2}}}\phi(-q),$$

connecting Weber's functions.

(iv) The u, v equation is obtained by rationalizing the relation between uv, $u'v'$.

23. From symmetry

$$(uv-1)^3+u'v'(3u^2v^2+auv+3)+(u'v')^2(3uv-3)+(u'v')^3 = 0$$

and, substituting the q-series, $a = -2$. The rational form is

$$u^{24}+v^{24}-2^{11}(u^{23}v^{23}+uv)+23\,.\,2^2\,.\,5\,.\,6421u^{12}v^{12}$$

$$+23\sum_{m=3}^{11} A_m(u^{24-m}v^{24-m}+u^m v^m)+23(u^8+v^8)\sum_{m=0}^{7} B_m(u^{8-m}v^{8-m}+u^{8+m}v^{8+m})$$

$$+23(u^{16}+v^{16})\sum_{m=0}^{3} C_m(u^{4-m}v^{4-m}+u^{4+m}v^{4+m}) = 0, \text{ with}$$

$A_3 = -2^{10}$, $A_4 = 2^8\,.\,3^2$, $A_5 = -2^7\,.\,47$,

$A_6 = 2^6\,.\,227$, $A_7 = -2^8\,.\,11^2$, $A_8 = 2^5\,.\,3^2\,.\,5^2\,.\,7$,

$A_9 = -2^4\,.\,4559$, $A_{10} = 2^3\,.\,3^2\,.\,1321$, $A_{11} = -2^4\,.\,7393$,

$B_0 = -\frac{1}{2}\,.\,3^2\,.\,2047$, $B_1 = 2^5\,.\,499$, $B_2 = -2^4\,.\,3\,.\,193$,

$B_3 = 2^5\,.\,7\,.\,29$, $B_4 = -2^4\,.\,3\,.\,113$, $B_5 = 2^6\,.\,3\,.\,17$,

$B_6 = -2^6\,.\,3^2$, $B_7 = 2^7$,

$C_0 = 3^2\,.\,199$, $C_1 = -2^3\,.\,3\,.\,97$, $C_2 = 2^2\,.\,3\,.\,7^2$,

$C_3 = -2^3\,.\,5$.

31. The irrational modular equation is

$$(uv-1)^4-u'v'(4u^3v^3+4)+(u'v')^2(6u^2v^2+6)-(u'v')^3(4uv+4)+(u'v')^4=0$$

and, rationalizing this, I find that

$$u^{32}+v^{32}-2^{15}(u^{31}v^{31}+uv)+31\sum_{m=1}^{14}A_m(u^{16+m}v^{16+m}+u^{16-m}v^{16-m})$$
$$+31(u^8+v^8)\sum_{m=1}^{11}B_m(u^{12+m}v^{12+m}+u^{12-m}v^{12-m})$$
$$+31(u^{16}+v^{16})\sum_{m=1}^{7}C_m(u^{8+m}v^{8+m}+u^{8-m}v^{8-m})$$
$$+31(u^{24}+v^{24})\sum_{m=1}^{3}D_m(u^{4+m}v^{4+m}+u^{4-m}v^{4-m})$$
$$+31\sum_{m=0}^{3}E_m(u^{8m}+v^{8m})u^{16-4m}v^{16-4m}=0$$

the numerical coefficients being

$$A_1=-2^7\cdot 132557,\quad A_2=2^6\cdot 168479,\quad A_3=-2^7\cdot 3\cdot 10931,$$
$$A_4=-2^6\cdot 23417,\quad A_5=2^9\cdot 5\cdot 1579,\quad A_6=-2^{10}\cdot 7\cdot 11\cdot 53,$$
$$A_7=2^8\cdot 13\cdot 709,\quad A_8=-2^8\cdot 5\cdot 11\cdot 23,\quad A_9=-2^{11}\cdot 19^2,$$
$$A_{10}=2^{11}\cdot 3^3\cdot 17,\quad A_{11}=-2^{11}\cdot 3\cdot 11^2,\quad A_{12}=2^{13}\cdot 3^2\cdot 5,$$
$$A_{13}=-2^{13}\cdot 13,\quad A_{14}=2^{14},$$
$$B_1=-2^5\cdot 250259,\quad B_2=2^4\cdot 363119,\quad B_3=-2^5\cdot 3^2\cdot 89\cdot 101,$$
$$B_4=-2^7\cdot 2041,\quad B_5=2^9\cdot 5\cdot 7\cdot 73,\quad B_6=-2^8\cdot 5479,$$
$$B_7=2^{10}\cdot 1093,\quad B_8=-2^{11}\cdot 277,\quad B_9=2^{10}\cdot 3\cdot 67,$$
$$B_{10}=-2^{10}\cdot 37,\quad B_{11}=2^{11},$$
$$C_1=-2^6\cdot 3\cdot 3529,\quad C_2=2^5\cdot 3\cdot 5083,\quad C_3=-2^6\cdot 3\cdot 7\cdot 13\cdot 23,$$
$$C_4=2^5\cdot 3^2\cdot 7\cdot 83,\quad C_5=-2^8\cdot 3\cdot 5\cdot 31,\quad C_6=2^8\cdot 3\cdot 23,$$
$$C_7=-2^7\cdot 3^2,$$
$$D_1=2^5\cdot 13\cdot 47,\quad D_2=2^4\cdot 241,\quad D_3=2^5\cdot 5,$$
$$E_0=7\cdot 1422299,\quad E_1=2^3\cdot 1147787,\quad E_2=2^2\cdot 231097,$$
$$E_3=2^3\cdot 4061.$$

(The larger coefficients have not been factorized.)

I also find the irrational equation of order 47 to be

$$(uv-1)^6-u'v'(6u^5v^5+10u^4v^4+10uv+6)+(u'v')^2(15u^4v^4-14u^2v^2+15)$$
$$-(u'v')^3(20u^3v^3+20)+(u'v')^4(15u^2v^2-10uv+15)$$
$$-(u'v')^5(6uv+6)+(u'v')^6=0.$$

MODULAR INVARIANTS EXPRESSIBLE IN TERMS OF QUADRATIC AND CUBIC IRRATIONALITIES

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1. In this paper are recorded the numerical values of Dedekind's invariant modular function $j(\omega)$, which satisfies the equations

$$j(\omega) = j(\omega+1) = j(-1/\omega)$$

$$= 1728 \times \text{Klein's } J = 256\,\frac{(1-k^2+k^4)^3}{k^4(1-k^2)^2} = \frac{1728 g_2^3}{g_2^3 - 27 g_3^2}$$

$$= q^{-2} + 744 + 1\ 96884 q^2 + 214\ 93760 q^4 + 8642\ 99970 q^6$$

$$+ 2\ 02458\ 56256 q^8 + 33\ 32026\ 40600 q^{10} + 425\ 20233\ 00096 q^{12} + \ldots,$$

in all cases where ω is a singular modulus and $j(\omega)$ an algebraic number, necessarily an integer, of the second or third degree. There are known to be 13 discriminants,

$$\Delta_1 = 3,\ 4,\ 7,\ 8,\ 11,\ 12,\ 16,\ 19,\ 27\ \ 28,\ 43,\ 67,\ 163,$$

such that $j(\frac{1}{2}i\sqrt{\Delta_1})$ or $j(\frac{1}{2}+\frac{1}{2}i\sqrt{\Delta_1})$, according as Δ_1 is even or odd, is a rational integer*. Due to each of 29 discriminants,

$$\Delta_2 = 15,\ 20,\ 24,\ 32,\ 35,\ 36,\ 40,\ 48,\ 51,\ 52,\ 60,\ 64,\ 72,\ 75,\ 88,\ 91,\ 99,\ 100,$$

$$112,\ 115,\ 123,\ 147,\ 148,\ 187,\ 232,\ 235,\ 267,\ 403,\ 427,$$

there are two non-equivalent singular moduli ω_2, ω_2', and $j(\omega_2)$, $j(\omega_2')$ are conjugate integers of a quadratic field. Three non-equivalent singular moduli ω_3, ω_3', ω_3'' belong to each of the 25 discriminants,

$$\Delta_3 = 23,\ 92,\ 31,\ 124;\ 108,\ 243;\ 44,\ 76,\ 172,\ 268,\ 652;\ 59,\ 83,\ 107,$$

$$139,\ 211,\ 283,\ 307,\ 331,\ 379,\ 499,\ 547,\ 643,\ 883,\ 907,$$

* H. Weber, *Lehrbuch der Algebra*, 3, ch. 19.

and $j(\omega_3)$, etc., are conjugate roots of an irreducible cubic equation with negative discriminant. In all these cases $j(\omega)$ and $j(\omega)-1728$ are here completely factorized in their respective fields. The norms of the algebraic prime factors of $j(\omega)$ and $j(\omega)-1728$ are of certain regular types which are noted in § 10. No attempt has been made on this occasion to prove general theorems, regarding the nature of these primes, which might be conjectured from the results now given. Klein's principal modulus* τ for transformations of orders 2, 3, 5, 7, 13 is also noted when it lies in the same number-field as $j(\omega)$.

2. A transformation of order n connects the modulus ω with $n+1$ moduli

$$n\omega, \quad \frac{\omega}{n}, \quad \frac{\omega+1}{n}, \quad \ldots, \quad \frac{\omega+n-1}{n}$$

and leads to an algebraic equation, the invariant equation, between $j(\omega)$, $j(n\omega)$, which can be calculated by substituting q-series for these functions†. Putting

$$J = j(\omega), \quad I = j(2\omega),$$

the invariant equation of the second order is

$$I^3+J^3-I^2J^2+2^4.3\,.\,31(I^2J+IJ^2)$$
$$-2^4.3^4.5^3(I^2+J^2)+3^4.5^3.4027\,.\,IJ$$
$$+2^8.3^7.5^6(I+J)-2^{12}.3^9.5^9 = 0,$$

whereas those of the third and fifth orders have been calculated by H. J. S. Smith‡ and the present writer§.

Whenever $\omega \sim n\omega$, ω is a singular modulus, $j(\omega) = j(n\omega)$, and hence an algebraic number. This requires that $\omega = (a+bi\sqrt{r})/c$, where a, b, c, r are rational integers, b, c, and r being positive.

Transformations associated with the quadratic values of j are of three types.

First, when $\Delta_2 = n^2\Delta_1$ each of the moduli ω_2, ω_2' belonging to Δ_2 is connected with ω_1, belonging to Δ_1, by a transformation of order n. In such cases the equation

$$\tau j(\omega_1)-f(\tau) = 0,$$

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* *Math. Annalen*, 14 (1879), 143.

† H. Weber, *Lehrbuch der Algebra*, 3 (1908), 237 seq.

‡ *Proc. London Math. Soc.* (1), 10 (1879), 87–91.

§ *Quart. J. Math.*, 47 (1916), 94–103.

τ being the principal modulus, has an irreducible quadratic factor with roots τ_1, τ_1' and, if $\tau_1\tau_2 = \tau_1'\tau_2' = c^*$,

$$j(\omega_2) = \frac{f(\tau_2)}{\tau_2}, \quad j(\omega_2') = \frac{f(\tau_2')}{\tau_2'}.$$

The possible values of (Δ_1, Δ_2) under this head are :

$n = 2$, (8, 32), (12, 48), (16, 64), (28, 112),
$n = 3$, (4, 36), (8, 72), (11, 99);
$n = 5$, (3, 75), (4, 100),
$n = 7$, (3, 147).

Other values of Δ_2 are such that a transformation of order n links ω_2, ω_2'. In such cases τ_2, τ_2' are conjugate integers of the quadratic field $[j]$; in fact, since $\tau_2\tau_2' = c$,

$$\tau_2 = \pm\sqrt{c}\epsilon^s, \quad \tau_2' = \pm\sqrt{c}\epsilon^{-s},$$

ϵ being the primitive unit. The values of Δ_2 are :

$n = 2$, $\Delta_2 = 20, 24, 36, 40, 52, 72, 88, 100, 148, 232,$
$n = 3$, $\Delta_2 = 15, 24, 48, 51, 60, 75, 123, 147, 267,$
$n = 5$, $\Delta_2 = 15, 20, 35, 40, 60, 115, 235,$
$n = 7$, $\Delta_2 = 35, 91, 112, 427,$
$n = 13$, $\Delta_2 = 52, 91, 403.$

Thirdly, each ω under $\Delta_2 = 15$ is connected by quadratic transformation with an ω under $\Delta_2 = 60$.

Excepting 187, for which a class-invariant has been given by Russell†, each Δ_2 appears at least once in the above lists. In fact, the transformations are enough to determine the principal moduli, and hence the j's.

3.

Δ_2	m	ϵ	$N(\epsilon)$
24, 32, 64, 88	2	$1+\sqrt{2}$	-1
36, 48	3	$2+\sqrt{3}$	$+1$
15, 20, 35, 40, 60, 75, 100, 115, 235	5	$\frac{1}{2}(1+\sqrt{5})$	-1
72	6	$5+2\sqrt{6}$	$+1$
112	7	$8+3\sqrt{7}$	$+1$
52, 91, 403	13	$\frac{1}{2}(3+\sqrt{13})$	-1

Δ_2	m	ϵ	$N(\epsilon)$
51, 187	17	$4+\sqrt{17}$	-1
147	21	$\frac{1}{2}(5+\sqrt{21})$	$+1$
232	29	$\frac{1}{2}(5+\sqrt{29})$	-1
99	33	$23+4\sqrt{33}$	$+1$
148	37	$6+\sqrt{37}$	-1
123	41	$32+5\sqrt{41}$	-1
427	61	$\frac{1}{2}(39+5\sqrt{61})$	-1
267	89	$500+53\sqrt{89}$	-1

* $c = 1, 1, 125, 49, 13$, according as $n = 2, 3, 5, 7, 13$.

† *Proc. London Math. Soc.* (1), 21 (1890), 351.

On page 55 are tabulated the 14 quadratic fields $[\sqrt{m}]$ in which $j(\omega_2)$, $j(\omega_2')$ lie, together with the primitive unit ϵ and its norm $N(\epsilon)$ for each. The class-number of each of these fields being unity, every complex integer therein is a unique product of prime integral factors.

The next table gives the values of Klein's principal moduli when they lie in the quadratic fields $[j]$, the notation being such that the positive value of the radical corresponds to

$$\omega_2 = \tfrac{1}{2}i\sqrt{\Delta_2} \quad \text{or} \quad \tfrac{1}{2}+\tfrac{1}{2}i\sqrt{\Delta_2}$$

according as Δ_2 is even or odd.

Δ_2	τ_2	τ_3	τ_5	τ_7	τ_{13}
15	$2^{-6}\epsilon^{-8}$	ϵ^2	$5\sqrt{5}\epsilon$	—	—
20	ϵ^6	—	$-5\sqrt{5}$	—	
24	$-\epsilon^4$	$-\epsilon^2$	—	—	—
32	$-8\epsilon^3$	—	—	—	—
35	—	—	$5\sqrt{5}\epsilon^3$	$-7\epsilon^2$	—
36	ϵ^4	$-3\sqrt{3}\epsilon$	—	-	—
40	$-\epsilon^{12}$	—	$-5\sqrt{5}\epsilon^3$	—	—
48	$-16\epsilon^3$	$-\epsilon^3$	—	—	—
51	—	ϵ^2	—	—	—
52	ϵ^6	—	—	—	$\sqrt{13}$
60	$64\epsilon^8$	$-\epsilon^{10}$	$-5\sqrt{5}\epsilon^5$	—	—
64	$-16\sqrt{2}\epsilon^6$	—	—	—	—
72	$-\epsilon^4$	-27ϵ	—	—	—
75	—	ϵ^{12}	$25\sqrt{5}\epsilon^3$	—	—
88	$-\epsilon^{12}$	—	—	—	—
91	—	—	—	$-7\epsilon^2$	$-\sqrt{13}\epsilon$
99	—	27ϵ	—	—	—
100	ϵ^{24}	—	$-5^3\epsilon^3$	—	—
112	$-64\epsilon^3$	—	—	7ϵ	—
115	—	—	$5\sqrt{5}\epsilon^9$	—	—
123	—	ϵ^2	—	—	—
147	—	ϵ^6	—	-49ϵ	—
148	ϵ^6	—	—	—	—
187	—	—	—	—	—
232	$-\epsilon^{12}$	—	—	—	-
235	—	—	$5\sqrt{5}\epsilon^{15}$	—	—
267	—	ϵ^2	—	—	—
403	—	—	—	—	$-\sqrt{13}\epsilon^3$
427	—	—	—	$-7\epsilon^2$	—

4. In tabulating $j(\omega_2)$ and $l(\omega_2) \equiv j(\omega_2) - 1728$ it is convenient to put

$$\gamma = \tfrac{1}{2}(1+\sqrt{m}) \quad \text{when} \quad m \equiv 5 \pmod 8.$$

The conjugate factors of a positive prime p are θ, θ' or $\theta\epsilon^s$, $\theta'\epsilon^{-s}$: in the present table s is so chosen that $p = \pi\pi'$, $\pi = \theta\epsilon^s$, where

$$N(\pi) = N(\pi') = -p, \quad \text{and} \quad \pi - \pi'$$

is least.

15
$$\begin{cases} j = -\epsilon^{-1}\{3\sqrt{5}\epsilon^2(3\gamma-1)\}^3, \\ l = -3\{3\,.\,7\epsilon^3(3\gamma-2)\}^2. \end{cases}$$

20
$$\begin{cases} j = \{2^2\,.\,\sqrt{5}\epsilon^3(3\gamma-2)\}^3, \\ l = \epsilon\{2^4\epsilon^2(3\gamma-1)(4\gamma-1)\}^2. \end{cases}$$

24
$$\begin{cases} j = \epsilon^{-1}\{2^2\,.\,3\epsilon^2(3\sqrt{2}-1)\}^3, \\ l = 3\{2^3\sqrt{2}\,.\,3\epsilon^3(4\sqrt{2}-3)\}^2. \end{cases}$$

32
$$\begin{cases} j = \{2\,.\,5\epsilon^3(4\sqrt{2}-3)\}^3, \\ l = \epsilon\{2\sqrt{2}\,.\,7\,.\,\epsilon^2(4\sqrt{2}+3)(4\sqrt{2}-1)\}^2. \end{cases}$$

35
$$\begin{cases} j = -\{2^5\,.\,\sqrt{5}\epsilon^4\}^3, \\ l = -7\{2^3\epsilon^7(4\gamma-3)(5\gamma-3)\}^2. \end{cases}$$

36
$$\begin{cases} j = \epsilon^{-1}\{2^2\,.\,\sqrt{3}\epsilon(2\sqrt{3}+1)(3\sqrt{3}+2)\}^3, \\ l = 2\sqrt{3}\{2^4\,.\,\sqrt{3}\,.\,7\epsilon^2(2\sqrt{3}-1)\}^2. \end{cases}$$

40
$$\begin{cases} j = \{2^2\,.\,3\sqrt{5}\epsilon^4(5\gamma-4)\}^3, \\ l = 2\{2^3\,.\,3^3\epsilon^5(5\gamma-2)\}^2. \end{cases}$$

48
$$\begin{cases} j = \tfrac{1}{2}\{2\,.\,3\sqrt{3}\,.\,5\epsilon^2(2\sqrt{3}-1)\}^3, \\ l = \epsilon\{2\,.\,3\sqrt{3}\,.\,\epsilon^2(2\sqrt{3}+1)(3\sqrt{3}+2)(4\sqrt{3}-1)\}^2 \end{cases}$$

51
$$\begin{cases} j = -\epsilon^{-1}\left\{2^5\,.\,3\,.\,\epsilon^2\left(\frac{\sqrt{17}-3}{2}\right)\right\}^3, \\ l = -3\{2^3\,.\,3\,.\,7\epsilon(4\sqrt{17}+15)\}^2. \end{cases}$$

52
$$\begin{cases} j = \{2^2\,.\,3\,.\,5\epsilon^2(3\gamma-4)\}^3, \\ l = \epsilon\{2^4\,.\,3^3\epsilon^2\gamma(4\gamma-5)\}^2. \end{cases}$$

60 $$\begin{cases} j = \epsilon\{3\sqrt{5}\epsilon^4(5\gamma-1)(6\gamma-1)\}^3, \\ l = 3\{3.7.11\epsilon^9(7\gamma-5)\}^2. \end{cases}$$

64 $$\begin{cases} j = \{\sqrt{2}.3\epsilon^4(4\sqrt{2}+3)(6\sqrt{2}-5)\}^3, \\ l = \sqrt{2}\{\sqrt{2}.3^3.7.11\epsilon^5\}^2. \end{cases}$$

72 $$\begin{cases} j = \epsilon^{-1}\{2^2.5\epsilon(3\sqrt{6}+5)(3\sqrt{6}-1)\}^3, \\ l = 2\{2^3.7\epsilon^2(2\sqrt{6}+1)(4\sqrt{6}-7)(4\sqrt{6}-5)\}^2. \end{cases}$$

75 $$\begin{cases} j = -\sqrt{5}\{2^5.3\epsilon^6(3\gamma-1)\}^3, \\ l = -3\{2^3.3\epsilon^{10}(3\gamma-2)(7\gamma-2)(8\gamma-7)\}^2. \end{cases}$$

88 $$\begin{cases} j = \{2^2.3.5\epsilon^2(3\sqrt{2}+1)(5\sqrt{2}+3)\}^3, \\ l = 11\{2^3\sqrt{2}.3^3.7\epsilon^5(8\sqrt{2}-7)\}^2. \end{cases}$$

91 $$\begin{cases} j = -\{2^5.3\epsilon^4(3\gamma-5)\}^3, \\ l = -7\{2^3.3^3.11\epsilon^5(\gamma-1)\}^2. \end{cases}$$

99 $$\begin{cases} j = -\epsilon^{-1}\left\{2^5.\epsilon\left(\frac{\sqrt{33}-5}{2}\right)(2\sqrt{33}+11)(\sqrt{33}+4)\right\}^3, \\ l = -\epsilon\{2^3.7.19\epsilon(2\sqrt{33}+7)\}^2. \end{cases}$$

100 $$\begin{cases} j = \{2^2.3\epsilon^4(3\gamma-1)(7\gamma-2)(8\gamma-1)\}^3, \\ l = \sqrt{5}\{2^5.3^3.7\epsilon^8(3\gamma-2)(4\gamma-1)\}^2. \end{cases}$$

112 $$\begin{cases} j = \{3.5\epsilon^2(3\sqrt{7}-2)(6\sqrt{7}-13)\}^3, \\ l = 7\epsilon\{3^3\epsilon(\sqrt{7}+2)(2\sqrt{7}-3)(5\sqrt{7}+12)(4\sqrt{7}+3)\}^2. \end{cases}$$

115 $$\begin{cases} j = -\{2^5.3\sqrt{5}\epsilon^{10}(3\gamma-2)\}^3, \\ l = -23\{2^3.3^3\epsilon^{13}(3\gamma-1)(8\gamma-3)\}^2. \end{cases}$$

123 $$\begin{cases} j = -\epsilon^{-1}\{2^5.3.5\epsilon^2(8\sqrt{41}-51)\}^3, \\ l = -3\{2^3.3.7\epsilon(8\sqrt{41}+51)(18\sqrt{41}-115)(6\sqrt{41}+37)\}^2. \end{cases}$$

147 $$\begin{cases} j = -3\sqrt{21}\{2^5.3.5\epsilon^3(\sqrt{21}-2)\}^3, \\ l = -3\{2^3.3.11\epsilon^6(4\sqrt{21}-17)(2\sqrt{21}+1)(6\sqrt{21}-25)\}^2. \end{cases}$$

148 $$\begin{cases} j = \{2^2.3.5\epsilon^2(3\gamma-10)(9\gamma-31)(12\gamma+29)\}^3, \\ l = \epsilon\{2^4.3^3.7(\gamma+2)(3\gamma+7)(4\gamma+7)(4\gamma-5)\}^2. \end{cases}$$

$$187\quad \begin{cases} j = -\{2^5.3.5\sqrt{17}\epsilon^2[\tfrac{1}{2}(\sqrt{17}+3)]^2\}^3, \\ l = -11\{2^3.3^3\epsilon^3(2\sqrt{17}+7)(2\sqrt{17}+5)(4\sqrt{17}+11)\}^2. \end{cases}$$

$$232\quad \begin{cases} j = \{2^3.3.5\epsilon^2(\gamma+1)(3\gamma-5)(10\gamma+19)(7\gamma-17)\}^3, \\ l = 2\{2^3.3^4.7.11\epsilon^5(3\gamma-8)(5\gamma-8)(7\gamma-15)\}^2. \end{cases}$$

$$235\quad \begin{cases} j = -\{2^5.3\sqrt{5}.11\epsilon^{14}(6\gamma-5)\}^3, \\ l = -47\{2^3.3^4.7\epsilon^{19}(4\gamma-1)(13\gamma-3)\}^2. \end{cases}$$

$$267\quad \begin{cases} j = -\epsilon^{-1}\{2^5.3.5[\tfrac{1}{2}(\sqrt{89}+9)]^2(30\sqrt{89}+283) \\ \qquad\qquad \times(3\sqrt{89}+28)(12\sqrt{89}-113)\}^3, \\ l = -3\{2^3.3.7\epsilon(30\sqrt{89}-283)(8\sqrt{89}+75)(348\sqrt{89}+3283) \\ \qquad\qquad \times(10\sqrt{89}+93)(104\sqrt{89}-981)\}^2. \end{cases}$$

$$403\quad \begin{cases} j = -\{2^5.3.5\epsilon^8(3\gamma+2)(3\gamma+1)(6\gamma-11)\}^3, \\ l = -31\{2^3.3^3.7\epsilon^{11}(3\gamma-4)(4\gamma+1)(5\gamma-1)(13\gamma+7)\}^2. \end{cases}$$

$$427\quad \begin{cases} j = -\{2^5.3.5.11\epsilon^2(18\gamma+61)(3\gamma-11)\}^3, \\ l = -7\{2^3.3^3.23\epsilon(\gamma+3)(4\gamma+13)(3\gamma+8)(19\gamma+64)(44\gamma+149)\}^2. \end{cases}$$

The moduli ω_2, ω_2', reduced to Klein's fundamental triangle for the modular group, for the 29 discriminants are as under, the positive values of $\sqrt{m}$, γ, $\epsilon-1$ being taken for ω_2 in the class-invariants.

Δ_2	ω_2	ω_2'
15	$\frac{1}{2}(-1+i\sqrt{15})$	$\frac{1}{4}(-1+i\sqrt{15})$
20	$i\sqrt{5}$	$\frac{1}{2}(-1+i\sqrt{5})$
24	$i\sqrt{6}$	$\frac{1}{2}i\sqrt{6}$
32	$2i\sqrt{2}$	$\frac{1}{3}(-1+2i\sqrt{2})$
35	$\frac{1}{2}(-1+i\sqrt{35})$	$\frac{1}{6}(-1+i\sqrt{35})$
36	$3i$	$\frac{1}{2}(-1+3i)$
40	$i\sqrt{10}$	$\frac{1}{2}i\sqrt{10}$
48	$2i\sqrt{3}$	$\frac{1}{2}i\sqrt{3}$
51	$\frac{1}{2}(-1+i\sqrt{51})$	$\frac{1}{6}(-3+i\sqrt{51})$
52	$i\sqrt{13}$	$\frac{1}{2}(-1+i\sqrt{13})$
60	$i\sqrt{15}$	$\frac{1}{3}i\sqrt{15}$
64	$4i$	$-\frac{1}{2}+i$
72	$3i\sqrt{2}$	$\frac{3}{2}i\sqrt{2}$
75	$\frac{1}{2}(-1+5i\sqrt{3})$	$\frac{1}{6}(-3+5i\sqrt{3})$
88	$i\sqrt{22}$	$\frac{1}{2}i\sqrt{22}$

Δ_2	ω_2	ω_2'
91	$\frac{1}{2}(-1+i\sqrt{91})$	$\frac{1}{10}(-3+i\sqrt{91})$
99	$\frac{1}{2}(-1+3i\sqrt{11})$	$\frac{1}{10}(-1+3i\sqrt{11})$
100	$5i$	$\frac{1}{2}(-1+5i)$
112	$2i\sqrt{7}$	$\frac{1}{2}i\sqrt{7}$
115	$\frac{1}{2}(-1+i\sqrt{115})$	$\frac{1}{10}(-5+i\sqrt{115})$
123	$\frac{1}{2}(-1+i\sqrt{123})$	$\frac{1}{6}(-3+i\sqrt{123})$
147	$\frac{1}{2}(-1+7i\sqrt{3})$	$\frac{1}{6}(-3+7i\sqrt{3})$
148	$i\sqrt{37}$	$\frac{1}{2}(-1+i\sqrt{37})$
187	$\frac{1}{2}(-1+i\sqrt{187})$	$\frac{1}{14}(-3+i\sqrt{187})$
232	$i\sqrt{58}$	$\frac{1}{2}i\sqrt{58}$
235	$\frac{1}{2}(-1+i\sqrt{235})$	$\frac{1}{10}(-5+i\sqrt{235})$
267	$\frac{1}{2}(-1+i\sqrt{267})$	$\frac{1}{6}(-3+i\sqrt{267})$
403	$\frac{1}{2}(-1+i\sqrt{403})$	$\frac{1}{22}(-9+i\sqrt{403})$
427	$\frac{1}{2}(-1+i\sqrt{427})$	$\frac{1}{14}(-7+i\sqrt{427})$

5. Our last table gives the norms of the algebraic prime factors appearing successively to the right of the unit in $\sqrt[3]{j}$ and $\sqrt{(j-1728)}$.

Δ_2	j	$j-1728$
15	11	11
20	11	11.19
24	17	23
32	23	23.31
35	1	19.31
36	11.23	11
40	29	31
48	11	11.23.47
51	2	47
52	23	3.43
60	29.41	59
64	23.47	1
72	29.53	23.47.71
75	11	11.59.71
88	17.41	79
91	17	3
99	2.11.17	83
100	11.59.71	11.19
112	59.83	3.19.31.103
115	11	11.79
123	23	23.59.107
147	17	47.83.131
148	11.47.107	3.11.67.139
187	2.2	19.43.151
232	5.53.149.173	23.151.223
235	41	19.199
267	2.2.11.17.47	11.71.167.251.263
403	17.23.53	23.43.79.367
427	41.47	3.19.47.103.283

6. The moduli whose absolute invariants involve a cubic irrationality are also associated with three types of transformations.

First when ω_1 is linked with each of the moduli ω_3, ω_3', ω_3'' by a transformation of order n the function

$$f(\tau)-\tau j(\omega_1)$$

has an irreducible cubic factor $(\tau-\tau_1)(\tau-\tau_1')(\tau-\tau_1'')$, and if

$$\tau_1\tau_3 = \tau_1'\tau_3' = \tau_1''\tau_3'' = c,$$

then $$j(\omega_3) = \frac{f(\tau_3)}{\tau_3}, \quad j(\omega_3') = \frac{f(\tau_3')}{\tau_3'}, \quad j(\omega_3'') = \frac{f(\tau_3'')}{\tau_3''}.$$

The eight pairs of values of (Δ_1, Δ_3) connected in this way are

$n = 2$, (27, 108); (11, 44), (19, 76), (43, 172), (67, 268), (163, 652);

$n = 3$, (12, 108), (27, 243).

Again, in certain cases, each two of the moduli ω_3, ω_3', ω_3'' are connected by a transformation of the n-th order,

e.g. $\Delta = 59$, $\omega_3 = \dfrac{1+i\sqrt{59}}{2}$, $\omega_3' = \dfrac{1+i\sqrt{59}}{6}$, $\omega_3'' = \dfrac{-1+i\sqrt{59}}{6}$,

$$\omega_3' = \frac{\omega_3}{3}, \quad \omega_3'' = \frac{\omega_3 - 1}{3}, \quad \omega_3'' = \frac{\omega_3' - 2}{\omega_3' + 1}.$$

On a previous occasion* I have shown how to obtain the values of τ which correspond to quintic transformations of this type. A similar method applies to transformations of orders 2, 3, 7, 13. Its application has led to the values of several class-invariants given in § 8. The possible values of Δ_3 are:

$n = 2$, 23, 31;

$n = 3$, 23, 92; 44; 59, 83, 107;

$n = 5$, 31, 124; 44, 76; 59, 139, 211, 331, 379, 499;

$n = 7$, 31, 124; 108, 243; 76; 59, 83, 139, 283, 307, 643;

$n = 13$, 23, 92; 108, 243; 172; 107, 139, 211, 283, 547, 883, 907.

Thirdly, quadratic transformation connects each modulus under 23 with a modulus under 92, and the discriminants 31, 124 are similarly related.

7. Excepting $[\sqrt[3]{2}]$ and $[\sqrt[3]{3}]$ each of the cubic fields $[j(\omega_3)]$ is represented by the standard equation $a(\theta) = 0$ defined by the author on a previous occasion†. D is the discriminant of the field, $(1, \theta, \lambda)$ the integral basis, and h the class-number.

* *Quart. J. of Math.*, 50 (1923), 24–26.

† *Proc. London Math. Soc.* (2), 12 (1913), 417.

Δ_3	$a(\theta)$	$D(\theta)$	D	λ	h
23, 92	$\theta^3-\theta-1$	-23	-23	θ^2	1
31, 124	$\theta^3-\theta^2-1$	-31	-31	θ^2	1
108	θ^3-2	-108	$-2^2.3^3$	θ^2	1
243	θ^5-3	-243	-3^5	θ^2	1
44	$\theta^3-\theta^2-\theta-1$	-44	$-2^2.11$	θ^2	1
76	$\theta^3-3\theta^2+\theta-1$	-76	$-2^2.19$	θ^2	1
172	$\theta^3-2\theta^2-2$	-172	$-2^2.43$	θ^2	1
268	$\theta^3-2\theta^2-2\theta-2$	-268	$-2^2.67$	θ^2	1
652	$\theta^3-3\theta^2-5\theta-3$	-652	$-2^2.163$	θ^2	1
59	$\theta^3-2\theta^2-1$	-59	-59	θ^2	1
83	$\theta^3-2\theta^2-2\theta-1$	-83	-83	θ^2	1
107	$\theta^3-\theta^2-3\theta-2$	-107	-107	θ^2	1
139	$\theta^3-3\theta^2-5\ -2$	-139	-139	θ^2	1
211	$\theta^3-3\theta^2+\theta-2$	-211	-211	θ^2	1
283	$\theta^3-4\theta^2-1$	-283	-283	θ^2	2
307	$\theta^3-5\theta^2-\theta-4$	-9.307	-307	$\frac{1}{3}(\theta^2-1)$	1
331	$\theta^3-4\theta^2-2\theta-1$	-331	-331	θ^2	2
379	$\theta^3-6\theta^2-5\theta-2$	-4.379	-379	$\frac{1}{2}(\theta^2+\theta)$	1
499	$\theta^3-4\theta^2-5\theta-4$	-4.499	-499	$\frac{1}{2}(\theta^2+\theta)$	1
547	$\theta^3-4\theta^2+2\theta-3$	-547	-547	θ^2	1
643	$\theta^3-10\theta^2-6\theta-1$	-643	-643	θ^2	2
883	$\theta^3-5\theta^2-5\theta-2$	-883	-883	θ^2	1
907	$\theta^3-5\theta^2+\theta-2$	-907	-907	θ^2	1

Of Klein's principal moduli only τ_2, τ_3 ever belong to the field $[j]$, and they in only twelve cases as below, ϵ (>1) being the primitive unit.

Δ_3	τ_2	τ_3
23	$2^{-6}\epsilon^{-24}$	—
92	$2^6\epsilon^{24}$	—
31	$2^{-6}\epsilon^{-24}$	—
124	$2^6\epsilon^{24}$	—
108	$2^2\epsilon^8$	$-3^2\epsilon^4$
243	—	$3^2\theta^2\epsilon^4$

Δ_3	τ_2
44	$2^2\epsilon^8$
76	$2^2\epsilon^8$
172	$2^2\epsilon^8$
268	$2^2\epsilon^8$
652	$2^2\epsilon^8$

8. It is now possible to tabulate further modular invariants, first giving more particulars of the fields in which they lie.

$-D$ (when prime) $= \alpha^2\beta$, $j(\omega_3) = \gamma_2^3$, $j(\omega_3)-1728 = (-)^{\Delta_3}\xi^2/(-D)$.

Δ_3	ϵ	ϵ^{-1}	α	β
23, 92	θ	θ^2-1	$3\theta^2-1$	$3\theta^2-4$
31, 124	ρ	$\theta^2-\theta$	$3\theta-2$	$3\theta+1$

$$23\begin{cases}\gamma_2 = -5(5\theta^2+11\theta+7) = -5\epsilon^4(2\theta-1)(3\theta+2),\\ \xi = 7(307\theta^2+379\theta+240)\\ \quad = 7\alpha\beta\epsilon^7(4\theta^2+2\theta-3)(3\theta+1).\end{cases}$$

$$92\begin{cases}\gamma_2 = 5(1075\theta^2+1434\theta+819)\\ \quad = 5\epsilon^{18}\alpha(4\theta^2+2\theta-3)(-3\theta+5),\\ \xi = 7\alpha\beta\epsilon^{16}(2\theta-1)(3\theta+1)(4\theta+3)(\theta^2+2\theta+4)(5\theta+2).\end{cases}$$

$$31\begin{cases}\gamma_2 = -3(36\theta^2+9\theta+23) = -3\epsilon^{11}(3\theta-4)(-\theta+3)(-\theta^2+2\theta+2),\\ \xi = 27(379\theta^2+163\theta+243)\\ \quad = 3^3\alpha\beta\epsilon(\theta+1)(9\theta^2+3\theta+4).\end{cases}$$

$$124\begin{cases}\gamma_2 = 3(11007\theta^2+5121\theta+7510)\\ \quad = 3\epsilon^{13}(9\theta^2+3\theta+4)(2\theta^2+3)(5\theta-6),\\ \xi = 3^3\alpha\beta\epsilon^{27}(\theta+1)(3\theta-4)(4\theta^2-5\theta+7)(-2\theta^2+4\theta+1).\end{cases}$$

Next for the moduli $3i\sqrt{3}$, $\frac{1}{2}(-1+9i\sqrt{3})$:

Δ_3	ϵ	ϵ^{-1}	
108	$\theta^2+\theta+1$	$\theta-1$	$3\epsilon = (\theta+1)^3$
243	$2\theta^2+3\theta+4$	θ^2-2	

$$108\begin{cases}j = \epsilon\{2\theta(\theta+1)\,5\epsilon^2(4\theta^2+6\theta+9)(2\theta+1)\}^3,\\ l = 3\{2^2\epsilon^8(-2\theta+3)(-\theta^2+3)(2\theta^2+3)(-2\theta^2+2\theta+3)(3\theta^2-1)\}^2.\end{cases}$$

$$243\begin{cases}j = -\theta\epsilon^{-1}\{2^5\cdot 5\epsilon^4(\theta+2)(2\theta-1)\}^3,\\ l = -\theta\{2^3\epsilon^5(\theta^2-2\theta+4)(2\theta^2+\theta+2)(3\theta^2-4)\\ \qquad\qquad \times(3\theta^2+4\theta+2)(2\theta^2+5\theta+8)\}^2.\end{cases}$$

The next five Δ_3's are solved by quadratic transformation of $\frac{1}{4}\Delta_3$ which $\equiv 3 \pmod 8$. In these

$$\tfrac{1}{4}\Delta_3 = \alpha^2\beta, \quad 2\epsilon = \eta^3, \quad j = \gamma_2^3, \quad j-1728 = \xi^2/(\tfrac{1}{4}\Delta_3).$$

Δ_3	ϵ	ϵ^{-1}	α	β	η
44	θ	$\theta^2-\theta-1$	$\theta^2+\theta-1$	$-\theta^2+4$	$\theta^2-\theta$
76	θ	$\theta^2-3\theta+1$	$2\theta+1$	$4\theta^2-12\theta+3$	$\theta-1$
172	θ^2+1	$-\theta^2+2\theta+1$	θ^2+3	$-2\theta^2+2\theta+7$	θ
268	$\theta^2+\theta+1$	$-\theta+3$	$\theta^2+2\theta+3$	$4\theta^2-14\theta+7$	θ
652	$3\theta^2+4\theta+2$	$\theta^2-2\theta-10$	$3\theta^2+2\theta+2$	$-26\theta^2+94\theta+79$	$\theta+1$

$$44\left\{\begin{aligned}\gamma_2 &= 2(94\theta^2+83\theta+49)\\ &= 2\epsilon^2\eta\alpha(2\theta^2-1)(2\theta^2-2\theta+1),\\ \xi &= 2^2.7\alpha\beta\epsilon^8(\theta^2+2)(-\theta+4).\end{aligned}\right.$$

$$76\left\{\begin{aligned}\gamma_2 &= 6(207\theta^2-45\theta+74)\\ &= 2.3\epsilon\eta(2\theta^2-1)(2\theta^2+2\theta+1),\\ \xi &= 2^2.3^3\alpha\beta\epsilon^7(\theta-2)(2\theta^2-2\theta+3).\end{aligned}\right.$$

$$172\left\{\begin{aligned}\gamma_2 &= 30(4230\theta^2+1521\theta+3586)\\ &= 2.3.5\epsilon^2\eta(\theta+1)(\theta^2+\theta-1)(3\theta^2-2\theta+1),\\ \xi &= 2^2.3^3.7\alpha\beta\epsilon^5(\theta-1)(2\theta-1)(3\theta^2+3\theta+1).\end{aligned}\right.$$

$$268\left\{\begin{aligned}\gamma^2 &= 30(78129\theta^2+71847\theta+53518)\\ &= 2.3.5\epsilon^2\eta(2\theta^2+\theta+3)(2\theta^2-4\theta+1)(2\theta^2-3),\\ \xi &= 2^2.3^3.7.11\alpha\beta\epsilon^5(\theta+1)^2(2\theta-5)(3\theta+1).\end{aligned}\right.$$

$$652\left\{\begin{aligned}\gamma_2 &= 30(5458\ 08924\theta^2+7197\ 19101\theta+3791\ 54407)\\ &= 2.3.5\epsilon^2\eta(4\theta^2-14\theta-13)(\theta-2)(2\theta^2-8\theta-1)\\ &\qquad\times(2\theta^2-4\theta-5)(2\theta^2-4\theta-1)(14\theta^2+17\theta+8),\\ \xi &= 2^2.3^3.7\alpha\beta\epsilon^3\theta(2\theta+1)(-4\theta^2+15\theta+10)(-\theta^2+5\theta-1)\\ &\quad\times(7\theta^2+8\theta+4)(2\theta^2-6\theta-5)(3\theta^2+\theta-1)(\theta^2+\theta+5)(7\theta^2+7\theta+5).\end{aligned}\right.$$

There remain fourteen prime values of Δ_3, $\equiv 3 \pmod 8$, for which $\Delta_3 = \alpha^2\beta$, $j = \gamma_2^3$, $j-1728 = -\xi^2/\Delta_3$.

Δ_3	ϵ	ϵ^{-1}	α	β
59	θ	$\theta^2-2\theta$	$3\theta-4$	$3\theta+2$
83	θ	$\theta^2-2\theta-2$	$-2\theta^2+7\theta$	$-2\theta^2+7\theta+2$
107	$\theta+1$	$\theta^2-2\theta-1$	$3\theta^2-2\theta-3$	$3\theta^2-2\theta-13$
139	$\theta+1$	$\theta^2-4\theta-1$	$4\theta^2-13\theta-13$	$4\theta^2-13\theta-11$
211	θ^2+1	$-\theta+3$	$3\theta^2-6\theta+1$	$3\theta^2-6\theta-5$
283	θ	$\theta^2-4\theta$	$3\theta-8$	$3\theta+4$
307	$2\lambda+1$	$-2\lambda+4\theta-3$	$4\lambda-5\theta-5$	$4\lambda-5\theta+3$
331	θ	$\theta^2-4\theta-2$	$-2\theta^2+11\theta-4$	$-2\theta^2+11\theta+4$
379	λ	$\lambda-3\theta-6$	$5\lambda-17\theta-10$	$5\lambda-17\theta-8$
499	$3\lambda+1$	$-3\lambda+9\theta+1$	$3\lambda-6\theta-11$	$3\lambda-6\theta+1$
547	$6\theta^2-2\theta+5$	$-2\theta^2+6\theta+5$	$3\theta^2-8\theta+2$	$3\theta^2-8\theta-8$
643	θ	$\theta^2-10\theta-6$	$3\theta^2-20\theta-6$	$3\theta^2-20\theta-124$
883	$9\theta^2+8\theta+3$	$-8\theta^2+43\theta+25$	$3\theta^2-10\theta-5$	$3\theta^2-10\theta-45$
907	$17\theta^2-2\theta+7$	$4\theta^2-23\theta+17$	$3\theta^2-10\theta+1$	$3\theta^2-10\theta-21$

59
$$\gamma_2 = -32(17\theta^2+3\theta+8) = -2^5\epsilon^4(\theta-1)(2\theta-1),$$
$$\zeta = 2^3\alpha\beta\epsilon^6(4\theta^2-6\theta-3)(2\theta^2-2\theta-1)(3\theta-2).$$

83
$$\gamma_2 = -160(8\theta^2+7\theta+3) = -2^5.5\epsilon^3(\theta+1),$$
$$\zeta = 2^3\alpha\beta\epsilon^3(2\theta-1)(2\theta+3)(4\theta+3)(3\theta-2).$$

107
$$\gamma_2 = -160(29\theta^2+44\theta+23) = -2^5.5\epsilon^3(\theta^2+1),$$
$$\zeta = 2^3.7\alpha\beta\epsilon^5(2\theta-3)(2\theta^2-3)(2\theta^2+2\theta+3).$$

139
$$\gamma_2 = -96(99\theta^2+126\theta+46) = -2^5.3\epsilon^4\theta(-\theta+5),$$
$$\zeta = 2^3.3^3\alpha\beta\epsilon^7(4\theta^2-14\theta-13)(2\theta-3).$$

211
$$\gamma_2 = -96(4806\theta^2-513\theta+3322) = -2^5.3\epsilon^2\theta^2(2\theta-1)(2\theta^2-2\theta+1),$$
$$\zeta = 2^3.3^3.7\alpha\beta\epsilon^5(2\theta^2+\theta+1)(2\theta^2-4\theta-3).$$

283
$$\gamma_2 = -480(5490\theta^2+333\theta+1352) = -2^5.3.5\epsilon^7(2\theta-3),$$
$$\zeta = 2^3.3^3\alpha\beta\epsilon^9[2\theta^2+\theta+2][-6\theta^2+22\theta+39].$$

307
$$\gamma_2 = -480(6273\theta^2+2061\theta+4709) = -2^5.3.5\epsilon^4(-\lambda+\theta+4)(\lambda-2),$$
$$\zeta = 2^3.3^3\alpha\beta\epsilon^2\lambda^2(\lambda+2)(9\lambda+4)(5\lambda-2\theta+4).$$

331
$$\gamma_2 = -96(864\,90\theta^2+427\,68\theta+192\,43)$$
$$= -2^5.3\epsilon^5[4\theta^2+2\theta-1](2\theta-1)(-\theta+6),$$
$$\zeta = 2^3.3^3.7\alpha\beta\epsilon^{10}[-4\theta^2+21\theta-10](\theta^2+2\theta+10).$$

379
$$\gamma_2 = -48(2\,885\,49\theta^2+2\,253\,15\theta+85106)$$
$$= -2^5.3\epsilon^2(-\theta^2+7\theta-1)(17\theta^2+13\theta+5)$$
$$\times(\theta^2-5\theta-3)(2\theta^2-12\theta-9),$$
$$\zeta = 2^3.3^3.7\alpha\beta\epsilon^5(3\theta^2-19\theta-9)(\theta^2-3\theta-3)$$
$$\times(\theta^2-\theta-1)(4\theta^2-22\theta-31).$$

499
$$\gamma_2 = -96(45\,759\,78\theta^2+51\,585\,93\theta+35\,698\,78)$$
$$= -2^5.3\epsilon\lambda(2\theta^2-12\theta+9)(17\theta^2+19\theta+13)(-\theta^2+5\theta+3)$$
$$\times(2\theta+3)(7\theta^2+7\theta+5),$$
$$\zeta = 2^3.3^3.7.11\alpha\beta\epsilon^4(6^2+\theta+1)^2(46^2-22\theta+13)(-6\theta^2+28\theta+15).$$

$$547\begin{cases}\gamma_2 = -48\ (68\,58504\theta^2 - 22\,08483\theta + 55\,94222)\\ \quad = -2^5 . 3 . 5\epsilon^3(2\theta-7)(2\theta-1)(6\theta^2-\theta+4),\\ \xi = 2^3 . 3^3 . 7\alpha\beta\epsilon^6(\theta^2-4\theta+2)(\theta^2-6\theta+14)(2\theta-5)(-2\theta^2+5\theta+10).\end{cases}$$

$$643\begin{cases}\gamma_2 = -8160(3\,53691\theta^2 + 2\,03814\theta + 33442)\\ \quad = -2^5 . 3 . 5 . 17\epsilon^4[28\theta^2+19\theta+4],\\ \xi = 2^3 . 3^3 . \alpha\beta\epsilon^{11}[4\theta+1](7\theta+2)[-53\theta^2+560\theta+6]\\ \qquad\qquad \times(2\theta-21)[-30\theta^2+322\theta+117].\end{cases}$$

$$883\begin{cases}\gamma_2 = -480(16795\,57833\theta^2 + 15186\,96378\theta + 5689\,34327)\\ \quad = -2^5 . 3 . 5\epsilon^3(\theta^2-4\theta-9)(\theta^2-4\theta-1)(2\theta^2-10\theta-7)(3\theta-1).\\ \xi = 2^3 . 3^3 . 7\alpha\beta\epsilon^6(\theta^2-6\theta+1)(\theta-1)(-\theta^2+6\theta+3)\\ \qquad\qquad \times(-11\theta^2+52\theta+77)(-2\theta^2+13\theta-5)(4\theta^2-20\theta-13).\end{cases}$$

$$907\begin{cases}\gamma_2 = -960(21931\,895880\theta^2 - 2652\,49539\theta + 8990\,21800)\\ \quad = -2^6 . 3 . 5\epsilon^2\theta(4\theta-1)(5\theta^2+3)(2\theta^2-4\theta+1),\\ \xi = 2^3 . 3^3 . 7 . 11\alpha\beta\epsilon^5(-\theta^2+6\theta-3)(4\theta^2-14\theta-17)(3\theta+1)(-\theta+9).\end{cases}$$

The real value of θ corresponds to $j(\frac{1}{2}i\sqrt{\Delta_3})$ or $j(-\frac{1}{2}+\frac{1}{2}i\sqrt{\Delta_3})$ according as Δ_3 is even or odd. It has not been determined which complex root of the cubic in θ belongs to ω_3' or ω_3'' in any of the cases.

9. The last table (page 67) gives, in order, the norms of the algebraic factors on the right of the units in $\sqrt[3]{j}$ and $\sqrt{(j-1728)}$ as tabulated in § 8. When $h = 1$ these algebraic factors are prime. If

$$2 = \gamma\delta, \quad N(\gamma) = 2, \quad N(\delta) = 2^2,$$

2 . 2 in the table of norms means that γ^2 is a factor, while 2^2 is given when δ is a factor. When $\Delta_3 = 283$, 331, and 643, $h = 2$ and each factor enclosed in a square bracket is the product of two non-principal prime ideals. The corresponding norm, a product of two factors, is bracketed below.

10. An inspection of the results of §§ 4, 5, 8, 9 suggests the possibility of several general theorems, unattempted here, concerning the factors of j and $j-1728$.

(i)
$$\Delta \equiv 3 \pmod 8, \quad j \equiv 0 \pmod{2^{15}};$$
$$\tfrac{1}{4}\Delta \equiv 3 \pmod 8, \quad j \equiv 0 \pmod{2^4};$$
$$\Delta \text{ or } \tfrac{1}{4}\Delta \text{ or } \tfrac{1}{16}\Delta \equiv 7 \pmod 8, \quad j \not\equiv 0 \pmod 2;$$

Δ_3	$\sqrt[3]{j}$	$\sqrt{(j-1728)}$
23	11.17	11^2.19
92	23.11^2.53	11.19.43.67.83
31	11.17.23	3.11^2
124	11^2.29.89	3.11.23^2.43
108	11^2.17	11.23.59.83.107
243	11.23	11^2.47.179.227.239
44	2.11.17.29	19.43
76	2.41.53	3.67
172	2.5.29.113	3.19.163
268	2.101.137.197	3.3.31.43
652	2.11^2.17.89.233.293.389	3.11.19.59.67.127.211.571.643
59	2.11	11^2.23.43
83	2	19.47.67.79
107	17	43.71.103
139	2.23	3^2.103
211	2.2.17.29	23.67
283	53	(3.31).(19^2.139)
307	2^2.47	3.3.23.163.271
331	(11.29).23.59	(3^2.11^2).59^2
379	2.2.11.17.53.71	7.11^2.31.47^2
499	2.17.23.41.71.83	3.3.71^2.463
547	17.23.101	3^2.31^2.59.223
643	(11.113)	(3.3).43.(11^2.67).71.(499.607)
883	11^2.41.89.113	3^2.11.23.43^2.307.739
907	2.131.137.167	47.67^2.79.331

$\Delta \equiv 0 \pmod 8$, excluding Δ's in the last line, $j \equiv 0 \pmod{2\sqrt{2}}$;

$\Delta \equiv 4 \pmod{16}$, $j \equiv 0 \pmod{2^6}$ and $j-1728 \equiv 0 \pmod{2^8}$.

(ii) $(-\Delta\,|\,3) = -1$, $j \equiv 0 \pmod{3^3}$;

$(-\Delta\,|\,3) = +1$, $N(j) \not\equiv 0 \pmod 3$;

$\Delta \equiv \pm 3 \pmod 9$, $j \equiv 0 \pmod{3^3}$.

(iii) $(-\Delta\,|\,5) = -1$, $j \equiv 0 \pmod{5^3}$;

$(-\Delta\,|\,5) = +1$, $N(j)\,N(j-1728) \not\equiv 0 \pmod 5$;

$\Delta \equiv 0 \pmod 5$ but $\not\equiv 0 \pmod{25}$, $j \equiv 0 \pmod{5\sqrt{5}}$.

(iv) $(-\Delta\,|\,7) = -1$, $j-1728 \equiv 0 \pmod{7^2}$;

$(-\Delta\,|\,7) = +1$, $N(j)\,N(j-1728) \not\equiv 0 \pmod 7$.

(v) $(-\Delta\,|\,11) = -1,\quad j^3(j-1728)^2 \equiv 0 \pmod{11};$

$(-\Delta\,|\,11) = +1,\quad N(j)N(j-1728) \equiv 0 \pmod{11}.$

The remaining results are inferred from the tables of norms in §§ 5, 9.

(vi) When $N(j) \equiv 0 \pmod{p}, \quad p \equiv 2 \pmod 3,$

and when $N(j-1728) \equiv 0 \pmod{p}, \quad p \equiv 3 \pmod 4.$

(vii) When an algebraic factor of an odd prime p, prime to Δ, divides j or $j-1728$, $(-\Delta\,|\,p) = -1$ and $p < \Delta$.

(viii) In the cubic fields the significance of (vii) is that $(p) = \mathfrak{pq}$ where $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of the first and second degrees respectively. Further, except when Δ or $\frac{1}{4}\Delta \equiv 7 \pmod 8$, $p = \Delta - a^2$ when $p > \frac{1}{4}\Delta$ and $N(j-1728) \equiv 0 \pmod{p}$; and every prime $\Delta - a^2$ $(> \frac{1}{4}\Delta)$ divides $\sqrt{N(j-1728)}$ just once.

(ix) In the quadratic fields $[\sqrt{m}]$ for which $(-1\,|\,m) = -1$ (*i.e.* when $m = 3, 6, 7, 21, 33$), the only primes p which occur are such that $(p\,|\,m) = -1$, and this, with $(m\,|\,p) = +1$, necessitates

$$p = mu^2 - v^2, \quad N(v + u\sqrt{m}) = -p.$$

On this account, when $(-1\,|\,m) = 1$, $N(\epsilon) = -1$, the factors in the table of § 4 have been chosen to have negative norms.

(x) When, in the quadratic fields, $N(j-1728) \equiv 0 \pmod{p}$, and $p > \frac{1}{4}\Delta$, then $p = \Delta - a^2$, and every prime of this type divides $\sqrt{N(j-1728)}$.

11. It remains to mention previous lists of class-invariants belonging to the discriminants of this paper. In a few cases j or J has been calculated before (hardly ever in factors) : in others, for which Δ is italicized here, material has been given for evaluating j by rational operations.

G. B. Mathews* gave the values of Klein's principal moduli arising from the solution of $j(\omega) = j(s\omega)$.

R. Russell's list† includes *44*, 52, *59*, 60, *76*, *83*, 91, 100, *107*, *108*, 112, *124*, 148, *172*, *187*, *268*, *331*, *427*, *907*.

G. Greenhill's last collection‡ gives *15*, *20*, *23*, *24*, *31*, 35, *36*, *40*, 51, *52*, 59, *72*, 75, 83, *88*, 91, 99, *100*, 107, 115, *147*, *148*, *187*, *211*, *232*, *283*, *331*, *427*, *907*.

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* *Proc. London Math. Soc.* (1), 21 (1890), 234–246.

† *Ibid.* (1891), 351–395.

‡ *Ibid.*, 403–422.

H. Weber's last list* contains *20*, *24*, *32*, *36*, *40*, *44*, *48*, 51, *52*, *60*, *64*, *72*, *76*, *88*, 91, *92*, 99, *100*, *108*, *112*, *124*, *148*, *172*, *232*, *268*, *652*.

W. E. H. Berwick† has given a table of values of τ_5 leading to j for 15, 20, 35, 40, 44, 59, 60, 75, 76, 100, 115, 139, 211, 235, 331, 379, 499.

It is believed that the class-invariants under $\Delta = 123$, 267, 307, 403, 547, 643, 883 are now given for the first time.

It has not been proved that the lists of Δ_2's and Δ_3's given above are complete, this being a well known crux in the theory of numbers; but, allowing for the fact that Gauss's tables‡ are not everywhere accurate, it seems unlikely that there are any others.

The author has carried out the heavier numerical work involved in the preparation of these results on a Trinks-Brunsviga calculating machine in the mathematical laboratory of Leeds University.

* *Lehrbuch der Algebra*, 3 (1908), 523, 721–726.

† *Quart. J. of Math.*, 50 (1923), 1–32.

‡ *Werke*, 2 (1876), 450–474.

ON AN INEQUALITY SATISFIED BY THE ZETA-FUNCTION OF RIEMANN

By E. C. TITCHMARSH.

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1. *Let σ be a fixed number in the range $\frac{1}{2} \leqslant \sigma < 1$. Then the inequality*

$$|\zeta(\sigma+it)| > \exp\{(\log t)^a\}$$

is satisfied for indefinitely large values of t, provided that

$$a < 1-\sigma.$$

This result has already been obtained by Littlewood* on the assumption that the Riemann hypothesis is true. It will now be proved independently of any hypothesis. It is obtained by considering integrals of the form

$$\int_0^\infty |\zeta(\sigma+it)|^{2k} e^{-\delta t}\, dt,$$

where k is a large positive integer and δ is small, k being ultimately made a function of δ.

Throughout the proof k is supposed large enough, and δ small enough, for any purpose that may be required. We take $\frac{1}{2} < \sigma < 1$ (except in § 7), and the constants $A_1, A_2, \ldots$, and those implied by the symbol O, are independent of k and δ, but may depend on σ, and on ϵ when it occurs. The case $\sigma = \frac{1}{2}$ is then deduced from the case $\sigma > \frac{1}{2}$.

We conclude with some further remarks on the functions $\zeta(1+it)$ and $\zeta(\frac{1}{2}+it)$.

2. LEMMA 1. *If $f(t)$ and $|f(t)|^2$ are integrable over $(-\infty, \infty)$, and*

$$\int_{-\infty}^{\infty} f(t)\, e^{-ixt}\, dt = \sqrt{(2\pi)}\, F(x),$$

then

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(x)|^2 dx.$$

This is well known.

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* J. E. Littlewood, *Proc. London Math. Soc.* (2), 24 (1925), 175–201.

3. LEMMA 2. *If* $\delta > 0$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\Gamma(\sigma+it)\,\zeta^k(\sigma+it)\,e^{(\frac{1}{2}\pi-\delta)t}|^2\,dt = \int_0^{\infty} |\phi_k(ixe^{-i\delta})|^2\,x^{2\sigma-1}\,dx,$$

where

$$\phi_k(x) = \sum_{n=1}^{\infty} d_k(n)\,e^{-nx} - \frac{1}{x}\left\{\lambda_1 - \lambda_2 \log x + \ldots + (-1)^{k-1}\,\frac{\lambda_k \log^{k-1} x}{(k-1)!}\right\},$$

$d_k(n)$ *being the number of decompositions of* n *into* k *factors, and the numbers* $\lambda_1, \lambda_2, \ldots$ *being the coefficients in the series*

$$\Gamma(s)\,\zeta^k(s) = \sum_{m=1}^{k} \frac{\lambda_m}{(s-1)^m} + \ldots.$$

We have, if $R(z) > 0$,

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(w)\,\zeta^k(w)\,z^{-w}\,dw = \sum_{n=1}^{\infty} d_k(n)\,e^{-nz}.$$

Moving the line of integration to $R(w) = \sigma$, and taking account of the residue at $w = 1$, we obtain

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(w)\,\zeta^k(w)\,z^{-w}\,dw = \phi_k(z).$$

Putting $z = ie^{x-i\delta}$ $(\delta > 0)$, we may write this in the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma(\sigma+it)\,\zeta^k(\sigma+it)\,\exp\left\{-(\sigma+it)(x+\tfrac{1}{2}i\pi-i\delta)\right\}dt = \phi_k(ie^{x-i\delta}).$$

Hence in Lemma 1 we may take

$$f(t) = (2\pi)^{-\frac{1}{2}}\,\Gamma(\sigma+it)\,\zeta^k(\sigma+it)\,\exp\left\{-i(\tfrac{1}{2}\pi-\delta)(\sigma+it)\right\},$$

$$F(x) = e^{\sigma x}\,\phi_k(ie^{x-i\delta}).$$

Consequently

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma(\sigma+it)\,\zeta^k(\sigma+it)\,e^{(\frac{1}{2}\pi-\delta)t}|^2\,dt = \int_{-\infty}^{\infty} e^{2\sigma x}\,|\phi_k(ie^{x-i\delta})|^2\,dx,$$

and the result stated follows.

4. LEMMA 3. $\qquad |\lambda_m| < \exp(A_1 k \log k) \quad (1 \leqslant m \leqslant k).$

We have

$$\sum_{m=1}^{k} \frac{\lambda_m}{(s-1)^m} + \ldots = \frac{1}{(s-1)^k}\sum_{n=0}^{\infty} c_n (s-1)^n \sum_{n=0}^{\infty} a_{n,k}(s-1)^n,$$

where

$$\Gamma(s) = \sum_{n=0}^{\infty} c_n(s-1)^n, \quad \xi^k(s) = (s-1)^{-k} \sum_{n=0}^{\infty} a_{n,k}(s-1)^n.$$

Now $$|c_n| < A_2^n, \quad |a_{n,1}| < A_3^n \quad (A_2 > 1, \ A_3 > 1).$$

Hence $|a_{n,k}|$ is less than the coefficient of $(s-1)^n$ in

$$\left\{\sum_{n=0}^{\infty} A_3^n(s-1)^n\right\}^k = \{1-A_3(s-1)\}^{-k} = \sum_{n=0}^{\infty} \frac{(k+n)!}{k!\,n!} A_3^n(s-1)^n.$$

Hence

$$|\lambda_m| = \left|\sum_{r+n=k-m} c_r a_{n,k}\right| < \sum_{n=0}^{k-m} A_2^{k-m+n} \frac{(k+n)!}{k!\,n!} A_3^n$$

$$< A_2^k A_3^k k \frac{(2k)!}{(k!)^2} < \exp(A_1 k \log k).$$

5. Lemma 4.

$$\frac{1}{\pi}\int_{-\infty}^{\infty} |\Gamma(\sigma+it)\,\xi^k(\sigma+it)\,e^{(\frac{1}{2}\pi-\delta)t}|^2\,dt$$

$$> \int_1^{\infty} \left|\sum_{n=1}^{\infty} d_k(n) \exp(-inxe^{-i\delta})\right|^2 x^{2\sigma-1}\,dx - \exp(A_4 k \log k).$$

By Lemma 2 the left-hand side is greater than

$$2\int_1^{\infty} |\phi_k(ixe^{-i\delta})|^2 x^{2\sigma-1}\,dx > \int_1^{\infty} \left|\sum_{n=1}^{\infty} d_k(n) \exp(-inxe^{-i\delta})\right|^2 x^{2\sigma-1}\,dx$$

$$-2\int_1^{\infty} |\psi_k(ixe^{-i\delta})|^2 x^{2\sigma-1}\,dx,$$

where $$\psi_k(x) = \frac{1}{x}\left\{\lambda_1 - \lambda_2 \log x + \ldots + (-1)^{k-1}\lambda_k \frac{\log^{k-1} x}{(k-1)!}\right\},$$

$$|\psi_k(ixe^{-i\delta})| < \frac{1}{x}\left\{|\lambda_1| + |\lambda_2|(\log x + \tfrac{1}{2}\pi) + \ldots + \frac{|\lambda_k|}{(k-1)!}(\log x + \tfrac{1}{2}\pi)^{k-1}\right\}$$

$$< \frac{(\log x + \frac{1}{2}\pi)^{k-1}}{x}\left\{|\lambda_1| + \ldots + \frac{|\lambda_k|}{(k-1)!}\right\}.$$

Now $$\int_1^{\infty} (\log x + \tfrac{1}{2}\pi)^{2k-2} x^{2\sigma-3}\,dx < \int_1^{\infty} (2\log x)^{2k-2} x^{2\sigma-3}\,dx + \int_1^{\infty} \pi^{2k-2} x^{2\sigma-3}\,dx$$

$$= \frac{\Gamma(2k-1)}{(2-2\sigma)^{2k-1}} + \frac{\pi^{2k-2}}{2-2\sigma},$$

and, by Lemma 3,

$$|\lambda_1|+\ldots+\frac{|\lambda_k|}{(k-1)!}<\exp(A_1 k\log k+1).$$

Hence the result.

6. LEMMA 5.

$$\int_1^\infty\left|\sum_{n=1}^{\infty}d_k(n)\exp(-inxe^{-i\delta})\right|^2 x^{2\sigma-1}\,dx>\frac{A_5}{\delta^{2\sigma}}\sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{2\sigma}}e^{-2n\sin\delta}$$
$$-A_6\log\frac{1}{\delta}\sum_{n=1}^{\infty}d_k^2(n)\,e^{-n\sin\delta}.$$

We have

$$\int_1^\infty\left|\sum_{n=1}^{\infty}d_k(n)\exp(-inxe^{-i\delta})\right|^2 x^{2\sigma-1}\,dx$$
$$=\int_1^\infty\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}d_k(m)\,d_k(n)\exp(imxe^{i\delta}-inxe^{-i\delta})\,x^{2\sigma-1}\,dx$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}d_k(m)\,d_k(n)\int_1^\infty\exp(imxe^{i\delta}-inxe^{-i\delta})\,x^{2\sigma-1}\,dx$$
$$=\sum_{n=1}^{\infty}d_k^2(n)\int_1^\infty e^{-2nx\sin\delta}x^{2\sigma-1}\,dx$$
$$+\sum\sum_{m\neq n}d_k(m)\,d_k(n)\int_1^\infty\exp\{-(m+n)x\sin\delta$$
$$+i(m-n)x\cos\delta\}\,x^{2\sigma-1}\,dx$$
$$=\Sigma_1+\Sigma_2,$$

say. Now

$$\int_1^\infty x^{2\sigma-1}e^{-2nx\sin\delta}\,dx=(2n\sin\delta)^{-2\sigma}\int_{2n\sin\delta}^\infty y^{2\sigma-1}e^{-y}\,dy,$$

and, for $2n\sin\delta\leqslant 1$,

$$\int_{2n\sin\delta}^\infty y^{2\sigma-1}e^{-y}\,dy\geqslant\int_1^\infty y^{2\sigma-1}e^{-y}\,dy=A_7>A_7e^{-2n\sin\delta},$$

while, for $2n\sin\delta>1$,

$$\int_{2n\sin\delta}^\infty y^{2\sigma-1}e^{-y}\,dy>\int_{2n\sin\delta}^\infty e^{-y}\,dy=e^{-2n\sin\delta}.$$

Hence

$$\Sigma_1>\frac{A_5}{\delta^{2\sigma}}\sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{2\sigma}}e^{-2n\sin\delta}.$$

Again, for $m > n$,

$$\left| \int_1^\infty \exp\{-(m+n)\,x\sin\delta + i(m-n)\,x\cos\delta\}\,x^{2\sigma-1}\,dx \right|$$

$$= \left| \int_0^\infty \exp\{-(m+n)(1+iy)\sin\delta + i(m-n)(1+iy)\cos\delta\}(1+iy)^{2\sigma-1}\,dy \right|$$

$$< 2^{2\sigma-1} e^{-(m+n)\sin\delta} \int_0^1 e^{-(m-n)y\cos\delta}\,dy + 2^{2\sigma-1} e^{-(m+n)\sin\delta} \int_1^\infty e^{-(m-n)y\cos\delta} y^{2\sigma-1}\,dy$$

$$< A_8 \frac{e^{-(m+n)\sin\delta}}{m-n} < A_8 \frac{e^{-m\sin\delta}}{m-n}.$$

Hence

$$|\Sigma_2| < 2A_8 \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} d_k(m)\,d_k(n)\,\frac{e^{-m\sin\delta}}{m-n}$$

$$= 2A_8 \sum_{r=1}^{\infty} \sum_{m=r+1}^{\infty} d_k(m)\,d_k(m-r)\,\frac{e^{-m\sin\delta}}{r}$$

$$= 2A_8 \sum_{r=1}^{\infty} \frac{e^{-\frac{1}{2}r\sin\delta}}{r} \sum_{m=r+1}^{\infty} d_k(m)\,e^{-\frac{1}{2}m\sin\delta}\,d_k(m-r)\,e^{-\frac{1}{2}(m-r)\sin\delta}$$

$$\leqslant 2A_8 \sum_{r=1}^{\infty} \frac{e^{-\frac{1}{2}r\sin\delta}}{r} \left\{ \sum_{m=r+1}^{\infty} d_k^2(m)\,e^{-m\sin\delta} \sum_{m=r+1}^{\infty} d_k^2(m-r)\,e^{-(m-r)\sin\delta} \right\}^{\frac{1}{2}}$$

$$< 2A_8 \sum_{r=1}^{\infty} \frac{e^{-\frac{1}{2}r\sin\delta}}{r} \sum_{m=1}^{\infty} d_k^2(m)\,e^{-m\sin\delta}$$

$$< A_6 \log\frac{1}{\delta} \sum_{m=1}^{\infty} d_k^2(m)\,e^{-m\sin\delta}.$$

This proves the lemma.

7. *The function* $F_k(s)$. We have the identity

$$\sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} = \prod_p \sum_{m=0}^{\infty} \frac{(k+m-1)!}{(k-1)!\,m!}\,\frac{1}{p^{ms}} \quad (\sigma > 1),$$

each side being equal to $\zeta^k(s)$. If we replace the coefficient of each term p^{-ms} by its square, the coefficient of each n^{-s} is replaced by its square. Hence

$$\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^s} = \prod_p \sum_{m=0}^{\infty} \left\{ \frac{(k+m-1)!}{(k-1)!\,m!} \right\}^2 \frac{1}{p^{ms}}. \tag{7.1}$$

We shall write this as

$$F_k(s) = \prod_p f_k(p^{-s}).$$

Now it is easily verified that

$$\frac{(k+m-1)!}{(k-1)!\,m!} \leqslant \frac{(k^2+m+1)!}{(k^2-1)!\,m!}.$$

Hence, for $0 < x < 1$,

$$f_k(x) \leqslant \sum_{m=0}^{\infty} \frac{(k^2+m-1)!}{(k^2-1)!\,m!}\, x^m = (1-x)^{-k^2}. \tag{7.2}$$

Again
$$f_k(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \sum_{m=0}^{\infty} \frac{(k+m-1)!}{(k-1)!\,m!}\, x^{\frac{1}{2}m}\, e^{im\phi} \right|^2 d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} |\,1-x^{\frac{1}{2}}\,e^{i\phi}\,|^{-2k}\,d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{d\phi}{(1-2\sqrt{x}\cos\phi+x)^k}. \tag{7.3}$$

[Comparing this with the formula

$$P_n(z) = \frac{1}{\pi}\int_0^{\pi} \frac{d\phi}{\{z-\sqrt{(z^2-1)}\cos\phi\}^{n+1}},$$

where $P_n(z)$ denotes, as usual, the Legendre polynomial of degree n, we see that

$$f_k(x) = \frac{1}{(1-x)^k}\, P_{k-1}\left(\frac{1+x}{1-x}\right).$$

But it seems most convenient to obtain the required properties of $f_k(x)$ directly.]

Hence

$$f_k(x) < (1-\sqrt{x})^{-2k}, \tag{7.4}$$

and also, for $0 < x < \frac{1}{2}$,

$$f_k(x) > \frac{1}{\pi}\int_0^{\pi/k} \frac{d\phi}{(1-2\sqrt{x}\cos\phi+x)^k}$$

$$= \frac{1}{(1-\sqrt{x})^{2k}}\,\frac{1}{\pi}\int_0^{\pi/k} \left\{1-\frac{2\sqrt{x}(1-\cos\phi)}{1-2\sqrt{x}\cos\phi+x}\right\}^k d\phi$$

$$= \frac{1}{(1-\sqrt{x})^{2k}}\,\frac{1}{\pi}\int_0^{\pi/k} \left\{1-O\left(\frac{1}{k^2}\right)\right\}^k d\phi$$

$$> \frac{1}{2k(1-\sqrt{x})^{2k}}. \tag{7.5}$$

Hence

$$\begin{aligned}\log|F_k(s)| &\leqslant \log|F_k(\sigma)| \\ &= \sum_{p^\sigma \leqslant k^2} \log|f_k(p^{-\sigma})| + \sum_{p^\sigma > k^2} \log|f_k(p^{-\sigma})| \\ &\leqslant 2k \sum_{p^\sigma \leqslant k^2} \log(1-p^{-\frac{1}{2}\sigma})^{-1} + k^2 \sum_{p^\sigma > k^2} \log(1-p^{-\sigma})^{-1} \\ &= O\Big(k \sum_{p^\sigma \leqslant k^2} p^{-\frac{1}{2}\sigma}\Big) + O\Big(k^2 \sum_{p^\sigma > k^2} p^{-\sigma}\Big) \\ &= O\{k(k^{2/\sigma})^{1-\frac{1}{2}\sigma}\} + O\{k^2(k^{2/\sigma})^{1-\sigma}\} = O\{k^{2/\sigma}\},\end{aligned}$$

or

$$|F_k(s)| < \exp\{A_9 k^{2/\sigma}\}. \tag{7.6}$$

On the other hand, for any N,

$$\begin{aligned}\log F_k(\sigma) &> \sum_{p<N} \log\{f_k(p^{-\sigma})\} \\ &> 2k \sum_{p<N} \log(1-p^{-\frac{1}{2}\sigma})^{-1} - \sum_{p<N} \log 2k \\ &> 2k \sum_{p<N} p^{-\frac{1}{2}\sigma} - A_{10} \frac{N}{\log N} \log k \\ &> A_{11} k \frac{N^{1-\frac{1}{2}\sigma}}{\log N} - A_{10} \frac{N}{\log N} \log k.\end{aligned}$$

Let

$$N = \Big(\frac{A_{11}}{2A_{10}} \frac{k}{\log k}\Big)^{2/\sigma}.$$

Then

$$\log F_k(\sigma) > A_{12}\Big(\frac{k}{\log k}\Big)^{2/\sigma},$$

$$F_k(\sigma) > \exp\Big\{A_{12}\Big(\frac{k}{\log k}\Big)^{2/\sigma}\Big\}. \tag{7.7}$$

8. LEMMA 6. *If* $0 < \epsilon < 2\sigma - 1$,

$$\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} e^{-2n\sin\delta} > \exp\Big\{A_{12}\Big(\frac{k}{\log k}\Big)^{1/\sigma}\Big\} - \delta^\epsilon \exp\{A_{13} k^{2/(2\sigma-\epsilon)}\}.$$

We have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} e^{-2n\sin\delta} &= \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \Gamma(w) \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} (2n\sin\delta)^{-w}\, dw \\ &= \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \Gamma(w) F_k(2\sigma+w)(2\sin\delta)^{-w}\, dw \\ &= F_k(2\sigma) + \frac{1}{2\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \Gamma(w) F_k(2\sigma+w)(2\sin\delta)^{-w}\, dw.\end{aligned}$$

Now, by (7 . 7),

$$F_k(2\sigma) > \exp\left\{A_{12}\left(\frac{k}{\log k}\right)^{1/\sigma}\right\},$$

and, by (7 . 6),

$$|F_k(2\sigma+iv)| \leqslant F_k(2\sigma-\epsilon) < \exp\left\{A_9 k^{2/(2\sigma-\epsilon)}\right\}.$$

Also
$$(2\sin\delta)^{-w} = O(\delta^{\epsilon}),$$

$$\int_{-\infty}^{\infty} |\Gamma(-\epsilon+iv)|\,dv = O(1).$$

Hence the result.

9. LEMMA 7.

$$\sum_{n=1}^{x} d_k^2(n)\, e^{-n\sin\delta} < \delta^{\epsilon-2\sigma} \exp\left\{A_{14} k^{2/(2\sigma-\epsilon)}\right\}.$$

For
$$\sum_{n=1}^{\infty} d_k^2(n)\, e^{-n\sin\delta} = \frac{1}{2\pi i}\int_{2\sigma-\epsilon-i\infty}^{2\sigma-\epsilon+i\infty} \Gamma(w)\, F_k(w)(\sin\delta)^{-w}\, dw,$$

and the result follows as in the previous lemma.

10. *Proof of the main theorem for* $\frac{1}{2} < \sigma < 1$. Collecting the results of the lemmas, we see that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} |\Gamma(\sigma+it)\,\zeta^k(\sigma+it)\,e^{(\frac{1}{2}\pi-\delta)t}|^2\,dt > \frac{A_5}{\delta^{2\sigma}}\exp\left\{A_{12}\left(\frac{k}{\log k}\right)^{1/\sigma}\right\} \tag{10 . 1}$$

$$-\frac{A_5}{\delta^{2\sigma-\epsilon}}\exp\left\{A_{13}k^{2/(2\sigma-\epsilon)}\right\} - \frac{A_6}{\delta^{2\sigma-\epsilon}}\log\frac{1}{\delta}\exp\left\{A_{14}k^{2/(2\sigma-\epsilon)}\right\}$$

$$-\exp(A_4 k\log k).$$

Let

$$\delta = \exp\left\{-\frac{A_{15}}{\epsilon}k^{2/(2\sigma-\epsilon)}\right\}, \quad A_{15} = \mathrm{Max}\,(A_{13}, A_{14}). \tag{10 . 2}$$

Then the last three terms on the right of (10 . 1) are small compared with the first, so that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} |\Gamma(\sigma+it)\,\zeta^k(\sigma+it)\,e^{(\frac{1}{2}\pi-\delta)t}|^2\,dt > \tfrac{1}{2}\frac{A_5}{\delta^{2\sigma}}\exp\left\{A_{12}\left(\frac{k}{\log k}\right)^{1/\sigma}\right\}.$$

Since
$$|\Gamma(\sigma+it)| \sim A_{16}|t|^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}\pi|t|},$$

the result may also be written

$$\int_0^{\infty} t^{2\sigma-1}|\zeta(\sigma+it)|^{2k}e^{-2\delta t}\,dt > \frac{A_{17}}{\delta^{2\sigma}}\exp\left\{A_{12}\left(\frac{k}{\log k}\right)^{1/\sigma}\right\}. \tag{10 . 3}$$

Suppose now that

$$|\xi(\sigma+it)| \leqslant \exp\{(\log t)^\alpha\} \quad (t \geqslant T).$$

Then

$$\int_0^\infty t^{2\sigma-1}|\xi(\sigma+it)|^{2k} e^{-2\delta t}\,dt < A_{18}T^{k+1} + \int_2^\infty t^{2\sigma-1}\exp(2k\log^\alpha t)\, e^{-2\delta t}\,dt.$$

Now

$$\exp(2k\log^\alpha t) < \exp\left\{A_{19}\left(\log\frac{1}{\delta}\right)^{\sigma-\frac{1}{2}\epsilon}\log^\alpha t\right\} < \exp\left\{A_{20}\left(\log\frac{1}{\delta}\right)^{\alpha+\sigma-\frac{1}{2}\epsilon}\right\}$$

$$(t < \delta^{-2}),$$

$$< \exp\{A_{21}(\log t)^{\alpha+\sigma-\frac{1}{2}\epsilon}\} \quad (t > \delta^{-2}).$$

Hence

$$\int_2^\infty t^{2\sigma-1}\exp(2k\log^\alpha t)\, e^{-2\delta t}\,dt < \exp\left\{A_{20}\left(\log\frac{1}{\delta}\right)^{\alpha+\sigma-\frac{1}{2}\epsilon}\right\}\int_2^{\delta^{-2}} t^{2\sigma-1}e^{-2\delta t}\,dt$$

$$+\int_{\delta^{-2}}^\infty t^{2\sigma-1}\exp\{A_{21}(\log t)^{\alpha+\sigma-\frac{1}{2}\epsilon}\}\, e^{-2\delta t}\,dt$$

$$< \frac{A_{22}}{\delta^{2\sigma}}\exp\left\{A_{20}\left(\log\frac{1}{\delta}\right)^{\alpha+\sigma-\frac{1}{2}\epsilon}\right\} + A_{23}.$$

Comparing this with (10.3), we see that

$$A_{17}\exp\left\{A_{12}\left(\frac{k}{\log k}\right)^{1/\sigma}\right\} < 2A_{22}\exp\left\{A_{20}\left(\log\frac{1}{\delta}\right)^{\alpha+\sigma-\frac{1}{2}\epsilon}\right\}$$

$$= 2A_{22}\exp\left\{A_{20}\left(\frac{A_{15}}{\epsilon}k^{2/(2\sigma-\epsilon)}\right)^{\alpha+\sigma-\frac{1}{2}\epsilon}\right\}.$$

This can only be true for large k if

$$\frac{1}{\sigma} \leqslant \frac{2(\alpha+\sigma-\frac{1}{2}\epsilon)}{2\sigma-\epsilon},$$

i.e.

$$\alpha \geqslant (1-\sigma)\left(1-\frac{\epsilon}{2\sigma}\right).$$

Since ϵ may be as small as we please, the result follows.

11. *The case* $\sigma = \frac{1}{2}$. Suppose that

$$\xi(\tfrac{1}{2}+it) = O\{\exp(\log^\beta t)\} \quad (0 < \beta < \tfrac{1}{2}).$$

Then the function $f(s) = \xi(s)\exp(-\log^\beta s)$

is bounded on the lines $\sigma = \frac{1}{2}$, $\sigma = 2$, $t > t_0$, and it is $O(t)$ uniformly

in this strip. Hence, by the method of Phragmén and Lindelöf, $f(s)$ is bounded in the strip, *i.e.*

$$\zeta(\sigma+it) = O\{\exp(\log^{\beta} t)\}$$

for $\frac{1}{2} < \sigma < 2$. But this is not true for $\frac{1}{2} < \sigma < 1-\beta$. Hence the result.

12. *The function* $\zeta(1+it)$. It was proved by Littlewood, in the memoir already referred to, that

$$\overline{\lim_{t\to\infty}} \frac{|\zeta(1+it)|}{\log\log t} \geqslant e^{\gamma},$$

where γ is Euler's constant. This may also be proved as follows. (The use of the integral with which we begin was suggested by Prof. Hardy.)

Let $\sigma > 1$. Then

$$\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)|\zeta(\sigma+it)|^{2k}\,dt = \int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\sum_{m=1}^{\infty}\frac{d_k(m)}{m^{\sigma+it}}\sum_{n=1}^{\infty}\frac{d_k(n)}{n^{\sigma-it}}\,dt$$

$$= \sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{2\sigma}}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)dt - \sum\sum_{m\neq n}\frac{d_k(m)\,d_k(n)}{(mn)^{\sigma}}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\left(\frac{n}{m}\right)^{it}dt$$

$$= T\sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{2\sigma}} + \sum\sum_{m\neq n}\frac{d_k(m)\,d_k(n)}{(mn)^{\sigma}}\,\frac{4\sin^2\{\frac{1}{2}T\log(m/n)\}}{T\log^2(m/n)}$$

$$\geqslant T\sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{2\sigma}} > T\prod_{p<\delta k} f_k(p^{-2\sigma}) \qquad (k^{-\frac{1}{2}} < \delta < 1)$$

$$> T\prod_{p<\delta k}\left\{\frac{1}{2k}\left(1-\frac{1}{p^{\sigma}}\right)^{-2k}\right\} > T\left(\frac{1}{2k}\right)^{2\delta k/\log(\delta k)}\prod_{p<\delta k}\left(1-\frac{1}{p^{\sigma}}\right)^{-2k}.$$

Let $\sigma = 1+u/\log k$, $0 < u < 1$. Then

$$\log\left\{\left(\frac{1}{2k}\right)^{2\delta k/\log(\delta k)}\right\} > -\frac{4\delta k}{\log k}\log 2k > -8\delta k,$$

and, as in Littlewood's proof,

$$\log\prod_{p<\delta k}\left(\frac{1-p^{-1}}{1-p^{-\sigma}}\right) > \log\prod_{p<k}\left(\frac{1-p^{-1}}{1-p^{-\sigma}}\right) > -Au.$$

Hence

$$\left\{\frac{2}{T}\int_0^T\left(1-\frac{t}{T}\right)|\zeta(\sigma+it)|^{2k}\,dt\right\}^{1/2k} > e^{-4\delta-Au}\prod_{p<\delta k}\left(1-\frac{1}{p}\right)^{-1}$$

$$= e^{-4\delta-Au+\gamma+o(1)}\log\delta k,$$

since

$$\prod_{p<x}\left(1-\frac{1}{p}\right)^{-1} \sim e^{\gamma}\log x.$$

Now $$\left\{\int_1^T \left(1-\frac{t}{T}\right)|\xi(\sigma+it)|^{2k}\,dt\right\}^{1/2k} \geqslant \left\{\int_0^T \left(1-\frac{t}{T}\right)|\xi(\sigma+it)|^{2k}\,dt\right\}^{1/2k} - \left\{\int_0^1 \left(1-\frac{t}{T}\right)|\xi(\sigma+it)|^{2k}\,dt\right\}^{1/2k},$$

and the last term is less than

$$\left\{\int_0^1 \left(1-\frac{t}{T}\right)\left(\frac{2}{\sigma-1}\right)^{2k} dt\right\}^{1/2k} < \frac{2}{\sigma-1} = \frac{2\log k}{u}.$$

Hence

$$\left\{\frac{2}{T}\int_1^T \left(1-\frac{t}{T}\right)|\xi(\sigma+it)|^{2k}\,dt\right\}^{1/2k} > e^{-4\delta - Au+\gamma+o(1)}\log \delta k - \left(\frac{2}{T}\right)^{1/2k}\frac{2\log k}{u}.$$

Taking $T = 2H^{2k}$, and choosing δ and u small enough, then H large enough, then k large enough, we obtain

$$\operatorname*{Max}_{1\leqslant t\leqslant T}|\xi(\sigma+it)| > e^{\gamma-\epsilon}\log\log T.$$

The result then follows from the usual Phragmén-Lindelöf argument.

13. *The mean value of* $|\xi(\frac{1}{2}+it)|^{2k}$. We may also note that for any *fixed* integer k, as $\delta\to 0$,

$$\int_0^\infty |\xi(\tfrac{1}{2}+it)|^{2k} e^{-\delta t}\,dt \geqslant \frac{C_k}{\delta}\log^{k^2}\frac{1}{\delta},$$

where C_k depends on k only. For

$$F_k(s) = \xi^{k^2}(s)\prod_p\left\{f_k\left(\frac{1}{p^s}\right)\left(1-\frac{1}{p^s}\right)^{k^2}\right\},$$

and this product is convergent for $\sigma > \frac{1}{2}$. Hence $F_k(s)$ is regular for $\sigma > \frac{1}{2}$, except for a pole of order k^2 at $s = 1$. Hence

$$\sum_{n=1}^\infty \frac{d_k^2(n)}{n} e^{-2n\sin\delta} = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\Gamma(s)\,F_k(s+1)(2\sin\delta)^{-s}\,ds \sim C_k\log^{k^2}\frac{1}{\delta}$$

(moving the contour to the left and evaluating the residue at $s = 0$).

Methods similar to those used above now lead to the result.

A NOTE ON CERTAIN APPROXIMATE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER, WITH AN APPLICATION TO THE MATHIEU EQUATION

By S. GOLDSTEIN.

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1. *A note on certain approximate solutions of linear differential equations of the second order.*

If χ is a uniform positive function of x, capable of assuming large values in the range of x to be considered, then the differential equation

$$y''-\chi y=0 \tag{1}$$

is approximately satisfied by

$$y=\chi^{-\frac{1}{4}}\exp(\pm M), \tag{2}$$

where

$$M=\int^{x}\chi^{\frac{1}{2}}dx. \tag{3}$$

If χ is negative and equal to $-\psi$, these solutions may be replaced by

$$y=\psi^{-\frac{1}{4}}\cos L,\quad \psi^{-\frac{1}{4}}\sin L, \tag{4}$$

where

$$L=\int^{x}\psi^{\frac{1}{2}}dx. \tag{5}$$

If we put $\chi=h^2\chi_0$, where h is a parameter independent of x, capable of assuming large values, then these solutions are really the first terms in asymptotic expansions of the form

$$y=\phi e^{h\omega}\left(1+\frac{f_1}{h}+\frac{f_2}{h^2}+\dots\right), \tag{6}$$

where ϕ, ω, f_1, f_2, ... are functions of x alone*.

* Jeffreys, *Proc. London Math. Soc.* (2), 23 (1925), 428 *et seq.*, and references there given.

The approximations fail near a zero of χ, and Jeffreys has given a method of finding what linear combinations of the solutions on one side of the zero are approximations to the same functions that, say, the standard solutions are on the other side. The method is here extended to a zero of any order. The results differ slightly from those obtained by Jeffreys, and it is worth noting that this difference does not affect the result of the application made by Jeffreys to Mathieu functions (*loc. cit.*, p 445).

1.1. Let x_0 be a zero of χ. When x is near x_0 put

$$x_0 - x = x_1, \quad x - x_0 = x'. \tag{1}$$

Consider first a zero of order $2n$, where χ is negative on both sides of the zero. In order to make the standard approximate solutions, $\psi^{-\frac{1}{4}} \cos L$ and $\psi^{-\frac{1}{4}} \sin L$, definite, let

$$L = \int_x^{x_0} \psi^{\frac{1}{2}} dx \quad (x < x_0), \qquad L = \int_{x_0}^{x} \psi^{\frac{1}{2}} dx \quad (x > x_0), \tag{2}$$

the positive branch of the integrand being taken in each case, so that L is essentially positive. Let

$$\left.\begin{aligned} \psi^{-\frac{1}{4}} \cos L &= G_1 \\ \psi^{-\frac{1}{4}} \sin L &= g_1 \end{aligned}\right\} \ (x < x_0), \qquad \left.\begin{aligned} \psi^{-\frac{1}{4}} \cos L &= G' \\ \psi^{-\frac{1}{4}} \sin L &= g' \end{aligned}\right\} \ (x > x_0). \tag{3}$$

In the neighbourhood of x_0,

$$\psi = \lambda^{2n+2} x_1^{2n} \quad \text{or} \quad \lambda^{2n+2} x'^{2n}, \tag{4}$$

where

$$\lambda^{2n+2} = \frac{1}{(2n)!} \left(\frac{d^{2n}\psi}{dx^{2n}}\right)_{x_0}, \tag{5}$$

and the differential equation becomes

$$\frac{d^2 y}{dx_1^2} + \lambda^{2n+2} x_1^{2n} y = 0. \tag{6}$$

Also, near x_0,

$$L = \theta_1 \quad (x < x_0), \qquad L = \theta' \quad (x > x_0), \tag{7}$$

where

$$\theta_1 = \lambda^{n+1} x_1^{n+1}/(n+1), \qquad \theta' = \lambda^{n+1} x'^{n+1}/(n+1); \tag{8}$$

and

$$\left.\begin{aligned} G_1 &= \lambda^{-\frac{1}{2}(n+1)} x_1^{-\frac{1}{2}n} \cos\theta_1, & G' &= \lambda^{-\frac{1}{2}(n+1)} x'^{-\frac{1}{2}n} \cos\theta' \\ g_1 &= \lambda^{-\frac{1}{2}(n+1)} x_1^{-\frac{1}{2}n} \sin\theta_1, & g' &= \lambda^{-\frac{1}{2}(n+1)} x'^{-\frac{1}{2}n} \sin\theta' \end{aligned}\right\}. \tag{9}$$

The solutions of (6), when x_1 is positive, are $x_1^{\frac{1}{2}} J_{1/(2n+2)}(\theta_1)$ and $x_1^{\frac{1}{2}} J_{-[1/(2n+2)]}(\theta_1)$. The continuations of these functions when x' is positive are $-x'^{\frac{1}{2}} J_{1/(2n+2)}(\theta')$ and $x'^{\frac{1}{2}} J_{-[1/(2n+2)]}(\theta')$.

Also

$$\lambda^{-\frac{1}{2}(n+1)} x_1^{-\frac{1}{2}n} \cos\theta_1 \sim \sqrt{(\tfrac{1}{2}\pi)}\ \operatorname{cosec}\frac{\pi}{2n+2}\ x_1^{\frac{1}{2}} \times\left[J_{-[1/(2n+2)]}(\theta_1)\sin\frac{(n+2)\pi}{4n+4} - J_{1/(2n+2)}(\theta_1)\sin\frac{n\pi}{4n+4}\right]$$

$$= \sqrt{(\tfrac{1}{2}\pi)}\ \operatorname{cosec}\frac{\pi}{2n+2}\ x'^{\frac{1}{2}}\left[J_{-[1/(2n+2)]}(\theta')\sin\frac{(n+2)\pi}{4n+4} + J_{1/(2n+2)}(\theta')\sin\frac{n\pi}{4n+4}\right]$$

$$\sim \operatorname{cosec}\frac{\pi}{2n+2}\ \lambda^{-\frac{1}{2}(n+1)} x'^{-\frac{1}{2}n}\left[\cos\theta' + \sin\theta'\cos\frac{\pi}{2n+2}\right]; \tag{10}$$

and

$$\lambda^{-\frac{1}{2}(n+1)} x_1^{-\frac{1}{2}n} \sin\theta_1 \sim \sqrt{(\tfrac{1}{2}\pi)}\ \operatorname{cosec}\frac{\pi}{2n+2}\ x_1^{\frac{1}{2}} \times\left[J_{1/(2n+2)}(\theta_1)\cos\frac{n\pi}{4n+4} - J_{-[1/(2n+2)]}(\theta_1)\cos\frac{(n+2)\pi}{4n+4}\right]$$

$$= -\sqrt{(\tfrac{1}{2}\pi)}\ \operatorname{cosec}\frac{\pi}{2n+2}\ x'^{\frac{1}{2}} \times\left[J_{1/(2n+2)}(\theta')\cos\frac{n\pi}{4n+4} + J_{-[1/(2n+2)]}(\theta')\cos\frac{(n+2)\pi}{4n+4}\right]$$

$$\sim -\operatorname{cosec}\frac{\pi}{2n+2}\ \lambda^{-\frac{1}{2}(n+1)} x'^{-\frac{1}{2}n}\left[\cos\theta'\cos\frac{\pi}{2n+2} + \sin\theta'\right]. \tag{11}$$

Hence
$$G_1 \longleftrightarrow G' \operatorname{cosec}\frac{\pi}{2n+2} + g' \cot\frac{\pi}{2n+2}, \tag{12}$$

$$g_1 \longleftrightarrow -G' \cot\frac{\pi}{2n+2} - g' \operatorname{cosec}\frac{\pi}{2n+2}, \tag{13}$$

where the sign $\longleftrightarrow$ is used, as by Jeffreys, to indicate that the functions it connects are asymptotic approximations to the same function in different circumstances.

1.2. Now consider a zero of order $2n+1$. Without loss of generality we may suppose that χ is positive for $x < x_0$, and negative for $x > x_0$.

The solutions are then, for $x < x_0$,

$$G_1 = \chi^{-\frac{1}{4}} \exp M, \quad g_1 = \chi^{-\frac{1}{4}} \exp(-M), \tag{1}$$

where

$$M = \int_x^{x_0} \chi^{\frac{1}{2}} dx; \tag{2}$$

and, for $x > x_0$,

$$G' = \psi^{-\frac{1}{4}} \cos L, \quad g' = \psi^{-\frac{1}{4}} \sin L, \tag{3}$$

where

$$L = \int_{x_0}^{x} \psi^{\frac{1}{2}} dx. \tag{4}$$

Now let Y, y be a pair of independent accurate solutions. Then, except in the neighbourhood of x_0, g_1/G_1 is small, and therefore, if, for $x < x_0$, G_1, g_1 are approximations to Y, y respectively, then $Y+Ay$, where A is any constant multiplier, is also approximately represented by G_1. But Y and $Y+Ay$ have different asymptotic expansions for $x > x_0$. Hence, to the left of x_0, G_1 is an approximation to any one of a single infinity of accurate solutions, all of which have different approximations, one from another, to the right of x_0. The connexions must therefore be, to some extent, indeterminate. This indeterminateness can be removed only by using some convention fixing the accurate solution to which G_1 is to approximate for $x < x_0$.

Now, beyond x_0, the approximate solutions are oscillatory; and it is always possible to choose the accurate solution to be represented by G_1 for $x < x_0$, so that the continuation of G_1, beyond x_0, differs in phase by $\frac{1}{2}\pi$ from the continuation of g_1. The accurate solutions asymptotically represented by G_1 and g_1 to the left of x have then, to the right of x_0, a phase difference asymptotically equal to $\frac{1}{2}\pi$. An example is afforded by the Bessel functions $J_n(nx)$, $J_{-n}(nx)$, whose character changes from hyperbolic to oscillatory when $x = 1$, the asymptotic expansions for $x > 1$ being the Debye expansions. The function whose phase differs asymptotically by $\frac{1}{2}\pi$ from that of J_n when $x > 1$ is Y_n.

This convention having been adopted, the procedure of (1.1) is easily carried out for this case, the connexions being made by means of the Bessel functions $I_{\pm[1/(2n+3)]}$, $J_{\pm[1/(2n+3)]}$. The results are

$$G_1 \longleftrightarrow \operatorname{cosec} \frac{\pi}{2(2n+3)} \psi^{-\frac{1}{4}} \cos(\tfrac{1}{4}\pi - L) = \frac{1}{2^{\frac{1}{2}}} (G'+g') \operatorname{cosec} \frac{\pi}{2(2n+3)}, \tag{5}$$

$$g_1 \longleftrightarrow 2 \sin \frac{\pi}{2(2n+3)} \psi^{-\frac{1}{4}} \sin(\tfrac{1}{4}\pi - L) = 2^{\frac{1}{2}} (G'-g') \sin \frac{\pi}{2(2n+3)}. \tag{6}$$

1.3. Finally, consider a zero of order $2n$, when χ is positive on both sides of x_0. The approximate solutions are then, for $x < x_0$,

$$G_1 = \chi^{-\frac{1}{4}} \exp M, \quad g_1 = \chi^{-\frac{1}{4}} \exp(-M), \tag{1}$$

where

$$M = \int_x^{x_0} \chi^{\frac{1}{2}}\, dx; \tag{2}$$

and, for $x > x_0$,

$$G' = \chi^{-\frac{1}{4}} \exp M, \quad g' = \chi^{-\frac{1}{4}} \exp(-M), \tag{3}$$

where

$$M = \int_{x_0}^{x} \chi^{\frac{1}{2}}\, dx. \tag{4}$$

In this case, both g_1/G_1 and g'/G' are small outside of the neighbourhood of x_0. As before, there is indeterminateness in the choice of the accurate solution to which G_1 is to approximate on the left of x_0. Also, if, on the right of x_0, any accurate solution is approximately given by G', this solution is also approximately given by G' plus any multiple of g'. There is thus a double indeterminateness in the connexions, and two conventions will be necessary to make the connexions determinate. These we proceed to fix.

Since g_1/G_1 is small, G_1 is, on the left of x_0, an asymptotic approximation to any one of a single infinity of accurate solutions; and, similarly, since g'/G' is small, G' is, to the right of x_0, an asymptotic approximation to any one of another single infinity of accurate solutions. On the other hand, the solutions to which g_1, g' respectively approximate are wholly determinate; and it is easy to see that they are not in a constant ratio, but may be taken as two independent solutions, for their asymptotic approximations involve G', G_1 respectively when continued across x_0. Hence we may take a multiple of the second of these as the solution to be represented by G_1 for $x < x_0$.

The accurate solutions represented by G_1 and g_1 are then such that, while to the left of x_0 one is large compared with the other, the position is completely reversed to the right of x_0.

With this convention, and also neglecting g' compared with G', we easily get, making the connexions by means of the functions $I_{\pm 1/(2n+2)}$,

$$G_1 \longleftrightarrow g' \sin \frac{\pi}{2n+2}, \tag{5}$$

$$g_1 \longleftrightarrow G' \operatorname{cosec} \frac{\pi}{2n+2}. \tag{6}$$

2. *Application to the Mathieu equation.*

An application of some interest is to the characteristic numbers of the Mathieu equation. This equation being taken in the form

$$y''+(4a-16q\cos 2x)\,y = 0, \tag{1}$$

it can be shown that there is, for each solution that is periodic in 2π, one positive value of q for which the corresponding value of $4a$ is equal to $16q$. The problem is to determine these values of q approximately.

With $4a = 16q$ the equation becomes

$$y''+k^2\sin^2 x\,.\,y = 0, \tag{2}$$

where

$$k^2 = 32q. \tag{3}$$

From (1 . 1) we see that approximate solutions are

$$\frac{\cos k(1-\cos x)}{|\sin x|^{\frac{1}{2}}} \quad \text{and} \quad \frac{\sin k(1-\cos x)}{|\sin x|^{\frac{1}{2}}}, \tag{4}$$

and that

$$\frac{\cos k(1-\cos x)}{|\sin x|^{\frac{1}{2}}}\ (x<0) \longleftrightarrow 2^{\frac{1}{2}}\frac{\cos k(1-\cos x)}{\sin^{\frac{1}{2}} x}+\frac{\sin k(1-\cos x)}{\sin^{\frac{1}{2}} x}\ (x>0), \tag{5}$$

$$\frac{\sin k(1-\cos x)}{|\sin x|^{\frac{1}{2}}}\ (x<0) \longleftrightarrow -\frac{\cos k(1-\cos x)}{\sin^{\frac{1}{2}} x}-2^{\frac{1}{2}}\frac{\sin k(1-\cos x)}{\sin^{\frac{1}{2}} x}\ (x>0). \tag{6}$$

Hence $\cos(k\cos x)/|\sin x|^{\frac{1}{2}}$, $\sin(k\cos x)/|\sin x|^{\frac{1}{2}}$ are approximate solutions, and if

$$\left.\begin{aligned}\frac{\cos(k\cos x)}{|\sin x|^{\frac{1}{2}}} &= P_1 \quad \text{for} \quad x<0\\ &= P' \quad \text{for} \quad x>0\end{aligned}\right\},$$
$$\left.\begin{aligned}\frac{\cos(k\cos x)}{|\sin x|^{\frac{1}{2}}} &= p_1 \quad \text{for} \quad x<0\\ &= p' \quad \text{for} \quad x>0\end{aligned}\right\}, \tag{7}$$

then

$$P' \longleftrightarrow P_1 2^{\frac{1}{2}}\cos 2k+p_1(2^{\frac{1}{2}}\sin 2k-1), \tag{8}$$

$$p' \longleftrightarrow P_1(2^{\frac{1}{2}}\sin 2k+1)-p_1 2^{\frac{1}{2}}\cos 2k. \tag{9}$$

Now the solutions which we require, being periodic in 2π, are known to be odd or even functions both of x and of $\frac{1}{2}\pi-x$. Those which are even in x are denoted by ce_0, ce_1, ce_2, ..., ce_n, ..., and they are odd or even functions of $\frac{1}{2}\pi-x$ according as n is odd or even. Those which are odd in x are denoted by se_1, se_2, ..., se_n, ..., and they are odd or even functions of $\frac{1}{2}\pi-x$ according as n is even or odd.

Now P' is an even function and p' an odd function of $\frac{1}{2}\pi - x$. Therefore, for $x > 0$, ce_{2n} and se_{2n+1} must be constant multiples of P', and ce_{2n+1} and se_{2n} constant multiples of p'.

Now ce_{2n} is an even function of x; therefore, if k is to have the value appropriate to ce_{2n}, P' and its continuation for $x < 0$ must have the same value for numerically equal values of x. But for numerically equal values of x, P' is equal to P_1, and therefore P_1 must be equal to the continuation of P' as given by (8). Hence, for ce_{2n}, we must have

$$2^{\frac{1}{2}} \cos 2k = 1, \quad 2^{\frac{1}{2}} \sin 2k - 1 = 0. \tag{10}$$

Similarly, for se_{2n+1} we must have

$$2^{\frac{1}{2}} \cos 2k = -1, \quad 2^{\frac{1}{2}} \sin 2k - 1 = 0; \tag{11}$$

for se_{2n}

$$2^{\frac{1}{2}} \cos 2k = 1, \quad 2^{\frac{1}{2}} \sin 2k + 1 = 0; \tag{12}$$

and for ce_{2n+1}

$$2^{\frac{1}{2}} \cos 2k = -1, \quad 2^{\frac{1}{2}} \sin 2k + 1 = 0. \tag{13}$$

We thus get the result that $4a = 16q$ when

$$k = \tfrac{1}{8}\pi, \ \tfrac{9}{8}\pi, \ \ldots, \ n\pi + \tfrac{1}{8}\pi, \ \ldots \quad \text{for} \quad ce_0, \ ce_2, \ \ldots, \ ce_{2n}, \ \ldots; \tag{14}$$

$$k = \tfrac{5}{8}\pi, \ \tfrac{13}{8}\pi, \ \ldots, \ n\pi + \tfrac{5}{8}\pi, \ \ldots \quad \text{for} \quad ce_1, \ ce_3, \ \ldots, \ ce_{2n+1}, \ \ldots; \tag{15}$$

$$k = \tfrac{7}{8}\pi, \ \tfrac{15}{8}\pi, \ \ldots, \ n\pi - \tfrac{1}{8}\pi, \ \ldots \quad \text{for} \quad se_2, \ se_4, \ \ldots, \ se_{2n}, \ \ldots; \tag{16}$$

$$k = \tfrac{3}{8}\pi, \ \tfrac{11}{8}\pi, \ \ldots, \ n\pi + \tfrac{3}{8}\pi, \ \ldots \quad \text{for} \quad se_1, \ se_3, \ \ldots, \ se_{2n+1}, \ \ldots. \tag{17}$$

This gives the following values for q:

ce_0	ce_1	ce_2	ce_3	ce_4	ce_5
0·005	0·120	0·390	0·814	1·39	2·12

se_1	se_2	se_3	se_4	se_5	se_6
0·043	0·236	0·583	1·08	1·74	2·55.

Interpolation from such tables as exist* shows these values to be in good agreement with those calculated accurately.

The proof that the approximate solutions are periodic in 2π, when k has the values above, can be completed by continuing the solutions across $x = \pi$.

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* Ince, *Proc. Royal Soc. Edinburgh*, 46 (1926), 28; 46 (1926), 321. Goldstein, *Trans. Camb. Phil. Soc.*, 23 (1927), following 336.

Then, when $4a = 16q$,

$$\left.\begin{aligned} ce_{2n} &\sim A_n \frac{\cos\{n+\frac{1}{8})\pi\cos x\}}{|\sin x|^{\frac{1}{2}}} \\ ce_{2n+1} &\sim B_n \frac{\sin\{(n+\frac{5}{8})\pi\cos x\}}{|\sin x|^{\frac{1}{2}}} \\ se_{2n} &\sim C_n \frac{\sin\{(n-\frac{1}{8})\pi\cos x\}}{|\sin x|^{\frac{1}{2}}} \\ se_{2n+1} &\sim D_n \frac{\cos\{n+\frac{3}{8})\pi\cos x\}}{|\sin x|^{\frac{1}{2}}} \end{aligned}\right\}, \tag{18}$$

where the A, B, C, D are constants.

These approximations will fail in the neighbourhoods of $x = 0$, $\pm\pi$, $\pm 2\pi$, etc.

3. *Note on the zeros in q of the associated Mathieu functions and their derivatives.*

In a paper recently communicated to the Cambridge Philosophical Society*, approximate formulae were given for the larger zeros in q of the associated Mathieu functions $Ce(x, q)$, $Se(x, q)$, defined by the relations

$$Ce(x, q) = ce(ix, q), \quad iSe(x, q) = se(ix, q),$$

and their x-derivatives. The following notation was used, being that commonly employed in physical applications :

$$\cosh x = e^{-1}, \quad 4\sqrt{(2q)} = \kappa ae. \tag{1}$$

(e is the eccentricity of the fundamental ellipse, a its semi-major axis, and κ is to be determined. In the physical applications, the fundamental ellipse is a boundary, and x is the value on the boundary of ξ, where ξ, η are the usual elliptic coordinates.) The formulae give approximate values of κa in terms of e, and indicate that these, and therefore also the values of q, become infinitely large as $e \to 1$. The formulae apply, however, only when q is large; and therefore if q remains small or moderate as $e \to 1$, the formulae do not hold.

Actually $q \to \infty$ as $e \to 1$ in all cases except those of the smallest zeros of the functions Ce'_n ($n \neq 0$). Physically, the reason for this excep-

* *Loc. cit.*, 326, 327.

tion is that the corresponding modes of vibration of an elliptical lake have no nodal ellipses, and become, as $e \to 1$, the longitudinal modes of an elliptical canal. A simple analytical explanation is given below.

3.1. The associated Mathieu functions satisfy the equation

$$y'' + (16q \cosh 2x - 4a)\, y = 0. \tag{1}$$

For each function, a is a single-valued function of q; and there is one positive value of q, say q_0, for which $a = 4q$. For $0 < q < q_0$, $a > 4q$; for $q > q_0$, $a < 4q$.

Ce_0 is an exception, since, for Ce_0, $q = 0$; and there are no positive values of q for which $a > 4q$.

3.11. Now let $y = Ce_n(x, q)$, $n \neq 0$. Let $q < q_0$, so that $a > 4q$, and let $x_0 = \frac{1}{2} \cosh^{-1} a/4q$. Then for $x > x_0$, y''/y is negative and continually decreases to $-\infty$ as x increases. Hence, for $x > x_0$, y oscillates with continually diminishing amplitude and wave-length.

Now suppose $y > 0$ for $x = 0$. (This involves no loss of generality, since the zeros and extrema are unaltered in position by a change of sign throughout; and, by definition, $y \neq 0$ for $x = 0$.) Also, since y is an even function of x, $y' = 0$ for $x = 0$. Hence we have

$$\left.\begin{aligned} y &> 0 \quad \text{for} \quad x = 0 \\ y' &= 0 \quad \text{for} \quad x = 0 \\ \text{and also} \quad y''/y &> 0 \quad \text{for} \quad x < x_0 \end{aligned}\right\}. \tag{1}$$

Therefore y is a minimum at $x = 0$, and both y and y' are positive and increasing for $x < x_0$.

For $x > x_0$, y oscillates, and since y and y' are both greater than zero at $x = x_0$, there must be a maximum of y at, say, $x = x_1$, before the first zero of y is reached.

Now suppose $q > q_0$, so that $a < 4q$. Then y''/y is negative for all x, and y oscillates with continually decreasing amplitude and wave-length right from $x = 0$ to $x = \infty$. Since $y > 0$ at $x = 0$, $y'' < 0$ at $x = 0$, so that y', and therefore also y, initially decreases. Thus there is now a maximum at $x = 0$, and the maximum at x_1 has completely disappeared.

When $q = q_0$, so that $a = 4q$, x_1 coincides with 0. For it is easy to show from the differential equation that in this case y'' and y''' are both zero at $x = 0$, and therefore $x = 0$ is a point of confluence of two extrema.

As $q \to 0$, a remains finite. Therefore, for q small, x_0, and therefore also x_1, is very large. So also, therefore, are all the other zeros and extrema of y. As q increases, x_0 and x_1 and all the other zeros and extrema of y decrease. When $q = q_0$, $x_1 = 0$; none of the other zeros and extrema vanish, but they all decrease indefinitely as q increases to ∞.

Hence, if x is any zero or extremum of y, $x \to 0$ and $q \to \infty$ together except for the first extremum. For it, $x \to 0$ and $q \to q_0$ together.

Now $e \to 1$ as $x \to 0$. Hence, as $e \to 1$, $q \to \infty$ for all the zeros and extrema except the first extremum; for it, $q \to q_0$ as $e \to 1$; and therefore $\kappa a \to 4\sqrt{(2q_0)} = k_0$, say.

Approximate values of k_0 were given in the preceding paragraph.

3.12. For Ce_0, $q_0 = 0$. Hence for any positive value of q, Ce_0 oscillates for all x; there are no inflexions except where $Ce_0 = 0$, and there is a maximum at $x = 0$. Hence there are no extrema other than $x = 0$ before the first zero; no zeros or extrema vanish for any value of q, and for any zero or extremum $q \to \infty$ as $e \to 1$.

3.13. A similar process can be carried out for Se_n. In this case $y = 0$ at $x = 0$. There is always an extremum smaller than the first zero other than $x = 0$; when q becomes greater than q_0, an inflexion disappears, but no extremum; and the result is that, for any zero or extremum, $q \to \infty$ as $e \to 1$.

3.14. Hence, as $e \to 1$, $\kappa a \to \infty$ for all the zeros and extrema except the first extremum of Ce_n $(n \neq 0)$. For it, $\kappa a \to k_0 = 4\sqrt{(2q_0)}$ where $a(q_0) = 4q_0$.

THE FREE OSCILLATIONS OF WATER IN A CANAL OF ELLIPTIC PLAN

By S. GOLDSTEIN.

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4. *The free oscillations of water in a canal of elliptic plan**.

We can now apply the results of §§ 2 and 3 to a discussion of the free oscillations of water in an elliptical canal, *i.e.* an elliptical lake whose eccentricity is practically unity.†

We give complete solutions for the first five normal modes of oscillation of an elliptical canal. Simple approximate formulae are given for all modes. The results of the accurate and approximate solutions are compared numerically, and the general agreement is good.

The results for an elliptical canal are compared with those for a uniform straight canal.

4.1. If ζ is the tide height in an elliptical lake of constant depth h whose boundary is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1,$$

* This paper is a continuation of the one preceding, and so paragraphs are numbered straight on throughout.

† Cp. Jeffreys, *Proc. London Math. Soc.*, 23 (1923), 462 *et seq.*

the normal modes of vibration are given by

$$\left.\begin{aligned} \xi &= Ce_n(\xi, q_s)\, ce_n(\eta, q_s) \cos(\sigma_s t + \epsilon_s) \\ \text{or} \quad & Se_n(\xi, q'_s)\, se_n(\eta, q'_s) \cos(\sigma'_s t + \epsilon'_s) \end{aligned}\right\}, \tag{2}$$

where

$$\left.\begin{aligned} x &= ae \cosh\xi \cos\eta \\ y &= ae \sinh\xi \sin\eta \end{aligned}\right\}; \tag{3}$$

q_s, q'_s are the s-th roots in q of

$$\left.\begin{aligned} Ce'_n(\xi_0, q) &= 0 \\ Se'_n(\xi_0, q) &= 0 \end{aligned}\right\}, \tag{4}$$

where

$$\cosh\xi_0 = e^{-1}; \tag{5}$$

$$\kappa_s = 4\sqrt{(2q_s)}/ae, \quad \kappa'_s = 4\sqrt{(2q'_s)}/ae; \tag{6}$$

$$\kappa_s, \kappa'_s = \sigma_s/V, \sigma'_s/V; \tag{7}$$

and

$$V = \sqrt{(gh)} \tag{8}$$

and is the velocity of a free wave in water whose depth is equal to that of the lake.

Then we have shown in § 3 that, as $e \to 1$, the speeds of all the modes become infinite except for the modes corresponding to the first zeros of Ce'_n ($n \neq 0$). These modes become the longitudinal modes of vibration of an elliptical canal. The speeds of these modes are found by taking $q = q_0$, the positive root of the equation

$$a(q) = 4q, \tag{9}$$

as explained in § 3.

As $e \to 1$ and $q \to q_0$, $\xi_0 \to 0$, and for $0 < \xi < \xi_0$, $\xi \to 0$; and therefore $Ce_n(\xi, q) \to Ce_n(0, q_0)$.

4.2. Hence the normal modes of vibration of an elliptical canal of length $2a$ (the origin being taken in the middle) are given by

$$\zeta = ce_n(\eta, q_{0,n}) \cos(\sigma_{0,n} t + \epsilon) \quad (n = 1, 2, 3, \ldots), \tag{1}$$

where

$$x = a\cos\eta; \tag{2}$$

$q_{0,n}$ is the positive root of

$$a_n(q) = 4q, \tag{3}$$

where $a_n(q)$ is the value of a corresponding to ce_n; and

$$\sigma_{0,n} = \kappa_{0,n} V, \tag{4}$$

where

$$\kappa_{0,n} a = 4\sqrt{(2q_{0,n})}. \tag{5}$$

When n is even, ξ is symmetrical about the middle; when n is odd, it is antisymmetrical.

4.21. From § 2 it follows that, approximately,

$$\kappa_{0,n} a = \tfrac{1}{2}n\pi + \tfrac{1}{8}\pi, \tag{1}$$

and

$$ce_n(\eta, q_{0,n}) = A_n \frac{\cos\{(n+\frac{1}{4})\pi x/2a\}}{(1-x^2/a^2)^{\frac{1}{4}}} \Bigg\} \tag{2}$$

or

$$B_n \frac{\sin\{(n+\frac{1}{4})\pi x/2a\}}{(1-x^2/a^2)^{\frac{1}{4}}} \Bigg\},$$

where the A and B are constants, according as n is even or odd.

4.22. The oscillation in any mode may be regarded as the resultant of two waves travelling in opposite directions with velocity V.

For the asymmetrical modes, on the approximate solution,

$$\xi \sim B_n \frac{\sin \kappa_{0,n} x \cos(\kappa_{0,n} Vt + \epsilon)}{(1-x^2/a^2)^{\frac{1}{4}}} \tag{1}$$

$$= B_n \frac{\sin \kappa_{0,n}(x - Vt - \epsilon/\kappa_{0,n}) + \sin \kappa_{0,n}(x + Vt + \epsilon/\kappa_{0,n})}{2(1-x^2/a^2)^{\frac{1}{4}}}, \tag{2}$$

and similarly for the symmetrical modes.

Thus, since $(1-x^2/a^2)^{\frac{1}{2}}$ is proportional to the breadth, the amplitude of a progressive wave is inversely proportional to the square root of the breadth, in accordance with the usual approximate solution for progressive waves.*

4.23. Now for any given value of q the values of a_n and of the coefficients in the Fourier series for ce_n can be calculated by known methods; and the value of $q_{0,n}$ can be found to any required degree of accuracy by similar methods. We put $a = 4q$ in the relations connecting a and q, and proceed to solve for q by a method essentially that of trial and error. For the details of the calculation, reference may be made to Ince, *Proc. Royal Soc. Edin.*, 46 (1925), 20–29, and to Goldstein, *Trans. Camb. Phil. Soc.*, 23 (1927), 312–316. It will be seen from the latter paper that the ratios

* Lamb, *Hydrodynamics* (1924), Chap. 8, 254, 255, § 185.

of the coefficients are found in the process of calculating $q_{0,n}$, and the calculation is completed by determining the coefficients so that*

$$\int_0^{2\pi} [ce_n(\eta, q)]^2 \, d\eta = \pi. \tag{1}$$

We thus find

$$q_{0,1} = 0{\cdot}1112275, \tag{2}$$

$$ce_1(\eta, q_{0,1}) = 0{\cdot}99241 \cos\eta - 0{\cdot}12288 \cos 3\eta + 0{\cdot}00471 \cos 5\eta - 0{\cdot}00009 \cos 7\eta + \ldots; \tag{3}$$

$$q_{0,2} = 0{\cdot}3798842, \tag{4}$$

$$ce_2(\eta, q_{0,2}) = 0{\cdot}39516 + 0{\cdot}79032 \cos 2\eta - 0{\cdot}24989 \cos 4\eta + 0{\cdot}02552 \cos 6\eta - 0{\cdot}00134 \cos 8\eta + 0{\cdot}00004 \cos 10\eta - \ldots; \tag{5}$$

$$q_{0,3} = 0{\cdot}8032327, \tag{6}$$

$$ce_3(\eta, q_{0,3}) = 0{\cdot}71339 \cos\eta + 0{\cdot}60237 \cos 3\eta - 0{\cdot}35232 \cos 5\eta + 0{\cdot}06370 \cos 7\eta - 0{\cdot}00604 \cos 9\eta + 0{\cdot}00036 \cos 11\eta - 0{\cdot}00001 \cos 13\eta + \ldots; \tag{7}$$

$$q_{0,4} = 1{\cdot}3809991, \tag{8}$$

$$ce_4(\eta, q_{0,4}) = 0{\cdot}32544 + 0{\cdot}65107 \cos 2\eta + 0{\cdot}41535 \cos 4\eta - 0{\cdot}42190 \cos 6\eta + 0{\cdot}11561 \cos 8\eta - 0{\cdot}01661 \cos 10\eta + 0{\cdot}00151 \cos 12\eta - 0{\cdot}00010 \cos 14\eta + 0{\cdot}00001 \cos 16\eta - \ldots; \tag{9}$$

$$q_{0,5} = 2{\cdot}1130925, \tag{10}$$

$$ce_5(\eta, q_{0,5}) = 0{\cdot}61238 \cos\eta + 0{\cdot}57615 \cos 3\eta + 0{\cdot}23319 \cos 5\eta - 0{\cdot}45463 \cos 7\eta + 0{\cdot}17534 \cos 9\eta - 0{\cdot}03485 \cos 11\eta + 0{\cdot}00441 \cos 13\eta - 0{\cdot}00039 \cos 15\eta + 0{\cdot}00003 \cos 17\eta - \ldots. \tag{11}$$

* Goldstein, *loc. cit.*, 303–306.

4.24. § 4.23 (2), (4), (6), (8), and (10) give

$$\kappa a = 1{\cdot}8866,\ 3{\cdot}4866,\ 5{\cdot}0699,\ 6{\cdot}6477,\ 8{\cdot}2231,\ \ldots; \qquad (1)$$

whilst the approximate formula § 4.21 (1) gives

$$\kappa a = 1{\cdot}9635,\ 3{\cdot}5343,\ 5{\cdot}1051,\ 6{\cdot}6759,\ 8{\cdot}2467,\ \ldots. \qquad (2)$$

The higher the mode the better is the approximation.

For a uniform straight canal, $\kappa a = \frac{1}{2}n\pi$, so that, for an elliptical canal, the speeds σ are increased by approximately the same amount $\pi V/8a$ for all the modes. The increase in speed is due to the narrowing at the ends, and the consequent reflexion of the two progressive waves of which the oscillation may be supposed to be the resultant. This makes the effective length shorter. For the n-th mode the speed of the elliptical canal is the same as that of the corresponding mode of a uniform straight canal of length $a(1+1/4n)^{-1}$. The higher the mode the less is the shortening effect of the curvature of the boundary, and for high modes this shortening is practically proportional to n^{-1}. Thus the change in the period is due to reflexion from a portion of the boundary that shrinks up towards the ends as the order of the mode is increased.

For a circular lake of radius a, the values of κa for the corresponding modes (modes with no nodal circles) are given by

$$\kappa a = 1{\cdot}841,\ 3{\cdot}054,\ 4{\cdot}201,\ \ldots,\ \tfrac{1}{2}n\pi - \tfrac{3}{4}\pi,\ \ldots. \qquad (3)$$

These values are not very different from those for the elliptical canal. Then if we start with a circle and gradually increase the eccentricity to unity, thus obtaining a series of elliptical lakes, the speeds of the oscillations here considered (those symmetrical about the major axis and having no nodal ellipses) will always lie between their values for the circle and their values for the elliptical canal, and so will change but slightly during the whole transition. All the other modes, on the other hand, become infinitely rapid as $e \to 1$.

4.25. The amplitudes, as calculated from formulae § 4.23 (3), (5), (7), (9), and (11), are given in the following tables. ζ_a gives the amplitude according to the approximate solution § 4.21 (2), and ζ_s the amplitude for the corresponding mode of a uniform straight canal.

The approximate solutions differ but little from the accurate solutions, except at the ends. The approximation gets better and better, and holds nearer and nearer to the ends, as the order of the mode is increased. Roughly speaking, the approximation is good except for about a tenth of the total length.

	First asymmetrical mode.				First symmetrical mode.			Second asymmetrical mode.		
η	x/a	ζ	ζ_a	ζ_s	ζ	ζ_a	ζ_s	ζ	ζ_a	ζ_s
0°	1	0·8742	∞	0·8742	0·9598	∞	0·9598	1·0214	∞	1·0214
15°	0·9662	0·8730	1·2659	0·8729	0·9553	1·2571	0·9543	1·0113	1·2618	1·0083
25°	0·9063	0·8650	1·0230	0·8647	0·9257	1·0194	0·9185	0·9454	1·0078	0·9235
30°	0·8660	0·8555	0·9533	0·8549	0·8904	0·9360	0·8760	0·8681	0·8908	0·8245
35°	0·8192	0·8401	0·8970	0·8392	0·8346	0·8503	0·8090	0·7486	0·7489	0·6724
40°	0·7660	0·8172	0·8460	0·8158	0·7535	0·7514	0·7120	0·5815	0·5705	0·4610
45°	0·7071	0·7852	0·7950	0·7833	0·6437	0·6288	0·5814	0·3682	0·3529	0·1935
50°	0·6428	0·7426	0·7400	0·7402	0·5044	0·4893	0·4163	0·1198	0·1048	−0·1147
55°	0·5736	0·6882	0·6781	0·6853	0·3381	0·3233	0·2199	−0·1415	−0·1539	−0·4339
60°	0·5000	0·6214	0·6074	0·6182	0·1511	0·1392	0	−0·3838	−0·3926	−0·7228
65°	0·4226	0·5420	0·5268	0·5386	−0·0461	−0·0538	−0·2310	−0·5699	−0·5752	−0·9324
70°	0·3420	0·4505	0·4364	0·4474	−0·2396	−0·2426	−0·4570	−0·6653	−0·6680	−1·0206
75°	0·2588	0·3484	0·3366	0·3457	−0·4135	−0·4121	−0·6596	−0·6475	−0·6484	−0·9591
80°	0·1736	0·2375	0·2291	0·2355	−0·5519	−0·5470	−0·8205	−0·5134	−0·5135	−0·7456
85°	0·0872	0·1203	0·1160	0·1193	−0·6411	−0·6340	−0·9241	−0·2837	−0·2836	−0·4078
90°	0	0	0	0	−0·6720	−0·6640	−0·9598	0	0	0

	Second symmetrical mode.			Third asymmetrical mode.		
η	ζ	ζ_a	ζ_s	ζ	ζ_a	ζ_s
0°	1·0705	∞	1·0705	1·1116	∞	1·1116
15°	1·0522	1·2527	1·0460	1·0826	1·2367	1·0721
25°	0·9350	0·9669	0·8903	0·8995	0·9017	0·8240
30°	0·8006	0·8010	0·7131	0·6957	0·6824	0·5510
35°	0·6000	0·5853	0·4506	0·4047	0·3828	0·1666
40°	0·3344	0·3150	0·1077	0·0483	0·0271	−0·2930
45°	0·0243	0·0063	−0·2850	−0·3155	−0·3311	−0·7409
50	−0·2883	−0·3017	−0·6678	−0·5973	−0·6069	−1·0494
55°	−0·5436	−0·5516	−0·9581	−0·7018	−0·7042	−1·0877
60°	−0·6775	−0·6808	−1·0705	−0·5708	−0·5696	−0·7860
65°	−0·6439	−0·6441	−0·9464	−0·2282	−0·2256	−0·1964
70°	−0·4370	−0·4357	−0·5850	+0·2053	+0·2076	+0·4889
75°	−0·1014	−0·1028	−0·0593	0·5470	0·5483	0·9951
80°	+0·2592	+0·2604	+0·4941	0·6358	0·6362	1·0879
85°	0·5401	0·5408	0·9139	0·4205	0·4206	0·7029
90°	0·6456	0·6460	1·0705	0	0	0

The increase of width in the middle tends to lower the amplitude in the middle of the canal as compared with that of a uniform straight canal, while reflexion from the boundary tends to increase it. The relative effect of these two causes can easily be seen from the figures. For the first asymmetrical mode the difference in amplitude between the elliptical and the uniform canal is very slight, but the amplitude for the elliptical canal is slightly the larger everywhere between the middle and the end, so that reflexion plays the greater part. For all the other modes the amplitude is less in the middle and larger at the ends; while

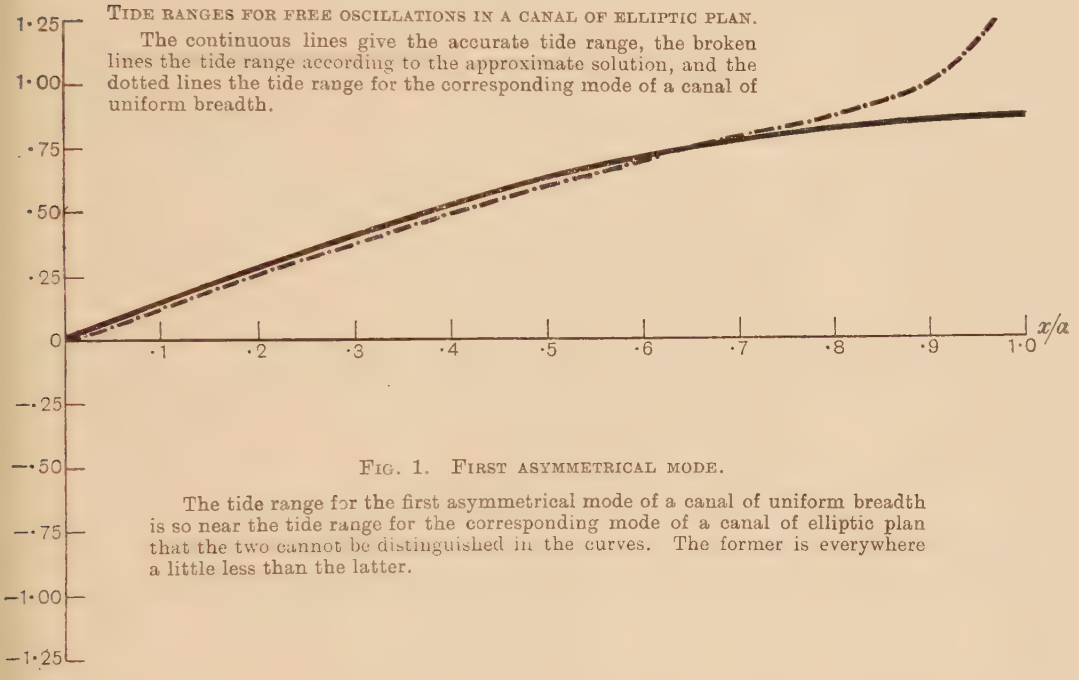

FIG. 1. FIRST ASYMMETRICAL MODE.

The tide range for the first asymmetrical mode of a canal of uniform breadth is so near the tide range for the corresponding mode of a canal of elliptic plan that the two cannot be distinguished in the curves. The former is everywhere a little less than the latter.

the range over which it is larger gets smaller and smaller as the order of the mode is increased, being effectively the distance from the last node to the end of the canal. Thus, again, we see that the portion of the boundary over which effective reflexion takes place shrinks up towards the ends as the order of the mode is increased.

4·26. According to the approximate solutions, the node for the first symmetrical mode is at

$$x/a = \tfrac{4}{9} = 0{\cdot}444\,; \qquad (1)$$

for the second symmetrical mode there are nodes at

$$x/a = \tfrac{4}{17} = 0{\cdot}235 \quad \text{and} \quad \tfrac{12}{17} = 0{\cdot}706\,; \qquad (2)$$

whilst for the s-th symmetrical mode the nodes are at

$$x/a = 4/(8s+1),\ 12/(8s+1),\ \ldots,\ (8s-4)/(8s+1). \tag{3}$$

For the second asymmetrical mode the node is, approximately, at

$$x/a = \tfrac{8}{13} = 0{\cdot}615\,; \tag{4}$$

for the third asymmetrical mode the nodes are at

$$x/a = \tfrac{8}{21} = 0{\cdot}381 \text{ and } x/a = \tfrac{16}{21} = 0{\cdot}762\,; \tag{5}$$

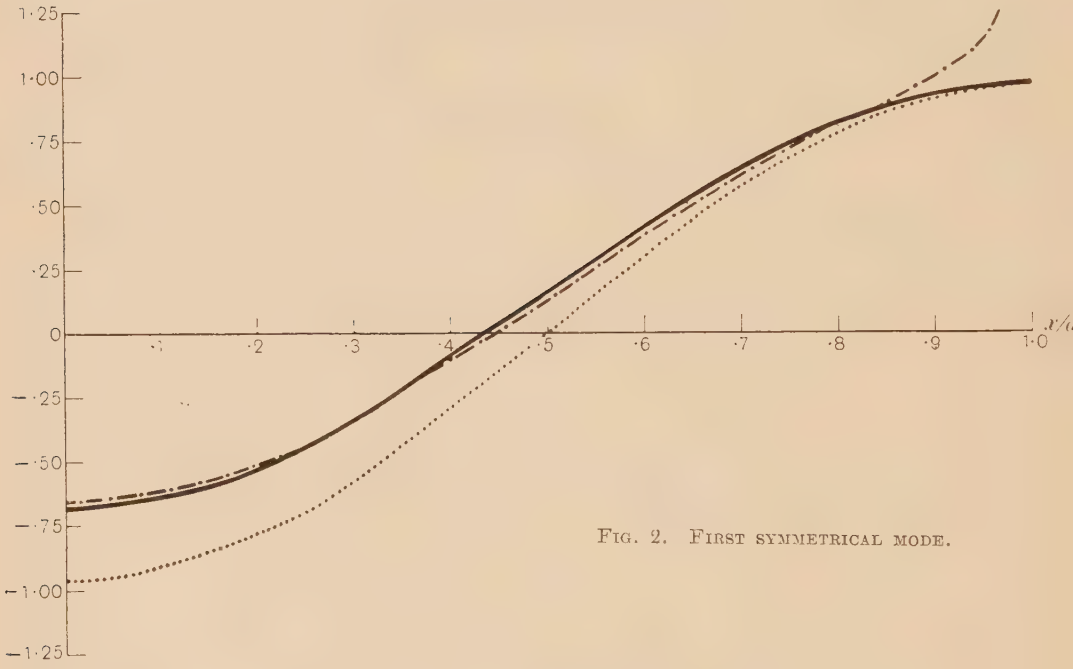

FIG. 2. FIRST SYMMETRICAL MODE.

while for the s-th asymmetrical mode the nodes are at

$$x/a = 8/(8s-3),\ 16/(8s-3),\ \ldots,\ (8s-8)/(8s-3). \tag{6}$$

By interpolation from the tables for ζ, the nodes are found to be at

$$x/a = 0{\cdot}441, \tag{7}$$

$$x/a = 0{\cdot}235 \text{ and } 0{\cdot}703 \tag{8}$$

for the first two symmetrical modes; and at

$$x/a = 0{\cdot}612, \tag{9}$$

$$x/a = 0{\cdot}381 \text{ and } 0{\cdot}760 \tag{10}$$

for the second and third asymmetrical modes.

Thus the approximate formulae for the position of the nodes are always correct to within less than 1 per cent., and the accuracy increases rapidly with the order of the mode except for the node nearest the end.

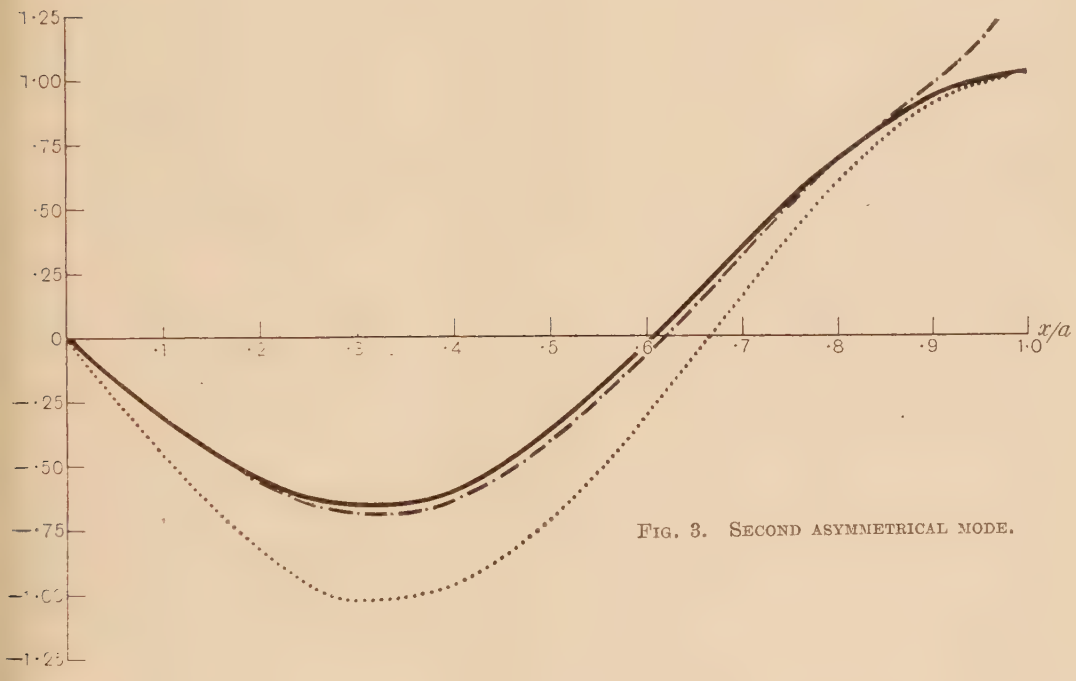

FIG. 3. SECOND ASYMMETRICAL MODE.

For a uniform straight canal the r-th node of the s-th symmetrical mode is at

$$x/a = (2r-1)/(2s), \tag{11}$$

and the r-th node of the s-th asymmetrical mode at

$$x/a = 2r/(2s-1). \tag{12}$$

Thus the nodes are all moved inwards; but the displacement of the nodes decreases rapidly as the order of the mode is increased, being ultimately proportional to s^{-2}.

4 . 27. If $|u|$ is the velocity-range, and $|\zeta|$ the tide range, then

$$|u| = \frac{g}{\sigma} \frac{\partial |\zeta|}{\partial x}, \tag{1}$$

and

$$\frac{\partial |\zeta|}{\partial x} = -\frac{ce_n'(\eta, q_{0,n})}{a \sin \eta}. \tag{2}$$

Hence $|u|$ can be calculated as accurately as is required from § 4 . 23 (3), (5), (7), (9), (11), etc. Its general character is easily seen

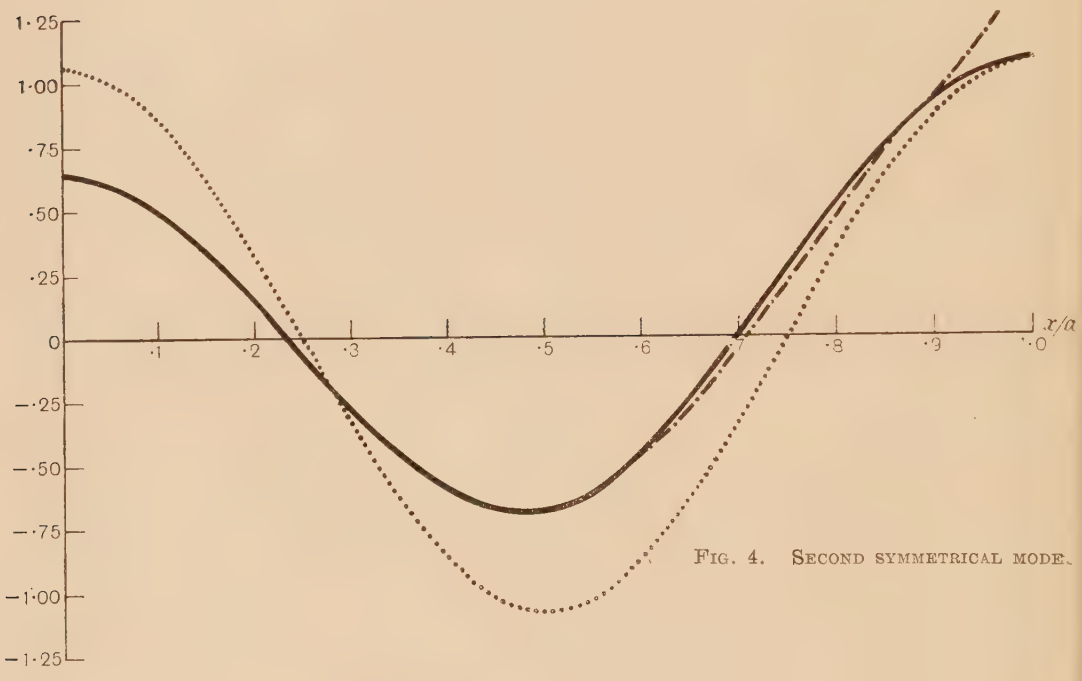

FIG. 4. SECOND SYMMETRICAL MODE.

from the ζ curves, and it will be found that, in general, the velocity range is less than for a uniform straight canal.

4 . 28. The "period" equation

$$a = 4q \tag{1}$$

may be derived from the fact that the normal velocity, or the normal gradient of ζ, must vanish on the boundary; and that at the ends of the

canal, where $\eta = 0$ or π, the normal gradient of ζ is $\pm \partial\zeta/\partial x$. Hence

$$\frac{\partial \zeta}{\partial x} = 0 \quad \text{at} \quad \eta = 0 \text{ and } \pi. \tag{2}$$

This gives

$$\frac{Ce_n'(\eta, q)}{\sin \eta} = 0 \quad \text{at} \quad \eta = 0 \text{ and } \pi. \tag{3}$$

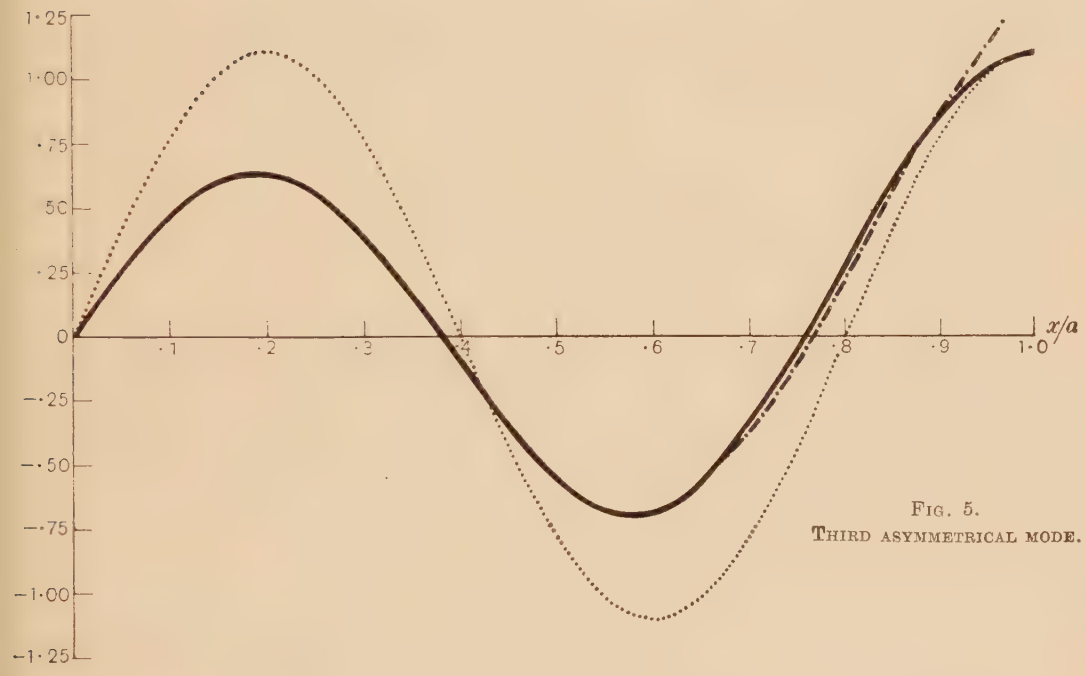

FIG. 5.
THIRD ASYMMETRICAL MODE.

Hence $Ce_n'(\eta, q)$ must have a double zero at these two points, and therefore we must have

$$Ce_n''(\eta, q) = 0 \quad \text{at} \quad \eta = 0 \text{ and } \pi. \tag{4}$$

But $Ce_n(\eta, q)$ does not vanish at these points. Therefore we must have

$$4\alpha = 16q \cos 2\eta \quad \text{at} \quad \eta = 0 \text{ and } \pi, \tag{5}$$

that is to say

$$\alpha = 4q.$$

ON THE THEORY OF THE OSCILLATIONS OF A ROTATING MECHANICAL SYSTEM OF INFINITE FREEDOM

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1. In a recent paper* the authors have discussed the oscillations of a rotating mechanical system of infinite freedom, and in the present paper this discussion is continued.

To secure the existence of certain limits a number of restrictions have to be placed on the functions which specify the system, and in the previous paper one of these restrictions is to the effect that

$$\sigma_n \to \infty \quad \text{as} \quad n \to \infty, \tag{1.1}$$

σ_n being the n-th fundamental speed† of oscillation, when these speeds are arranged in increasing order of magnitude. The object of the present paper is to show that the condition (1 . 1) may be replaced by

$$\nu_s \to \infty \quad \text{as} \quad s \to \infty, \tag{1.2}$$

where the sequence (ν) is that introduced in (1 . 21) of the previous paper. The advantage of (1 . 2) over (1 . 1) is that all the restrictions may now be explicitly made in defining the system, and for applications this is an important gain.

We shall show that the condition (1 . 1) follows as a consequence of (1 . 2) and the other restrictions, but before we can do this it is necessary to obtain by new methods some of the formulae of the previous paper.

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* *Proc. London Math. Soc.* (2), 27 (1928), 1–38.

† Speed of oscillation = 2π/period.

We shall refer to the previous paper as (O) and to its Appendix of purely mathematical theorems as (A). The same notation will be used throughout, and so need not be again defined.

In (O) the condition (1 . 1) enters only through Theorem 8 of (A), and the first equations to depend on it are (6 . 31). We may therefore quote any formulae of (O) earlier than (6 . 31), and such of the formulae subsequent to (6 . 31) as do not depend on (1 . 1) above.

2. *Boundedness and convergence.*

From (1 . 25) of (O) we have

$$\left.\begin{aligned} \sum_{s=1}^{N} \nu_s^2 a_{r,s}^2 &\leqslant \frac{A^2}{\nu_r^2} \\ \sum_{s=1}^{N} c_{r,s}^2 &\leqslant C^2 \end{aligned}\right\} \quad (r = 1, 2, \ldots, N), \qquad (2 . 11)$$

and from (1 . 24) and (1 . 21)

$$\left.\begin{aligned} \sum_{r=1}^{N} \nu_r^2 \left| \frac{\partial T_N(x)}{\partial x_r} \right|^2 &\leqslant \frac{2A^2}{a} T_N(x, \bar{x}) \\ \left| \sum_{r=1}^{N} \sum_{s=1}^{N} \beta_{r,s} x_r y_s \right|^2 &\leqslant B^2 \sum_{r=1}^{N} \frac{|x_r|^2}{\nu_r^2} \sum_{s=1}^{N} |y_s|^2 \end{aligned}\right\}, \qquad (2 . 12)$$

whatever the sequences (x) and (y) may be; from (2 . 41) and (2 . 42) of (O) we have

$$\left.\begin{aligned} \sum_{r=1}^{N} |q_r(N)|^2 &\leqslant \frac{2E}{c} \\ \sum_{r=1}^{N} \frac{1}{\nu_r^2} |\dot{q}_r(N)|^2 &\leqslant \frac{2E}{a} \\ \sum_{r=1}^{N} \nu_r^2 \left| \frac{\partial T\{\dot{q}(N)\}}{\partial \dot{q}_r(N)} \right|^2 &\leqslant \frac{2A^2 E}{a} \end{aligned}\right\}, \qquad (2 . 13)$$

while from (4 . 33) and (6 . 11)

$$\left.\begin{aligned} a_{n,r}(N) &\to a_{n,r} \\ \frac{\partial T\{a_n(N)\}}{\partial a_{n,r}(N)} &\to \frac{\partial T(a_n)}{\partial a_{n,r}} \\ \frac{\partial V\{a_n(N)\}}{\partial a_{n,r}(N)} &\to \frac{\partial V(a_n)}{\partial a_{n,r}} \end{aligned}\right\}, \qquad (2 . 14)$$

as $N \to \infty$, for any particular values of n and r. Also, from (4 . 31). (6 . 12), (6 . 46), and (5 . 32) of (O) we have

$$(N) \qquad \left\{ \begin{array}{c} \sum_{r=1}^{N} |a_{n,r}|^2 \leqslant \dfrac{2}{c} \\ \sigma_n^2 \sum_{r=1}^{N} \nu_r^2 \left| \dfrac{\partial T(a_n)}{\partial a_{n,r}} \right|^2 \leqslant \dfrac{2A^2}{a} \\ \sum_{r=1}^{N} \left| \dfrac{\partial V(a_n)}{\partial a_{n,r}} \right|^2 \leqslant \dfrac{2C^2}{c} \end{array} \right\}, \tag{2 . 15}$$

$$(N) \qquad \left\{ \begin{array}{c} \sum_{n=1}^{N} |a_{n,s}|^2 \leqslant \dfrac{1}{c} \\ \sum_{n=1}^{N} \sigma_n^2 |a_{n,s}|^2 \leqslant \dfrac{\nu_s^2}{a} \end{array} \right\}, \tag{2 . 16}$$

$$(N) \qquad \left\{ \begin{array}{c} \sum_{n=1}^{N} \left| \dfrac{\partial V(a_n)}{\partial a_{n,s}} \right|^2 = c_{s,s} \leqslant C^2 \\ \sum_{n=1}^{N} \sigma_n^2 \left| \dfrac{\partial T(a_n)}{\partial a_{n,s}} \right|^2 = a_{s,s} \leqslant \dfrac{A^2}{\nu_s^2} \end{array} \right\}. \tag{2 . 17}$$

In view of (1 . 2) and the first of relations (2 . 15) we deduce from the first of (2 . 14) by Theorem 8 of (A) that

$$\lim_{N \to \infty} \sum_{r=1}^{N} \frac{1}{\nu_r^2} |a_{n,r}(N)|^2 = \sum_{r=1}^{\infty} \frac{1}{\nu_r^2} |a_{n,r}|^2. \tag{2 . 2}$$

Again, the equations of free motion of the sectional system give

$$(N) \qquad -\sigma_n^2 \frac{\partial T(a_n)}{\partial a_{n,r}} + i\sigma_n \sum_{s=1}^{N} \beta_{r,s} a_{n,s} + \frac{\partial V(a_n)}{\partial a_{n,r}} = 0,$$

for $r = 1, 2, \ldots, N$; on multiplying this by

$$(N) \qquad \nu_r \frac{\partial V(a_{-n})}{\partial a_{-n,r}},$$

adding for $r = 1, 2, \ldots, N$, and rearranging, we obtain

$$(N) \qquad \sum_{r=1}^{N} \nu_r \left| \frac{\partial V(a_n)}{\partial a_{n,r}} \right|^2 = \sigma_n^2 \sum_{r=1}^{N} \nu_r \frac{\partial T(a_n)}{\partial a_{n,r}} \frac{\partial V(a_{-n})}{\partial a_{-n,r}}$$

$$- i\sigma_n \sum_{r=1}^{N} \sum_{s=1}^{N} \beta_{r,s} \nu_r \frac{\partial V(a_{-n})}{\partial a_{-n,r}} a_{n,s}. \tag{2 . 31}$$

Now from Theorem 1 of (A) and (2. 15) we have

$$(N) \qquad \left| \sigma_n \sum_{r=1}^{N} \nu_r \frac{\partial T(a_n)}{\partial a_{n,r}} \frac{\partial V(a_{-n})}{\partial a_{-n,r}} \right|^2 \leqslant \sigma_n^2 \sum_{r=1}^{N} \nu_r^2 \left| \frac{\partial T(a_n)}{\partial a_{n,r}} \right|^2 \sum_{r=1}^{N} \left| \frac{\partial V(a_{-n})}{\partial a_{-n,r}} \right|^2$$

$$\leqslant \frac{4A^2C^2}{ac}, \tag{2.32}$$

while from (2 . 12) and (2 . 15)

$$(N) \qquad \left| \sum_{r=1}^{N} \sum_{s=1}^{N} \beta_{r,s} \nu_r \frac{\partial V(a_{-n})}{\partial a_{-n,r}} a_{n,s} \right|^2 \leqslant B^2 \sum_{r=1}^{N} \left| \frac{\partial V(a_{-n})}{\partial a_{-n,r}} \right|^2 \sum_{s=1}^{N} |a_{n,s}|^2$$

$$\leqslant \frac{4B^2C^2}{c^2}. \tag{2.33}$$

It follows from (2 . 31), (2 . 32), and (2 . 33) that

$$\sum_{r=1}^{N} \nu_r \left| \frac{\partial V\{a_n(N)\}}{\partial a_{n,r}(N)} \right|^2 \leqslant 2\sigma_n(n) \frac{AC}{(ac)^{\frac{1}{2}}} + 2\sigma_n(n) \frac{BC}{c}, \tag{2.34}$$

since $\sigma_n(N) \leqslant \sigma_n(n)$ by (3 . 23) of (O). In view of (2 . 34), we deduce from the third of relations (2 . 14) by Theorem 8 of (A) that

$$\lim_{N \to \infty} \sum_{r=1}^{N} \left| \frac{\partial V\{a_n(N)\}}{\partial a_{n,r}(N)} \right|^2 = \sum_{r=1}^{\infty} \left| \frac{\partial V(a_n)}{\partial a_{n,r}} \right|^2. \tag{2.35}$$

3. *The transformation.*

Let ξ_n, for $\pm n = 1, 2, 3, \ldots$, be quantities satisfying the relation

$$\sum_{n=-\infty}^{\infty} |\xi_n|^2 \leqslant E, \tag{3.11}$$

and for any value of N take

$$\left.\begin{aligned} q_s(N) &= \sum_{n=-N}^{N} a_{n,s}(N)\, \xi_n \\ \dot{q}_s(N) &= \sum_{n=-N}^{N} i\sigma_n(N)\, a_{n,s}(N)\, \xi_n \end{aligned}\right\}, \tag{3.12}$$

for $s = 1, 2, \ldots, N$. These equations are consistent whatever functions (ξ_n) may be of n, and in view of (2 . 16) and (3 . 11) we deduce from (3 . 12) by the corollary to Theorem 7 of (A) that

$$q_s(N) \to q_s, \quad \dot{q}_s(N) \to \dot{q}_s, \tag{3.13}$$

where

$$\left.\begin{aligned} q_s &= \sum_{n=-\infty}^{\infty} a_{n,s}\, \xi_n \\ \dot{q}_s &= \sum_{n=-\infty}^{\infty} i\sigma_n\, a_{n,s}\, \xi_n \end{aligned}\right\}, \tag{3.14}$$

for $s = 1, 2, 3, \ldots$.

We have also

$$\left.\begin{aligned} \sum_{s=1}^{N} c_{r,s}\, q_s(N) &= \sum_{n=-N}^{N} \frac{\partial V\{a_n(N)\}}{\partial a_{n,r}(N)}\, \xi_n \\ \sum_{s=1}^{N} a_{r,s}\, \dot{q}_s(N) &= \sum_{n=-N}^{N} i\sigma_n(N)\, \frac{\partial T\{a_n(N)\}}{\partial a_{n,r}(N)}\, \xi_n \end{aligned}\right\}, \tag{3.21}$$

and, as in § 5 of (O),

$$V\{q(N), \bar{q}(N)\} + T\{\dot{q}(N), \bar{\dot{q}}(N)\} = \sum_{n=-N}^{N} |\xi_n|^2. \tag{3.22}$$

It follows from (3.22) and (3.11) that the quantities $q_s(N)$, $\dot{q}_s(N)$ given by (3.12) will satisfy the relations (2.13). From (3.21), and in view of (2.11), (2.13), (2.17), and (3.11), we deduce, by the corollary to Theorem 7 of (A), that

$$\left.\begin{aligned} \sum_{s=1}^{\infty} c_{r,s}\, q_s &= \sum_{n=-\infty}^{\infty} \frac{\partial V(a_n)}{\partial a_{n,r}}\, \xi_n \\ \sum_{s=1}^{\infty} a_{r,s}\, \dot{q}_s &= \sum_{n=-\infty}^{\infty} i\sigma_n\, \frac{\partial T(a_n)}{\partial a_{n,r}}\, \xi_n \end{aligned}\right\}. \tag{3.23}$$

Again, by § 5 of (O), we have from (3.12)

$$\xi_n = V\{a_{-n}(N), q(N)\} - i\sigma_n(N)\, T\{a_{-n}(N), \dot{q}(N)\}, \tag{3.31}$$

which may be written

$$2\xi_n = \sum_{r=1}^{N} q_r(N)\, \frac{\partial V\{a_{-n}(N)\}}{\partial a_{-n,r}(N)} - i\sigma_n(N) \sum_{r=1}^{N} a_{-n,r}(N)\, \frac{\partial T\{\dot{q}(N)\}}{\partial \dot{q}_r(N)}, \tag{3.32}$$

for $\pm n = 1, 2, \ldots, N$. By virtue of (2.13), (2.2), and (2.35) we deduce from Theorem 7 of (A) that

$$2\xi_n = \sum_{r=1}^{\infty} q_r\, \frac{\partial V(a_{-n})}{\partial a_{-n,r}} - i\sigma_n \sum_{r=1}^{\infty} a_{-n,r}\, \frac{\partial T(\dot{q})}{\partial \dot{q}_r}, \tag{3.33}$$

or

$$\xi_n = V(a_{-n}, q) - i\sigma_n T(a_{-n}, \dot{q}), \tag{3.34}$$

for $\pm n = 1, 2, 3, \ldots$. We now see that, by proper choice of ξ_n, the quantities q_s, $\dot{q}_s$ given by (3.14) will correspond to any state of the system satisfying the fundamental restrictions. The equations (3.14) and (3.34) constitute the transformation.

On taking $\quad q_r = 0 \quad (r \neq s), \qquad \dot{q}_r = 0,$

and then $\quad q_r = 0, \qquad \dot{q}_r = 0 \quad (r \neq s),$

we deduce from (3 . 14) and (3 . 33) that

$$\left.\begin{aligned}
&\tfrac{1}{2}\sum_{n=-\infty}^{\infty} a_{n,r}\frac{\partial V(a_{-n})}{\partial a_{-n,s}} = 1 \ (r=s), \quad = 0 \ (r\neq s)\\
&\tfrac{1}{2}\sum_{n=-\infty}^{\infty} \sigma_n a_{n,r}\frac{\partial V(a_{-n})}{\partial a_{-n,s}} = 0\\
&\tfrac{1}{2}\sum_{n=-\infty}^{\infty} \sigma_n a_{n,r}\frac{\partial T(a_{-n})}{\partial a_{-n,s}} = 0\\
&\tfrac{1}{2}\sum_{n=-\infty}^{\infty} \sigma_n^2 a_{n,r}\frac{\partial T(a_{-n})}{\partial a_{-n,s}} = 1 \ (r=s), \quad = 0 \ (r\neq s)
\end{aligned}\right\}, \tag{3.41}$$

and from (3 . 23) and (3 . 33) that

$$\left.\begin{aligned}
&\tfrac{1}{2}\sum_{n=-\infty}^{\infty} \frac{\partial V(a_n)}{\partial a_{n,r}}\frac{\partial V(a_{-n})}{\partial a_{-n,s}} = c_{r,s}\\
&\tfrac{1}{2}\sum_{n=-\infty}^{\infty} \sigma_n\frac{\partial V(a_n)}{\partial a_{n,r}}\frac{\partial T(a_{-n})}{\partial a_{-n,s}} = 0\\
&\tfrac{1}{2}\sum_{n=-\infty}^{\infty} \sigma_n^2\frac{\partial T(a_n)}{\partial a_{n,r}}\frac{\partial T(a_{-n})}{\partial a_{-n,s}} = a_{r,s}
\end{aligned}\right\}, \tag{3.42}$$

for $r, s = 1, 2, 3, \ldots$. The equations (3 . 41) and (3 . 42) are identical respectively with (6 . 5) and (6 . 32) of (O).

It is from equations (3 . 42) that all the results of §§ 8 and 9 of (O) are deduced, and, in particular, the equation

$$T(\dot{q},\dot{\bar{q}})+V(q,\bar{q}) = \sum_{n=-\infty}^{\infty} |\dot{\xi}_n|^2. \tag{3.5}$$

4. *Unboundedness of free speeds.*

In the formulae of the preceding section take

$$\dot{\xi}_n = -\tfrac{1}{2}i\sigma_n a_{-n,r} \tag{4.11}$$

for $\pm n = 1, 2, 3, \ldots$, and any particular value of r. The condition (3 . 11) is satisfied, since it follows from (2 . 16) and Theorem 6 of (A) that

$$\sum_{n=1}^{\infty} \sigma_n^2 |a_{n,r}|^2 \leqslant \frac{\nu_r^2}{a}. \tag{4.12}$$

We now see that equations (3 . 14) and (3 . 23) give

$$\dot{q}_s = \tfrac{1}{2}\sum_{n=-\infty}^{\infty} \sigma_n^2 a_{n,s} a_{-n,r} \tag{4.13}$$

and $$\sum_{\rho=1}^{\infty} a_{s,\rho}\,\dot{q}_\rho = \tfrac{1}{2}\sum_{n=-\infty}^{\infty} \sigma_n^2 \frac{\partial T(a_n)}{\partial a_{n,s}}\, a_{-n,r} \tag{4.14}$$

respectively. From (3.41) and (4.14) we deduce that

$$\frac{\partial T(\dot{q})}{\partial \dot{q}_s} = 0 \quad (s \neq r), \quad = 1 \quad (s = r), \tag{4.21}$$

so that $$2T(\dot{q}) = \dot{q}_r. \tag{4.22}$$

From (2.12) we have

$$\sum_{s=1}^{\infty} \nu_s^2 \left| \frac{\partial T(\dot{q})}{\partial \dot{q}_s} \right|^2 \leqslant \frac{2A^2}{a}\, T(\dot{q}), \tag{4.31}$$

and therefore $$\nu_r^2 \leqslant \frac{A^2}{a}\,\dot{q}_r, \tag{4.32}$$

or $$\sum_{n=1}^{\infty} \sigma_n^2 |\, a_{n,r}\,|^2 \geqslant \frac{a}{A^2}\,\nu_r^2. \tag{4.33}$$

But from (2.16) and Theorem 6 of (A) we have

$$\sum_{n=1}^{\infty} |\, a_{n,r}\,|^2 \leqslant \frac{1}{c}, \tag{4.34}$$

for all values of r. In view of (4.33) and (4.34) it follows that, as ν_r^2 is unbounded, σ_n^2 must also be unbounded.

ON A SPECIAL FORM OF INTEGRAL EQUATION WITH ASSOCIATED EXPANSION THEOREM

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1. The integral equation to be considered is

$$\int_0^1 \{-\sigma^2 A(x, s)+i\sigma B(x, s)\}\, q(s)\, ds+q(x) = Q(x), \qquad (1.0)$$

where σ is a parameter, $A(x, s)$, $B(x, s)$, $Q(x)$ are given functions, and $q(x)$ is the unknown function. The functions $A(x, s)$, $B(x, s)$, and $Q(x)$ are supposed to be defined in the ranges $0 \leqslant x \leqslant 1$, $0 \leqslant s \leqslant 1$, to be independent of σ, and to be continuous in these ranges. The function $A(x, s)$ is supposed to be real, of positive definite type, and symmetrical, while the function $B(x, s)$ is supposed to be real and skew-symmetrical.

When $Q(x)$ is replaced by zero, so that the equation (1.0) becomes homogeneous, we shall show that there exists an infinity of real values of σ of the form

$$\pm\sigma_1, \ \pm\sigma_2, \ \ldots, \ \pm\sigma_n, \ \ldots, \qquad (1.11)$$

such that

$$\sum_{n=1}^{\infty} \frac{1}{\sigma_n^2} \leqslant \int_0^1 A(x, x)\, dx+\tfrac{1}{2}\int_0^1\int_0^1 \{B(x, s)\}^2\, dx\, ds, \qquad (1.12)$$

and for each of which the equation (1.0) possesses a solution $q(x)$ whose square is integrable. We shall denote by $q_n(x)$ the solution corresponding to σ_n and by $q_{-n}(x)$ its conjugate, which is the solution corresponding to $-\sigma_n = \sigma_{-n}$ (say). We shall show that we may take

$$\tfrac{1}{2}\int_0^1 q_m(x)\, q_n(x)\, dx-\tfrac{1}{2}\sigma_m\sigma_n\int_0^1\int_0^1 A(x, s)\, q_m(x)\, q_n(s)\, dx\, ds$$
$$= 0 \quad (m+n \neq 0),$$
$$= 1 \quad (m+n = 0). \qquad (1.13)$$

Suppose next that $q(x)$, $\dot{q}(x)$ are given functions, defined in the range $0 \leqslant x \leqslant 1$, and integrable in this range together with their squares, and take

$$\xi_n = \tfrac{1}{2}\int_0^1 q(x)\, q_{-n}(x)\, dx - \tfrac{1}{2}i\sigma_n \int_0^1\int_0^1 A(x, s)\, \dot{q}(x)\, q_{-n}(s)\, dx\, ds, \quad (1.21)$$

for $\pm n = 1, 2, 3, \ldots$. We shall show that

$$\left.\begin{aligned} \int_0^1 A(x, s)\, q(s)\, ds &= \sum_{n=-\infty}^{\infty} \xi_n \int_0^1 A(x, s)\, q_n(s)\, ds \\ \int_0^1 A(x, s)\, \dot{q}(s)\, ds &= \sum_{n=-\infty}^{\infty} i\sigma_n \xi_n \int_0^1 A(x, s)\, q_n(s)\, ds \end{aligned}\right\}, \quad (1.22)$$

and

$$\tfrac{1}{2}\int_0^1 \{q(x)\}^2\, dx + \tfrac{1}{2}\int_0^1\int_0^1 A(x, s)\, \dot{q}(x)\, \dot{q}(s)\, dx\, ds = \sum_{n=-\infty}^{\infty} \xi_n \xi_{-n}. \quad (1.23)$$

Finally, let σ be given and be such that the homogeneous equation obtained from (1.0) has no solution, and let $q(x)$ be the solution of the equation (1.0) itself. Also, let

$$\left.\begin{aligned} &q_1, \ q_2, \ \ldots, \ q_s, \ \ldots \\ &a_{n,1}, \ a_{n,2}, \ \ldots, \ a_{n,s}, \ \ldots \end{aligned}\right\} \quad (1.31)$$

denote the coefficients of the Hilbert expansions of $q(x)$ and $q_n(x)$ respectively, with the nucleus $A(x, s)$. Then we shall show that

$$\left.\begin{aligned} q_s &= \tfrac{1}{2}\sum_{n=-\infty}^{\infty} \frac{-\sigma_n}{\sigma - \sigma_n}\, a_{n,s} \int_0^1 Q(x)\, q_{-n}(x)\, dx \\ \sigma q_s &= \tfrac{1}{2}\sum_{n=-\infty}^{\infty} \frac{-\sigma_n^2}{\sigma - \sigma_n}\, a_{n,s} \int_0^1 Q(x)\, q_{-n}(x)\, dx \end{aligned}\right\}. \quad (1.32)$$

The theorems enunciated above were suggested by the theory of the oscillations of a rotating mechanical system, and the notation has been chosen so as to correspond with the notation usual in that theory. In establishing the theorems we shall quote from two recent papers, by the present authors, on the theory of such oscillations. It will be convenient to refer to the first* and second† of these papers as (O) and (T) respectively, and to the Appendix to (O) as (A).

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* "On the oscillations of a rotating mechanical system of infinite freedom", *Proc. London Math. Soc.* (2), 27 (1927), 1–33.

† "On the theory of the oscillations of a rotating mechanical system of infinite freedom". Preceding paper in these *Proceedings*.

2. *Reciprocal relation.*

Fredholm's theory shows that, under the conditions supposed, the equation (1 . 0) possesses a solution when the corresponding homogeneous equation does not. Let $q'(x)$ denote the solution when σ, $Q(x)$ are replaced by σ', $Q'(x)$, satisfying similar conditions. We shall first obtain a relation corresponding to the general energy-relation (2 . 33) of (O).

Multiply (1 . 0) by $\sigma' q'(x)$ and integrate from $x=0$ to $x=1$; we obtain

$$\int_0^1\int_0^1 \{-\sigma^2\sigma' A(x,s)+i\sigma\sigma' B(x,s)\}\, q(s)\, q'(x)\, ds\, dx+\sigma'\int_0^1 q(x)\, q'(x)\, dx = \sigma'\int_0^1 Q(x)\, q'(x)\, dx.$$

Similarly, we may obtain

$$\int_0^1\int_0^1 \{-\sigma'^2\sigma A(x,s)+i\sigma'\sigma B(x,s)\}\, q'(s)\, q(x)\, ds\, dx+\sigma\int_0^1 q'(x)\, q(x)\, dx = \sigma\int_0^1 Q'(x)\, q(x)\, dx,$$

and then, on adding, we have

$$(\sigma+\sigma')\left\{\int_0^1 q(x)\, q'(x)\, dx-\sigma\sigma'\int_0^1\int_0^1 A(x,s)\, q(x)\, q'(s)\, dx\, ds\right\} = \int_0^1 \{\sigma' Q(x)\, q'(x)+\sigma Q'(x)\, q(x)\}\, dx, \quad (2.1)$$

since
$$A(s,x)=A(x,s), \quad B(s,x)=-B(x,s) \quad (2.2)$$

from the supposed conditions of symmetry.

3. *Auxiliary integral equation.*

Our general method will be to translate results from the corresponding problems for an infinite set of linear algebraic equations. For this purpose we shall utilize some of the theorems associated with the integral equation of symmetrical nucleus

$$-\nu^2\int_0^1 A(x,s)\, \phi(s)\, ds+\phi(x)=0, \quad (3.1)$$

where ν is a parameter.

Let $$\nu_1^2,\ \nu_2^2,\ \ldots,\ \nu_s^2,\ \ldots$$

denote the characteristic constants of (3 . 1) arranged in increasing order of magnitude, and

$$\phi_1(x),\ \phi_2(x),\ \ldots,\ \phi_s(x),\ \ldots$$

the corresponding normalized characteristic functions of this equation. It is known* that ν_s^2 is positive and that

$$\sum_{s=1}^{\infty} \frac{1}{\nu_s^2} = \int_0^1 A(x, x)\, dx, \tag{3 . 21}$$

so that $$\nu_s^2 \to \infty \quad \text{as} \quad s \to \infty . \tag{3 . 22}$$

If $q(x)$ and $q'(x)$ are any two functions which are integrable in the range $0 \leqslant x \leqslant 1$, together with their squares, and if we take

$$\left.\begin{aligned} q_s &= \int_0^1 q(x)\, \phi_s(x)\, dx \\ q_s' &= \int_0^1 q'(x)\, \phi_s(x)\, dx \end{aligned}\right\} \quad (s = 1, 2, 3, \ldots), \tag{3 . 31}$$

Parseval's theorem gives

$$\sum_{s=1}^{\infty} q_s q_s' = \int_0^1 q(x)\, q'(x)\, dx, \tag{3 . 32}$$

the series being absolutely convergent. On taking $q(x)$ and $q'(x)$ to be conjugate complexes, we have

$$\sum_{s=1}^{\infty} |q_s|^2 = \int_0^1 |q(x)|^2\, dx. \tag{3 . 33}$$

On using (3 . 1) and (2 . 2) we have

$$\int_0^1 \int_0^1 A(x, s)\, q(s)\, \phi_s(x)\, dx\, ds = \frac{1}{\nu_s^2} \int_0^1 q(s)\, \phi_s(s)\, ds = \frac{1}{\nu_s^2} q_s, \tag{3 . 34}$$

and, therefore, (3 . 32) gives

$$\sum_{s=1}^{\infty} \frac{1}{\nu_s^2} q_s q_s' = \int_0^1 \int_0^1 A(x, s)\, q(x)\, q'(s)\, dx\, ds. \tag{3 . 35}$$

Now take $$\beta_{r,s} = \int_0^1 \int_0^1 B(x, s)\, \phi_r(x)\, \phi_s(s)\, dx\, ds \tag{3 . 41}$$

for $r, s = 1, 2, 3, \ldots$, so that

$$\beta_{s,r} = -\beta_{r,s} \tag{3 . 42}$$

* J. Mercer, *Phil. Trans. Royal Soc.* (A), 209 (1908), 439–446.

from (2 . 2). We deduce from (3 . 32) and (3 . 41) that

$$\sum_{s=1}^{\infty} \beta_{r,s} q_s = \int_0^1 \int_0^1 B(x, s)\, \phi_r(x)\, q(s)\, dx\, ds, \tag{3 . 43}$$

and

$$\sum_{s=1}^{\infty} \beta_{r,s}^2 = \int_0^1 \left\{ \int_0^1 B(x, s)\, \phi_r(x)\, dx \right\}^2 ds, \tag{3 . 44}$$

while, from (3 . 32) and (3 . 43), we have

$$\sum_{r,s=1}^{\infty} \beta_{r,s} q_s q_r' = \int_0^1 \int_0^1 B(x, s)\, q(s)\, q'(x)\, dx\, ds. \tag{3 . 45}$$

By Bessel's Inequality we have

$$\sum_{r=1}^{N} \left\{ \int_0^1 B(x, s)\, \phi_r(x)\, dx \right\}^2 \leqslant \int_0^1 \{B(x, s)\}^2\, dx,$$

whatever integer N may be, so that, after integrating from $x = 0$ to $x = 1$, we see that

$$\sum_{r=1}^{\infty} \int_0^1 \left\{ \int_0^1 B(x, s)\, \phi_r(x)\, dx \right\}^2 ds \leqslant \int_0^1 \int_0^1 \{B(x, s)\}^2\, dx\, ds,$$

or, from (3 . 44),

$$\sum_{r,s=1}^{\infty} \beta_{r,s}^2 \leqslant B^2, \tag{3 . 46}$$

where

$$B^2 = \int_0^1 \int_0^1 \{B(x, s)\}^2\, dx\, ds. \tag{3 . 47}$$

Two applications of Schwarz's Inequality give

$$\left| \int_0^1 \int_0^1 B(x, s)\, q(x)\, q'(s)\, dx\, ds \right|^2 \leqslant \int_0^1 \int_0^1 \{B(x, s)\}^2\, dx\, ds \int_0^1 \int_0^1 |q(x)\, q'(s)|^2\, dx\, ds$$

$$\leqslant B^2 \int_0^1 |q(x)|^2\, dx \int_0^1 |q'(s)|^2\, ds,$$

so that, on using (3 . 33) and (3 . 45),

$$\left| \sum_{r,s=1}^{\infty} \beta_{r,s} q_r q_s' \right|^2 \leqslant B^2 \sum_{r=1}^{\infty} |q_r|^2 \sum_{s=1}^{\infty} |q_s'|^2. \tag{3 . 5}$$

4. *Algebraic equations.*

If we multiply (1 . 0) by $\phi_r(x)$ and integrate from $x = 0$ to $x = 1$ we obtain

$$\int_0^1 \int_0^1 \{-\sigma^2 A(x, s) + i\sigma B(x, s)\}\, q(s)\, \phi_r(x)\, ds\, dx + \int_0^1 q(x)\, \phi_r(x)\, dx$$

$$= \int_0^1 Q(x)\, \phi_r(x)\, dx, \tag{4 . 11}$$

for $r = 1, 2, 3, \ldots$. By (3 . 31), (3 . 34), and (3 . 43) this may be written as

$$-\frac{\sigma^2}{\nu_r^2} q_r + i\sigma \sum_{s=1}^{\infty} \beta_{r,s} q_s + q_r = Q_r, \tag{4 . 12}$$

where

$$Q_r = \int_0^1 Q(x)\, \phi_r(x)\, dx \tag{4 . 13}$$

for $r = 1, 2, 3, \ldots$.

Conversely, if we start with (4 . 12) and (4 . 13), and know that

$$\sum_{s=1}^{\infty} |q_s|^2 \tag{4 . 2}$$

is convergent, the Riesz-Fischer Theorem shows the existence of a function $q(x)$ satisfying (3 . 31). The equations (3 . 34) and (3 . 43) then follow, and (4 . 12) may be written in the form (4 . 11). But by Hilbert's theory regarding the functions ϕ_r, it follows that, since (4 . 11) is true for all the values of r, the equation (1 . 0) is also true.

On comparing the above equations with those of the mechanical problem of the two papers quoted at the end of § 1, we see that (4 . 12) are a particular case of the equations of motion. The particularity is that the kinetic and potential energy function $T(q)$ and $V(q)$ are represented by the sums of squares

$$\tfrac{1}{2} \sum_{s=1}^{\infty} \frac{1}{\nu_s^2} q_s^2, \quad \tfrac{1}{2} \sum_{s=1}^{\infty} q_s^2 \tag{4 . 3}$$

respectively. With these special forms the restrictive assumptions of the two earlier papers reduce to two. One of these is identical with (3 . 22) above, and is therefore satisfied in the present paper. The other is the second of the relations (1 . 21) of (O), which restricts the magnitude of $\beta_{r,s}$. This is now replaced by (3 . 46) above, the condition (3 . 5) being less stringent than that of (O) and (T).

The results of §§ 3, 4, 5 of (O) are independent of these restrictions, and will be quoted below. The arguments of §§ 8, 9 (i) of (O) depend only on the results of § 3 of (T), which, in turn, depend on the relation restricting $\beta_{r,s}$ only in so far as this relation is used in the establishment of (2 . 2) and (2 . 35) of (T). But, in § 6 below, we shall show that (2 . 2) and (2. 35) of (T) may be deduced from the restrictions of the present paper. It will then follow that we may quote any of the results of §§ 8, 9 (i) of (O) and § 3 of (T), and this we shall do freely in §§ 7, 8 below.

5. *Characteristic constants.*

From §§ 3, 4 of (O) there exist quantities

$$\sigma_n(N), \quad a_{n,s}(N),$$

such that

$$\sigma_{-n}(N) = -\sigma_n(N),$$

and

$$-\frac{1}{\nu_r^2}\sigma_n^2(N)\,a_{n,r}(N)+i\sigma_n(N)\sum_{s=1}^{N}\beta_{r,s}\,a_{n,s}(N)+a_{n,r}(N)=0, \quad (5.11)$$

for $\pm n = 1, 2, \ldots, N$, and $r = 1, 2, \ldots, N$. Also, from § 5 of (O), these quantities may be made to satisfy

$$\tfrac{1}{2}\sum_{-1}^{N} a_{m,r}(N)\,a_{n,r}(N)-\tfrac{1}{2}\sigma_m(N)\,\sigma_n(N)\sum_{r=1}^{N}\frac{1}{\nu_r^2}\,a_{m,r}(N)\,a_{n,r}(N)$$
$$= 0 \quad (m+n\neq 0),$$
$$= 1 \quad (m+n=0), \quad (5.12)$$

for $\pm m$, $\pm n = 1, 2, \ldots, N$, and $a_{n,s}(N)$, $a_{-n,s}(N)$ are conjugate complexes. Further, from (5.31) of (O), we have

$$\left.\begin{aligned}&\tfrac{1}{2}\sum_{n=-N}^{N} a_{n,r}(N)\,a_{-n,s}(N) = 0 \quad (r\neq s), \quad = 1 \quad (r=s)\\ &\tfrac{1}{2}\sum_{n=-N}^{N}\sigma_n(N)\,a_{n,r}(N)\,a_{-n,s}(N) = 0\end{aligned}\right\}, \quad (5.13)$$

for $r, s = 1, 2, \ldots, N$.

By multiplying (5.11) by $a_{-n,s}(N)/\sigma_n(N)$ and adding for $n = -N, \ldots, N$, we obtain

$$2i\beta_{r,s}+\sum_{n=-N}^{N}\frac{1}{\sigma_n(N)}\,a_{n,r}(N)\,a_{-n,s}(N)=0, \quad (5.21)$$

on using (5.13); by multiplying (5.11) by $a_{-n,r}(N)/\sigma_n^2(N)$ and adding for $n = -N, \ldots, N$, we obtain, on using (5.13), (5.21), and rearranging,

$$\tfrac{1}{2}\sum_{n=-N}^{N}\frac{1}{\sigma_n^2(N)}\,|a_{n,r}(N)|^2=\frac{1}{\nu_r^2}+\sum_{s=1}^{N}\beta_{r,s}^2. \quad (5.22)$$

From (5.12) we have

$$\tfrac{1}{2}\sum_{r=1}^{N}\frac{1}{\nu_r^2}\,|a_{n,r}(N)|^2+\frac{1}{2\sigma_n^2(N)}\sum_{r=1}^{N}|a_{n,r}(N)|^2=\frac{1}{\sigma_n^2(N)}, \quad (5.23)$$

and from (5.13)

$$\tfrac{1}{2}\sum_{n=-N}^{N}|a_{n,r}(N)|^2=1. \quad (5.24)$$

On adding (5 . 23) for $n = -N, \ldots, N$, using (5 . 22), (5 . 24), and rearranging, we obtain

$$\sum_{n=1}^{N} \frac{1}{\sigma_n^2(N)} = \sum_{r=1}^{N} \frac{1}{\nu_r^2} + \tfrac{1}{2} \sum_{r,s=1}^{N}\sum \beta_{r,s}^2. \tag{5 . 25}$$

Now, by § 3 of (O), there exist limits σ_n such that

$$\sigma_n(N) \to \sigma_n \quad \text{as} \quad N \to \infty, \tag{5 . 31}$$

for $n = 1, 2, 3, \ldots$, and, from (5 . 25), we have, therefore,

$$\sum_{n=1}^{\infty} \frac{1}{\sigma_n^2} \leqslant \sum_{r=1}^{\infty} \frac{1}{\nu_r^2} + \tfrac{1}{2} \sum_{r,s=1}^{\infty}\sum \beta_{r,s}^2. \tag{5 . 32}$$

From this, with (3 . 21), (3 . 46), and (3 . 47), the result (1 . 12) follows.

6. *Certain limits.*

From § 4 of (O) there exist limits $a_{n,r}$ such that

$$a_{n,r}(N) \to a_{n,r} \quad \text{as} \quad N \to \infty, \tag{6 . 11}$$

and

$$-\frac{\sigma_n^2}{\nu_r^2} a_{n,r} + i\sigma_n \sum_{s=1}^{\infty} \beta_{r,s}\, a_{n,s} + a_{n,r} = 0, \tag{6 . 12}$$

for all the values of n and r. Also, from (5 . 12) above, we have

$$\tfrac{1}{2} \sum_{r=1}^{N} |a_{n,r}(N)|^2 \leqslant 1. \tag{6 . 13}$$

From this and (3 . 22), (3 . 46), it follows by the theory* of completely continuous quadratic and bilinear forms that

$$\lim_{N\to\infty} \sum_{r=1}^{N} \frac{1}{\nu_r^2} |a_{n,r}(N)|^2 = \sum_{r=1}^{\infty} \frac{1}{\nu_r^2} |a_{n,r}|^2, \tag{6 . 21}$$

$$\lim_{N\to\infty} \sum_{r,s=1}^{N}\sum \beta_{r,s}\, a_{n,s}(N)\, a_{-n,r}(N) = \sum_{r,s=1}^{\infty}\sum \beta_{r,s}\, a_{n,s}\, a_{-n,r}. \tag{6 . 22}$$

On multiplying (5 . 11) by $a_{-n,r}(N)$, adding for $r = 1, 2, \ldots, N$, and rearranging, we have

$$\sum_{r=1}^{N} |a_{n,r}(N)|^2 = \sigma_n^2(N) \sum_{r=1}^{N} \frac{1}{\nu_r^2} |a_{n,r}(N)|^2 - i\sigma_n(N) \sum_{r,s=1}^{N}\sum \beta_{r,s}\, a_{n,s}(N)\, a_{-n,r}(N), \tag{6 . 31}$$

* D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, 147–164.

and in a similar manner, from (6 . 12), it follows that

$$\sum_{r=1}^{\infty} |a_{n,r}|^2 = \sigma_n^2 \sum_{r=1}^{\infty} \frac{1}{\nu_r^2} |a_{n,r}|^2 - i\sigma_n \sum_{r,s=1}^{\infty}\sum \beta_{r,s}\, a_{n,s}\, a_{-n,r}. \qquad (6 . 32)$$

From (6 . 21), (6 . 22), (6 . 31), (6 . 32) we deduce that

$$\lim_{N\to\infty} \sum_{r=1}^{N} |a_{n,r}(N)|^2 = \sum_{r=1}^{\infty} |a_{n,r}|^2. \qquad (6 . 33)$$

The equations (6 . 21) and (6 . 33) correspond respectively to (2 . 2) and (2 . 35) of (T).

7. *Homogeneous equation.*

From (6 . 11), (6 . 13) we deduce

$$\tfrac{1}{2} \sum_{s=1}^{\infty} |a_{n,s}|^2 \leqslant 1, \qquad (7 . 11)$$

and on comparing this and (6 . 12) with (4 . 2) and (4 . 12) respectively, we see that a function $q_n(x)$ exists such that

$$a_{n,s} = \int_0^1 q_n(x)\, \phi_s(x)\, dx, \qquad (7 . 12)$$

and

$$\int_0^1 \{-\sigma_n^2 A(x, s) + i\sigma_n B(x, s)\}\, q_n(s)\, ds + q_n(x) = 0, \qquad (7 . 13)$$

for all the values of n and r.

From (3 . 32) and (3 . 35) we have

$$\left.\begin{aligned} \sum_{s=1}^{\infty} a_{m,s}\, a_{n,s} &= \int_0^1 q_m(x)\, q_n(x)\, dx \\ \sum_{s=1}^{\infty} \frac{1}{\nu_s^2} a_{m,s}\, a_{n,s} &= \int_0^1 \int_0^1 A(x, s)\, q_m(x)\, q_n(s)\, dx\, ds \end{aligned}\right\}. \qquad (7 . 21)$$

But, by (9 . 1) of (O), we have

$$\begin{aligned} \tfrac{1}{2} \sum_{s=1}^{\infty} a_{m,s}\, a_{n,s} - \tfrac{1}{2}\sigma_m\sigma_n \sum_{s=1}^{\infty} \frac{1}{\nu_s^2} a_{m,s}\, a_{n,s} &= 0 \quad (m+n \neq 0), \\ &= 1 \quad (m+n = 0), \end{aligned} \qquad (7 . 22)$$

and hence we deduce the relations (1 . 13). This proves the first of the theorems enunciated in § 1.

8. *Expansion theorem.*

On taking the quantities ξ_n to be given by (1 . 21) and q_r, $\dot{q}_r$ according to (3 . 31), we have, by (3 . 32) and (3 . 35),

$$\xi_n = \tfrac{1}{2} \sum_{s=1}^{\infty} a_{-n,s}\, q_s - \tfrac{1}{2} i\sigma_n \sum_{s=1}^{\infty} \frac{1}{\nu_s^2}\, a_{-n,s}\, \dot{q}_s, \tag{8 . 11}$$

for $\pm n = 1, 2, 3, \ldots$. It then follows from § 8 of (O) that

$$\left.\begin{aligned} q_s &= \sum_{n=-\infty}^{\infty} a_{n,s}\, \xi_n \\ \dot{q}_s &= \sum_{n=-\infty}^{\infty} i\sigma_n\, a_{n,s}\, \xi_n \end{aligned}\right\}, \tag{8 . 12}$$

for $s = 1, 2, 3, \ldots$, and

$$\sum_{n=-\infty}^{\infty} |\xi_n|^2 = \tfrac{1}{2} \sum_{s=1}^{\infty} |q_s|^2 + \tfrac{1}{2} \sum_{s=1}^{\infty} \frac{1}{\nu_s^2} |\dot{q}_s|^2, \tag{8 . 13}$$

$$\sum_{n=-\infty}^{\infty} \xi_n \xi_{-n} = \tfrac{1}{2} \sum_{s=1}^{\infty} q_s^2 + \tfrac{1}{2} \sum_{s=1}^{\infty} \frac{1}{\nu_s^2} \dot{q}_s^2. \tag{8 . 14}$$

From (8 . 14), with the aid of (3 . 32) and (3 . 35), we deduce the equation (1 . 23).

By Hilbert's Expansion-theorem we have

$$\left.\begin{aligned} \sum_{s=1}^{\infty} \frac{1}{\nu_s^2}\, a_{n,s}\, \phi_s(x) &= \int_0^1 A(x, s)\, q_n(s)\, ds \\ \sum_{s=1}^{\infty} \frac{1}{\nu_s^2}\, q_s\, \phi_s(x) &= \int_0^1 A(x, s)\, q(s)\, ds \\ \sum_{s=1}^{\infty} \frac{1}{\nu_s^2}\, \dot{q}_s\, \phi_s(x) &= \int_0^1 A(x, s)\, \dot{q}(s)\, ds \end{aligned}\right\}, \tag{8 . 21}$$

the series being absolutely convergent, and, on using (8 . 12), we deduce that

$$\left.\begin{aligned} \sum_{n=-\infty}^{\infty} \xi_n \left\{ \sum_{s=1}^{N} \frac{1}{\nu_s^2}\, a_{n,s}\, \phi_s(x) \right\} &= \sum_{s=1}^{N} \frac{1}{\nu_s^2}\, q_s\, \phi_s(x) \\ \sum_{n=-\infty}^{\infty} i\sigma_n\, \xi_n \left\{ \sum_{s=1}^{N} \frac{1}{\nu_s^2}\, a_{n,s}\, \phi_s(x) \right\} &= \sum_{s=1}^{N} \frac{1}{\nu_s^2}\, \dot{q}_s\, \phi_s(x) \end{aligned}\right\}. \tag{8 . 22}$$

Now

$$\tfrac{1}{2} \sum_{n=-M}^{M} \left| \sum_{s=1}^{N} \frac{1}{\nu_s^2}\, a_{n,s}\, \phi_s(x) \right|^2 \leqslant \tfrac{1}{2} \sum_{n=-\infty}^{\infty} \left\{ \sum_{s=1}^{N} \frac{1}{\nu_s^2}\, a_{n,s}\, \phi_s(x) \right\} \left\{ \sum_{s=1}^{N} \frac{1}{\nu_s^2}\, a_{-n,s}\, \phi_s(x) \right\}, \tag{8 . 23}$$

and $\frac{1}{2}\sum_{n=-M}^{M}\sigma_n^2\left|\sum_{s=1}^{N}\frac{1}{\nu_s^2}a_{n,s}\phi_s(x)\right|^2$

$$\leqslant \tfrac{1}{2}\sum_{n=-\infty}^{\infty}\sigma_n^2\left\{\sum_{s=1}^{N}\frac{1}{\nu_s^2}a_{n,s}\phi_s(x)\right\}\left\{\sum_{s=1}^{N}\frac{1}{\nu_s^2}a_{-n,s}\phi_s(x)\right\}, \quad (8.24)$$

whatever integer M may be. But from (3 . 41) of (T)

$$\left.\begin{aligned}\tfrac{1}{2}\sum_{n=-\infty}^{\infty}a_{n,r}a_{-n,s}&=0 \quad (r\neq s), \quad =1 (r=s)\\ \tfrac{1}{2}\sum_{n=-\infty}^{\infty}\sigma_n^2 a_{n,r}a_{-n,s}&=0 \quad (r\neq s), \quad =\nu_s^2 (r=s)\end{aligned}\right\}, \quad (8.25)$$

and hence the series on the left-hand sides of (8 . 23) and (8 . 24) do not exceed the series

$$\sum_{s=1}^{\infty}\frac{1}{\nu_s^4}\{\phi_s(x)\}^2, \quad \sum_{s=1}^{\infty}\frac{1}{\nu_s^2}\{\phi_s(x)\}^2, \quad (8.26)$$

respectively, and these series are known* to be absolutely convergent. In view of this and (8 . 13) it follows, by Theorem 1 of (A), that the series on the left-hand side of (8 . 22) are absolutely convergent. We may therefore deduce from (8 . 22) by the corollary to Theorem 7 of (A) that

$$\left.\begin{aligned}\sum_{n=-\infty}^{\infty}\xi_n\left\{\sum_{s=1}^{\infty}\frac{1}{\nu_s^2}a_{n,s}\phi_s(x)\right\}&=\sum_{s=1}^{\infty}\frac{1}{\nu_s^2}q_s\phi_s(x)\\ \sum_{n=-\infty}^{\infty}i\sigma_n\xi_n\left\{\sum_{s=1}^{\infty}\frac{1}{\nu_s^2}a_{n,s}\phi_s(x)\right\}&=\sum_{s=1}^{\infty}\frac{1}{\nu_s^2}\dot{q}_s\phi_s(x)\end{aligned}\right\}. \quad (8.27)$$

From (8 . 21) and (8 . 27) the equations (1 . 22) now follow, and this proves the second of the theorems enunciated in § 1.

9. *Non-homogeneous equation.*

In the formulae of the preceding section let us take

$$\dot{q}(x)=i\sigma q(x), \quad (9.11)$$

and suppose that $q(x)$ is the solution of the general equation (1 . 0). Then we have

$$\xi_n=\tfrac{1}{2}\int_0^1 q(x)\,q_{-n}(x)\,dx+\tfrac{1}{2}\sigma\sigma_n\int_0^1\int_0^1 A(x,s)\,q(x)\,q_{-n}(s)\,dx\,ds \quad (9.12)$$

for $\pm n=1, 2, 3, \ldots$.

* J. Mercer, *loc. cit.*

In the reciprocal relation (2 . 1) let us take

$$Q'(x) = 0, \quad \sigma' = -\sigma_n, \quad q'(x) = q_{-n}(x),$$

so that it becomes

$$(\sigma - \sigma_n)\left\{\int_0^1 q(x)\, q_{-n}(x)\, dx + \sigma\sigma_n \int_0^1 \int_0^1 A(x, s)\, q(x)\, q_{-n}(s)\, dx\, ds\right\} = -\sigma_n \int_0^1 Q(x)\, q_{-n}(x)\, dx. \quad (9 . 2)$$

From (9 . 12) and (9 . 2) we have

$$\xi_n = \tfrac{1}{2} \frac{-\sigma_n}{\sigma - \sigma_n} \int_0^1 Q(x)\, q_{-n}(x)\, dx \quad (9 . 3)$$

for $\pm n = 1, 2, 3, \ldots$.

On substituting from (9 . 11) and (9 . 3) into (8 . 12) we have the results (1 . 32), and this proves the third theorem enunciated in § 1.

SOME RESULTS CONCERNING THE INTEGRALS OF MODULI OF REGULAR FUNCTIONS ALONG CURVES OF CERTAIN TYPES

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The main object of this paper is to give a proof of the following theorem :

If $f(z)$ is a regular function of z for z inside and on a circle Γ, and C is any convex curve inside Γ, and $\lambda > 0$, then

$$\int_C |f(z)|^\lambda |dz| \leqslant 2 \int_\Gamma |f(z)|^\lambda |dz|.$$

Further the coefficient 2 on the right-hand side is best possible.

It was the quest of results of this type which led to my theorems concerning integrals along lines and circles, published some while ago*.

If K is any curve, we write

$$I(K) = \int_K |f(z)|^\lambda |dz|,$$

and we take λ positive throughout the paper.

LEMMA 1. *Suppose that C and K are curves lying in a domain D, and that a transformation regular and "schlicht" in D puts K on K', C on C', and D on D'. If $I(C) \leqslant A \, . \, I(K)$ for all $f(z)$ regular in D, then $I(S') \leqslant A \, . \, I(K')$ for all $f(z)$ regular in D'.*

Let $w = w(z)$ be the transformation of the theorem, and let $f(w)$ be a function of w regular in D'. Then

$$f(w) = f\{w(z)\} = \phi(z)$$

say, where ϕ is a regular function of z for z in D.

* See *Journal London Math. Soc.*, 2 (1927), 112–117.

Now dw/dz is a regular function of z and is not zero for z in D, by the *schlichtheit* property. Hence $\phi(z)(dw/dz)^{1/\lambda}$ is a function regular for z in D. This implies

$$\int_C |\phi(z)|^\lambda |dw/dz||dz| \leqslant A \int_K |\phi(z)|^\lambda |dw/dz||dz|,$$

i.e.

$$\int_{C'} |f(w)|^\lambda |dw| \leqslant A \int_{K'} |f(w)|^\lambda |dw|,$$

and this is the lemma.

THEOREM 1.1*. *If L is a straight line enclosed by a simple closed curve K symmetrical about L, then, for $f(z)$ regular inside and on K, $I(L) \leqslant \frac{1}{2} I(K)$ for $\lambda = 2$.*

We may take L to be a portion of the real axis. Let K_1, K_2 be the parts of K on either side of the real axis. Consider $\bar{f}(z)$ equal to the conjugate of $f(\bar{z})$, where $\bar{z}$ is the conjugate of z. By a well known theorem $\bar{f}(z)$ is regular inside and on K.

Let L cut K in points A, B. Then

$$\begin{aligned}
\int_{AB} |f(z)|^2 |dz| &= \left| \int_A^B f(z)\bar{f}(z)\,dz \right| = \left| \int_{K_1} f(z)\bar{f}(z)\,dz \right| \\
&\leqslant \int_{K_1} |f(z)|\,|\bar{f}(z)|\,|dz| \\
&\leqslant \left(\int_{K_1} |f(z)|^2 |dz| \right)^{\frac{1}{2}} \left(\int_{K_1} |\bar{f}(z)|^2 |dz| \right)^{\frac{1}{2}} \\
&= \left(\int_{K_1} |f(z)|^2 |dz| \right)^{\frac{1}{2}} \left(\int_{K_2} |f(z)|^2 |dz| \right)^{\frac{1}{2}} \\
&\leqslant \tfrac{1}{2} \left\{ \int_{K_1} |f(z)|^2 |dz| + \int_{K_2} |f(z)|^2 |dz| \right\} \\
&= \tfrac{1}{2} \int_{K_1+K_2} |f(z)|^2 |dz|.
\end{aligned}$$

This proves the theorem.

In the last theorem we may take L to be a line AB, and K_1, K_2 to be equal circular arcs AXB, AYB on opposite sides of L. By Lemma 1, with the use of the inversion transformation, we have

* Cf. F. Riesz and L. Fejér, "Über einige Funktionentheoretische Ungleichungen", *Math. Zeitschrift*, 11 (1921), 305–314.

THEOREM 1.2. *If AOB is a circular arc, and AXB, AYB are circular arcs cutting AOB at equal angles on opposite sides of AOB, and if $f(z)$ is regular in the closed domain bounded by AXB, AYB, and $\lambda = 2$, then $I(AOB) \leqslant \frac{1}{2}\{I(AXB)+I(AYB)\}$.*

Consider a circle Γ divided into two arcs Γ_1, Γ_2 by points A, B. Let C be any circular arc inside Γ, joining A, B and cutting Γ_1, Γ_2 at angles $\theta_1\pi$, $\theta_2\pi$.

THEOREM 2.1. *If $f(z)$ is regular inside and on Γ, then*

$$I(C) \leqslant \theta_2 I(\Gamma_1)+\theta_1 I(\Gamma_2).$$

We shall first suppose that $\lambda = 2$. The theorem is certainly true for $\theta_1 = 0$ and for $\theta_1 = 1$.

We shall prove that if the theorem is true for two values of θ_1, it is also true for their arithmetic mean. Assume the theorem, then, for $\theta_1 = \alpha_1, \beta_1$, corresponding to C_α, C_β. Let $\theta_1 = \frac{1}{2}(\alpha_1+\beta_1)$ correspond to C. Then

$$\begin{aligned} I(C) &\leqslant \tfrac{1}{2}\{I(C_\alpha)+I(C_\beta)\} \\ &\leqslant \tfrac{1}{2}\{\alpha_2 I(\Gamma_1)+\alpha_1 I(\Gamma_2)+\beta_2 I(\Gamma_1)+\beta_1 I(\Gamma_2)\} \\ &= \tfrac{1}{2}(\alpha_2+\beta_2) I(\Gamma_1)+\tfrac{1}{2}(\alpha_1+\beta_1) I(\Gamma_2) \\ &= \theta_2 I(\Gamma_1)+\theta_1 I(\Gamma_2), \end{aligned}$$

by Theorem 1.2 and the assumption concerning C_α, C_β. Thus, if the theorem is true for $\theta_1 = \alpha_1, \beta_1$, it is also true for $\theta_1 = \frac{1}{2}(\alpha_1+\beta_1)$. Now the theorem is true for $\theta_1 = 1$ and for $\theta_1 = 0$, and is therefore true for $\theta_1 = r/2^n \leqslant 1$ where r and n are two integers. The theorem for general $\theta_1 \leqslant 1$ follows by continuity.

We proceed to remove the condition $\lambda = 2$. If $f(z)$ is *wurzelfrei* inside and on Γ the extension to general $\lambda > 0$ presents no difficulty, for $\phi^{\frac{1}{2}\lambda}(z)$ is regular inside and on Γ. Since $f(z)$ is regular inside and on Γ, the number of zeros inside Γ is finite. Let these zeros be $a_1, a_2, \ldots, a_n$, allowing for multiple zeros by repetition, and suppose that $f(z) \neq 0$ on Γ. Consider

$$\psi(z) = \prod_{r=1}^{n} \left(\frac{z-a_r}{\bar{a}_r z-1}\right),$$

where $\bar{a}_r$ is the conjugate of a_r. Then $|\psi(z)| = 1$ on Γ, $|\psi(z)| < 1$ inside Γ. If $f_1(z) = f(z)/\psi(z)$, then $f_1(z)$ is regular and *wurzelfrei* inside and on Γ, and thence the theorem for general λ holds for $f_1(z)$. Now $|f(z)| = |f_1(z)|$ on Γ, and $|f(z)| \leqslant |f_1(z)|$ on C, and hence the theorem

holds for $f(z)$ with general λ. We can deal with the case where $f(z)$ has zeros on Γ by continuity.

In particular, we may take C to be a chord L. By a simple addition this extends to

THEOREM 2.2. *With the notation and conditions of the last theorem, $I(P) \leqslant 2I(\Gamma)$ if P is any simple convex polygon with its vertices on Γ.*

Corresponding to each side L_r of P there exist two arcs of Γ, say Γ_r, Γ'_r, joining the extremities of L_r. One of these, Γ_r, say, does not contain any vertices of P in its interior; the other may be written

$$\Gamma'_r = \sum_{s \neq r} \Gamma_s.$$

Let L_r cut Γ_r at an angle $\theta_r \pi$. Then

$$\begin{aligned} I(L_r) &\leqslant (1-\theta_r)\, I(\Gamma_r) + \theta_r I(\Gamma'_r) \\ &= (1-\theta_r)\, I(\Gamma_r) + \theta_r \sum_{s \neq r} I(\Gamma_s), \\ I(P) &= \sum_r I(L_r) \\ &\leqslant \sum_r (1-\theta_r)\, I(\Gamma_r) + \sum_r \theta_r \sum_{s \neq r} I(\Gamma_s) \\ &\leqslant \sum_r I(\Gamma_r) \{1-\theta_r + \sum_{s \neq r} \theta_s\} \\ &\leqslant \sum_r I(\Gamma_r) \{1+\sum_s \theta_s\} \\ &= I(\Gamma) \{1+\sum_s \theta_s\}. \end{aligned}$$

By the property of the inscribed polygon $\Sigma\theta_s = 1$. Hence $I(P) \leqslant 2I(\Gamma)$.

To develop this further we require another mode of attack. We shall prove that if P is *any* convex simple polygon of a fixed number of sides and lying inside Γ, then $\operatorname{Max} I(P)$ for fixed $f(z)$ can be obtained with all the vertices of P on the boundary.

THEOREM 3.1. *If K is a convex curve bounding a finite domain D and $f(z)$ is regular inside and on K, then if A, B are fixed points and P a variable point all lying inside the closed domain $D' = D+K$,*

$$\operatorname{Max} \{I(AP)+I(PB)\}$$

occurs when P is on the boundary K.

Since $f(z)$ is regular in D', given ϵ

$$\left| |f(z)|^\lambda - |f(z_0)|^\lambda \right| < \epsilon$$

for all z, z_0 in D' provided that $|z-z_0|<\delta(\epsilon)$, where $\delta(\epsilon)$ is independent of z and z_0.

Let l be the length of the longest chord of D', and let

$$N = N(\epsilon) = [l/\delta]+1,$$

where $[l/\delta]$ is the integral part of l/δ. Now for any point P of D', AP and BP lie wholly in D' by the convexity of K. Divide AP, BP into N equal parts by points ξ_r, η_r respectively. Each subdivision is of length less than δ and lies in D'. Hence

$$\left|\int_{AP+PB}|f(z)|^\lambda|dz|-\sum_1^N|f(\xi_r)|^\lambda|\xi_r-\xi_{r-1}|-\sum_1^N|f(\eta_r)|^\lambda|\eta_r-\eta_{r-1}|\right|\leqslant 2\epsilon l.$$

Let P be the point ζ and A, B, the points α, β.

$$\xi_r=[r\alpha+(N-r)\zeta]/N,\quad \eta_r=[r\beta+(N-r)\zeta]/N,\quad r=(0,1,2,\ldots,N),$$

$$S_N(\zeta)=\sum_1^N|f(\xi_r)|^\lambda|\xi_r-\xi_{r-1}|+\sum_1^N|f(\eta_r)|^\lambda|\eta_r-\eta_{r-1}|.$$

We now appeal to an extended form of a result given by Pólya and Szegö, *Aufgaben u. Lehrsätze aus der Analysis*, 1, 143 (302). The extended result, which is easily proved by a method similar to that which they use, is as follows:

If functions $\chi_1(z)$, $\chi_2(z)$, ..., $\chi_n(z)$; $\phi_1(z)$, $\phi_2(z)$, ..., $\phi_n(z)$ *are all regular for* z *in a closed domain* D, *and* p_1, p_2, ..., p_n; q_1, q_2, ..., q_n *are all positive numbers, then* $F(z)=\sum_1^n|\phi_r(z)|^{p_r}|\chi_r(z)|^{q_r}$ *attains its maximum for* z *in* D, *when* z *takes at least one value on the boundary of* D.

The function $f(\xi_r)$, where $\xi_r=[r\alpha+(N-r)\zeta]/N$, is a regular function of ζ for ζ in D', for if ζ is in D', then ξ_r is also in D'. Taking all the p's equal to λ and all the q's equal to unity, then by a suitable choice of χ's and ϕ's we have the result that $S_N(\zeta)\leqslant M$ for ζ on the boundary of D' implies $S_N(\zeta)\leqslant M$ for ζ inside D'.

Thus we have proved that if $I(AP+PB)\leqslant M$ for P on the boundary, then $S_N(\xi)\leqslant M+2l\epsilon$ for ζ on the boundary, and hence that

$$S_N(\zeta)\leqslant M+2l\epsilon$$

for ζ inside D. This implies

$$I(AP+PB)\leqslant M+4l\epsilon$$

for all P in D'. Hence $I(AP+PB)\leqslant M$ for P inside D'. This is the theorem.

Next consider a convex domain D bounded by a straight line AB and a convex curve Γ joining A, B, and suppose that π_n is any n-sided simple convex polygon joining A, B, and lying in the closed domain $D' = D + AB + \Gamma$.

THEOREM 3.2. *With the conditions of the last theorem* $\operatorname{Max} I(\pi_n)$ *is obtained for some polygon* π_n *with all its vertices lying on* Γ.

We must allow the vertices to coincide if necessary, so that the polygon giving the maximum may be AB itself.

The theorem for $n = 2$ is included in the last theorem. Suppose that the theorem is true for $n < m$. We shall show that this implies the truth of the theorem for $n = m$.

Consider any π_m $(A = P_0, P_1, P_2, \ldots, P_m = B)$. Suppose that P_1 does not lie on Γ; let $P_3 P_2$ produced cut Γ in C. Let P_1 vary in the closed convex domain bounded by $P_0 P_2$, $P_2 C$ and the arc CP_0 of Γ. By the last theorem, $\operatorname{Max} I(P_0 P_1 + P_1 P_2)$ occurs when P_1 lies on the boundary, at P_1' say. Replace the original P_1 by P_1'. This does not decrease $I(\pi_m)$. Either (i) P_1' lies on $P_0 P_2$, or (ii) on $P_3 P_2$ produced, or (iii) on Γ. In cases (i) and (ii) the polygon π_m has been reduced to a π_{m-1} and for this the theorem is known; so that we could obtain as great a value for the integral with all the vertices on the boundary as for the original π_m. In case (iii) consider the $(m-1)$-sided polygon joining P_1', B. We can obtain at least as large a value for the integral by special choice of the vertices on the arc $P_1' B$ of Γ. The case in which P_1' lies on Γ is covered by this.

THEOREM 3.3. *If* K *is any convex curve and* π_n *is a simple* n*-sided convex closed polygon lying inside* K *and with one vertex on* K, *then* $\operatorname{Max} I(\pi_n)$ *is obtained when all the vertices lie on* K, *with the proviso about the coincidence of vertices as before.*

For this take A, B coincident in Theorem 3.2.

THEOREM 4. *If* C *is any convex curve inside a circle* Γ *and* $f(z)$ *is regular inside and on* Γ, $I(C) \leqslant 2I(\Gamma)$.

Take, first, C to be any convex polygon π_n with n sides, and let Γ be $|z| = 1$. Now let r be $\operatorname{Max}|z|$ for z on π_n. By the last theorem we can increase $I(\pi_n)$ in the wide sense, by taking all the vertices on the circle Γ' $|z| = r$. By Theorem 2.2, $\operatorname{Max} I(\pi_n) \leqslant 2I(\Gamma')$. But

$I(\Gamma') \leqslant I(\Gamma)$, since $\int_{|z|=r} |f(z)|^\lambda |dz|$

is an increasing function of r. Hence Max $I(\pi_n) \leqslant 2I(\Gamma)$.

We can approximate to any convex curve C by means of convex polygons and therefore, by continuity, $I(C) \leqslant 2I(\Gamma)$.

In Theorem 4 the coefficient 2 is best possible.

For given $f(z)$ regular in Γ, there exists a circle $\Gamma' = \Gamma_1' + \Gamma_2'$ with the following properties :

(i) Γ_1' inside Γ, Γ_2' outside Γ.

(ii) $I(\Gamma_1') > (1-\epsilon) I(\Gamma)$, where ϵ is any positive number.

Inverting with respect to a point of Γ' we have the result that, given ϵ, we can obtain $f(z)$ regular in Γ and L a chord of Γ such that $I(L) > (1-\epsilon) I(\Gamma)$. Now L, taken twice over, is a convex curve. This establishes the result.

DIVERGENT CONTINUED FRACTIONS AND POLAR SINGULARITIES

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PART III. *Several boundary poles.*

1. *Introduction.* In two preceding papers† the behaviour of the continued fractions associated with the rows of the Padé table‡ has been considered at points of divergence and at points of the circle which bounds the representation, for a function meromorphic in a region containing the origin. The theorem defining the region of representation of these continued fractions, due to Montessus de Ballore§, states that the continued fraction belonging to the ν-th row of the table converges at all points within the circle passing through the ν-pole (in order of proximity to the origin), except at enclosed poles, and diverges at all points outside this circle. The theorem holds whether the poles are simple, multiple, or with index-points of equal moduli, provided that they are counted according to multiplicity. In the case of a multiple boundary pole, or of several boundary poles, the bounding circle remains unchanged for a certain number of rows of the table equal to the total multiplicity of the boundary pole or poles. The first paper dealt with the case of a simple boundary pole, and in the second paper the solution was extended to include a

† R. Wilson, "Divergent continued fractions and polar singularities", *Proc. London Math. Soc.* (2), 26 (1927), 159-168. References to this paper will bear the asterisk *. R. Wilson, "Divergent continued fractions and polar singularities, Part 2", *Proc. London Math. Soc.* (2), 27 (1928), 497–512. References to this paper will bear the double asterisk **.

‡ H. Padé, "Sur la représentation approchée d'une fonction par des fonctions rationnelles, *Annales Sci. de l'Ecole Normale* (3), 9 (1892), 1-93, Supplément. O. Perron, *Die Lehre von den Kettenbrüchen*, Ch. 10.

§ R. Montessus de Ballore, "Sur les fractions continues algébriques", *Rend. di Palermo*, 19 (1905), 1-73.

multiple boundary pole. With a few exceptions, the results were direct generalizations of those obtained in the first paper, although the methods employed were different.

The object of this paper is to extend the discussion further to include a number of boundary poles. Although, in a certain sense, generalizations, the results are of a different character from those arising previously. The existence of several boundary poles introduces a degree of indetermination which is entirely absent from the precise results occurring with a single boundary pole.

On account of its simplicity the case of a number of simple boundary poles is first considered; the extension to several boundary poles, of which some or all are multiple, follows directly; finally, an attempt is made to discuss the case of continued fractions of preceding rows which possess the same bounding circle (as in § 8**).

2. *Several boundary poles and the abnormal case.* Consider a function having poles of order $\ldots, m, \ldots, p, \ldots, k_1, k_2, \ldots, k_l, \ldots$ at the points $\ldots, 1/\beta_s, \ldots, 1/\beta_r, \ldots, 1/\bar{\beta}_1, 1/\bar{\beta}_2, \ldots, 1/\bar{\beta}_l, \ldots$ of which the l last mentioned are points on the circle $|z\bar{\beta}| = 1$, and having no other singularities within or on this circle. Then, as in the second paper, a function of the simpler form

$$f(z) \equiv \sum_s \frac{B_s}{(1-\beta_s z)^m} + \sum_t \frac{\bar{B}_t}{(1-\bar{\beta}_t z)^{k_t}} + \phi(z) \tag{1}$$

will first be considered, where $\phi(z)$ is holomorphic within and on a circle of radius $1/\rho$, and $\rho < |\bar{\beta}| < \ldots < |\beta_r| < \ldots < |\beta_s| < \ldots$, the number of poles, counted according to multiplicity, being $(\nu+1)$.

Let the Taylor expansion of $f(z)$ in the neighbourhood of the origin be

$$f(z) \equiv c_0 + c_1 z + c_2 z^2 + \ldots + c_m z^m + \ldots.$$

The algebraic rational fraction $P_{n,\nu}(z)/Q_{n,\nu}(z)$, of degrees n in the numerator and ν in the denominator, approximating to the expansion to the first $(n+\nu+1)$ terms, is given by

$$P_{n,\nu}(z) \equiv \begin{vmatrix} \psi_n(z), & z\psi_{n-1}(z), & \ldots, & z^\nu \psi_{n-\nu}(z) \\ c_{n+1}, & c_n, & \ldots, & c_{n-\nu+1} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ c_{n+\nu}, & c_{n+\nu-1}, & \ldots, & c_n \end{vmatrix}, \quad Q_{n,\nu}(z) \equiv \begin{vmatrix} 1, & z, & \ldots, & z^\nu \\ c_{n+1}, & c_n, & \ldots, & c_{n-\nu+1} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ c_{n+\nu}, & c_{n+\nu-1}, & \ldots, & c_n \end{vmatrix}$$

where $\psi_{n-t}(z) = c_0 + c_1 z + \ldots + c_{n-t} z^{n-t}$ and $c_{-m} = 0$ $(m > 0)$.

$R_{n,\nu}(z) \equiv P_{n,\nu}(z)/Q_{n,\nu}(z)$ is the approximant of the $(\nu+1)$-th row and $(n+1)$-th column of the Padé table.

The continued fraction of the $(\nu+1)$-th row converges if $\lim_{n\to\infty} R_{n,\nu}(z)$ exists and is finite. From Montessus de Ballore's theorem this occurs at all points within the circle $|z\bar{\beta}| = 1$, except at the enclosed poles.

The methods of the second paper are applied to evaluate $P_{n,\nu}(z)$ and $Q_{n,\nu}(z)$ for large values of n, and $\lim_{n\to\infty} R_{n,\nu}(z)$ is discussed at

(*a*) the enclosed poles;

(*b*) the different boundary poles;

(*c*) other points on the boundary circle.

It is easy to prove, as in § 7**, that the results obtained for (1) are true for the more general function as first defined. Consequently all calculations will be made with functions of the simpler form given by (1).

A matter concerning the abnormal case†, not arising in the first two papers, occurs here. It was stated in a footnote to the first paper that the Padé table of a function, meromorphic in a suitable region containing the origin, is *normal* along the ν-th row (ν finite) for sufficiently large values of n. This statement requires some qualification. It is true for a single boundary pole, whether simple or multiple. In the case of several boundary poles there exists a class of exceptions arising from functions in which all the singularities are cyclically distributed, *i.e.* in which the singularities are symmetrically placed on circles concentric with the origin, the poles on any one circle being of the same multiplicity‡. The existence of such abnormal cases, however, will not affect the discussion, the only modification in the proof being that, instead of proceeding to the limiting value of $R_{n,\nu}(z)$ through successive integral values of n, we shall proceed through the sequence of increasing values of n given by the first corners of the successive squares of abnormality, for it is known that the approximants at all other points of a square of abnormality reduce to the *determinate* fraction of the first corner†.

The methods of the second paper are employed in full detail in the following development and will be assumed to be known. To avoid

† H. Padé, O. Perron, *loc. cit.* See also R. Wilson, "A note on the polynomial factor occurring in the abnormal case of the approximation to a function by means of a rational fraction", *Messenger of Math.*, 53 (1925), 177–184.

‡ The function $2/(e^z + e^{-z})$ provides an example.

repetition, calculations are made only in so far as they introduce points of difference from the earlier papers.

3. *A lemma.* The less precise nature of the results obtained in the ensuing paragraphs, as compared with those of the previous papers, depends on the occurrence in $R_{n,\nu}(z)$ of a factor of the form $\sum_1^k C_r e^{in\theta_r}$ or $\left(\sum_1^k C_r e^{in\theta_r}\right)^{-1}$, where the θ's are real and the C's are, in general, complex. It is thus necessary to discuss the behaviour of the expression $\sum_1^k C_r e^{in\theta_r}$ as $n \to \infty$.

Consider the rational algebraic function

$$\sum_1^k \frac{C_r}{1-ze^{i\theta_r}} \equiv a_0+a_1z+a_2z^2+\ldots+a_nz^n+\ldots.$$

The problem reduces to a discussion of the existence of a limiting value for a_n as $n \to \infty$.

Cauchy's theorem on the radius of convergence of a power series gives the result

$$\overline{\lim_{n\to\infty}} |a_n|^{1/n} = 1,$$

which, however, is not sufficiently refined for our purpose.

Darboux† has shown that, for a function having polar singularities on the circle of convergence of its Taylor series,

$$\overline{\lim_{n\to\infty}} |a_n| > 0.$$

Since in this case

$$|a_n| = \left|\sum_1^k C_r e^{in\theta_r}\right| \leqslant \sum_1^k |C_r| = \text{a finite constant},$$

the upper limit is finite.

In regard to the lower limit, it is easy to construct examples for which this is zero, and others for which it is non-zero.

An example of the first case is the function

$$\frac{1}{1-z}-\frac{1}{1+z}.$$

† G. Darboux, "Mémoire sur les fonctions de grands nombres", *J. de Math.* (3). 4 (1878), 5-36, 377-416.

If $|C_1| > \sum_2^k |C_r|$, then the second case occurs.

Hence, in the following discussions, these results will be used freely:

$$\overline{\lim_{n\to\infty}} \left| \sum_1^k C_r e^{in\theta_r} \right| > 0 \quad \textit{and} \quad \underline{\lim_{n\to\infty}} \left| \left(\sum_1^k C_r e^{in\theta_r} \right)^{-1} \right| \quad \textit{is finite and non-zero.}$$

4. *Several simple boundary poles. Value at an enclosed pole.* In this case each of the indices $k_1, k_2, \ldots, k_l$ is unity.

As in (4)** and (6)**, $Q_{n,\nu}(1/\beta_r)$, $P_{n,\nu}(1/\beta_r)$ are evaluated at an enclosed pole of order p at $(1/\beta_r)$, giving

$$Q_{n,\nu}(1/\beta_r) = \frac{\Delta_{1Q}\,\Delta_{2Q}}{B_r \beta_r^{\nu}} \left[1 + \sum_{t=1}^{l} F_t (\rho/\bar{\beta}_t)^n \right]$$

$$= \frac{1}{B_r \beta_r^{\nu}} \begin{vmatrix} \ldots, 0, & \ldots, 0, & \ldots, 0, \\ \ldots, {}^{-1}c_0 \beta_s^{n-\nu+1}, & \ldots, {}^{-1}c_{m-1} \beta_s^{n-\nu+1}, & \ldots, {}^{-1}c_0 \beta_r^{n-\nu+1}, \\ \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots \\ \ldots, {}^{-\nu}c_0 \beta_s^n, & \ldots, {}^{-\nu}c_{m-1} \beta_s^n, & \ldots, {}^{-\nu}c_0 \beta_r^n, \end{vmatrix}$$

$$\begin{vmatrix} \ldots, 1, & \ldots, 0, & \ldots, 0 \\ \ldots, {}^{-1}c_{p-1} \beta_r^{n-\nu+1}, & \ldots, \bar{\beta}_1^{n-\nu+1}, & \ldots, \bar{\beta}_l^{n-\nu+1} \\ \ldots \quad \ldots & \ldots \quad \ldots & \ldots \quad \ldots \\ \ldots \quad \ldots & \ldots \quad \ldots & \ldots \quad \ldots \\ \ldots, {}^{-\nu}c_{p-1} \beta_r^n, & \ldots, \bar{\beta}_1^n, & \ldots, \bar{\beta}_l^n \end{vmatrix}$$

$$\times \begin{vmatrix} \ldots, {}^{-m}c_{n-\nu} B_s, & \ldots, {}^{-1}c_{n-\nu} B_s, & \ldots, {}^{-p}c_{n-\nu} B_r, \\ \ldots, {}^{-m}c_{n-\nu+1} B_s \beta_s, & \ldots, {}^{-1}c_{n-\nu+1} B_s \beta_s, & \ldots, {}^{-p}c_{n-\nu+1} B_r \beta_r, \\ \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots \\ \ldots, {}^{-m}c_n B_s \beta_s^{\nu}, & \ldots, {}^{-1}c_n B_s \beta_s^{\nu}, & \ldots, {}^{-p}c_n B_r \beta_r^{\nu} \end{vmatrix}$$

$$\begin{vmatrix} \ldots, {}^{-1}c_{n-\nu} B_r, & \ldots, \bar{B}_1, & \ldots, \bar{B}_l \\ \ldots, {}^{-1}c_{n-\nu+1} B_r \beta_r, & \ldots, \bar{B}_1 \bar{\beta}_1, & \ldots, \bar{B}_l \bar{\beta}_l \\ \ldots \quad \ldots & \ldots \quad \ldots & \ldots \quad \ldots \\ \ldots \quad \ldots & \ldots \quad \ldots & \ldots \quad \ldots \\ \ldots, {}^{-1}c_n B_r \beta_r^{\nu}, & \ldots, \bar{B}_1 \bar{\beta}_1^{\nu}, & \ldots, \bar{B}_l \bar{\beta}_l^{\nu} \end{vmatrix}$$

$$\times \left[1 + \sum_{t=1}^{l} F_t (\rho/\bar{\beta}_t)^n \right], \qquad (2)$$

where every $|F_t| < F$, a finite constant, and

$$ {}^{-m}c_n \equiv \frac{m(m+1)\dots(m+n-1)}{n!}. $$

Similarly

$$ P_{n,\nu}(1/\beta_r) = \frac{\Delta_{1P}\Delta_{2P}}{\beta_r^{\nu}}\left[1+\sum_{t=1}^{l} G_t(\rho/\bar{\beta}_t)^n + \sum_{t=1}^{l} H_t(\bar{\beta}_t/\beta_r)^n\right], $$

$$ = \frac{1}{\beta_r^{\nu}} \begin{vmatrix} \dots, 0, & \dots, 0, & \dots, 0, \\ \dots, {}^{-1}c_0\beta_s^{n-\nu+1}, & \dots, {}^{-1}c_{m-1}\beta_s^{n-\nu+1}, & \dots, {}^{-1}c_0\beta_r^{n-\nu+1}, \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots, {}^{-\nu}c_0\beta_s^n, & \dots, {}^{-\nu}c_{m-1}\beta_s^n, & \dots, {}^{-\nu}c_0\beta_r^n, \end{vmatrix} $$

$$ \begin{vmatrix} \dots, 0, & \dots, 0, & \dots, 0, & 1 \\ \dots, {}^{-1}c_{p-1}\beta_r^{n-\nu+1}, & \dots, \bar{\beta}_1^{n-\nu+1}, & \dots, \bar{\beta}_l^{n-\nu+1}, & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots, {}^{-\nu}c_{p-1}\beta_r^n, & \dots, \bar{\beta}_1^n & \dots, \bar{\beta}_l^n, & 0 \end{vmatrix} $$

$$ \times \begin{vmatrix} \dots, {}^{-m}c_{n-\nu}B_s, & \dots, {}^{-1}c_{n-\nu}B_s, & \dots, {}^{-p}c_{n-\nu}B_r, \\ \dots, {}^{-m}c_{n-\nu+1}B_s\beta_s, & \dots, {}^{-1}c_{n-\nu+1}B_s\beta_s, & \dots, {}^{-p}c_{n-\nu+1}B_r\beta_r, \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots, {}^{-m}c_nB_s\beta_s^{\nu}, & \dots, {}^{-1}c_nB_s\beta_s^{\nu}, & \dots, {}^{-p}c_nB_r\beta_r^{\nu}, \end{vmatrix} $$

$$ \begin{vmatrix} \dots, {}^{-1}c_{n-\nu}B_r, & \dots, \bar{B}_1, & \dots, \bar{B}_l, & {}^{-p-1}c_{n-\nu}B_r \\ \dots, {}^{-1}c_{n-\nu+1}B_r\beta_r, & \dots, \bar{B}_1\bar{\beta}_1, & \dots, \bar{B}_l\bar{\beta}_l, & {}^{-p-1}c_{n-\nu+1}B_r\beta_r \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots, {}^{-1}c_nB_r\beta_r^{\nu}, & \dots, \bar{B}_1\bar{\beta}_1^{\nu}, & \dots, \bar{B}_l\bar{\beta}_l^{\nu}, & {}^{-p-1}c_nB_r\beta_r^{\nu} \end{vmatrix} $$

$$ \times\left[1+\sum_{t=1}^{l} G_t\left(\frac{\rho}{\bar{\beta}_t}\right)^n + \sum_{t=1}^{l} H_t\left(\frac{\bar{\beta}_t}{\beta_r}\right)^n\right], \qquad (3) $$

where $|G_t| < G$, $|H_t| < H$, two finite constants.

This matrix product can be summed as $\nu+2$ products of corresponding pairs of determinants, of which the l survivors are those from which a $\bar{\beta}$ column is omitted. As in (10)* or (7)*, $Q_{n,\nu}(1/\beta_r)$, $P_{n,\nu}(1/\beta_r)$

can be evaluated exactly, giving

$$\frac{P_{n,\nu}(1/\beta_r)}{Q_{n,\nu}(1/\beta_r)} = \Big[\sum_{t=1}^{l} C_t \Big(\frac{\beta_r}{\bar{\beta}_t}\Big)^n\Big](1+J_n\rho'^n),$$

where $|J_n| < J$, a finite constant, $|\rho'| < 1$, and

$$C_t = \frac{B_r^2}{\bar{B}_t}\,\frac{\bar{\beta}_t^{\nu-1}}{\beta_r^{\nu-2p}}\,\frac{\prod_s{}' (\beta_r-\beta_s)^{2m}}{\prod_s{}' (\bar{\beta}_t-\beta_s)^{2m}}\,(\beta_r-\bar{\beta}_t)^{1-2p},$$

the factor $(\beta_r \sim \bar{\beta}_t)$ being omitted from Π', in which the product extends to all other poles, including those on the boundary.

Hence $|R_{n,\nu}(1/\beta_r)|/|(\beta_r/\bar{\beta})|^n$ does *not* tend to a limit as $n \to \infty$, but, from § 3, it has a finite non-zero upper limit. (4)

5. *Value at a boundary pole.* Consider the value of $P_{n,\nu}(1/\bar{\beta}_t)$, $Q_{n,\nu}(1/\bar{\beta}_t)$. For the former, the terms in the right-hand *column* of Δ_{2P} of (3), $^{-p-1}c_{n-\nu}B_r$, ..., $^{-p-1}c_nB_r\beta_r^\nu$, are replaced by $^{-2}c_{n-\nu}\bar{B}_t$, ..., $^{-2}c_n\bar{B}_t\bar{\beta}_t^\nu$. Of the l surviving determinants in Δ_{2P}, all except one contain the columns

$$\begin{array}{llll} \bar{B}_t, & {}^{-2}c_{n-\nu}\bar{B}_t & \text{which reduce to} & \bar{B}_t, \quad 0 \\ \bar{B}_t\bar{\beta}_t, & {}^{-2}c_{n-\nu+1}\bar{B}_t\bar{\beta}_t & & \bar{B}_t\bar{\beta}_t, \quad \bar{B}_t\bar{\beta}_t \\ \cdots & \cdots \quad \cdots & & \cdots \quad \cdots \\ \cdots & \cdots \quad \cdots & & \cdots \quad \cdots \\ \bar{B}_t\bar{\beta}_t^\nu, & {}^{-2}c_n\bar{B}_t\bar{\beta}_t^\nu & & \bar{B}_t\bar{\beta}_t^\nu, \quad \nu\bar{B}_t\bar{\beta}_t^\nu. \end{array}$$

The determinant from which the *column* $\bar{B}_t$, ..., $\bar{B}_t\bar{\beta}_t^\nu$ is omitted retains the *column* $^{-2}c_{n-\nu}\bar{B}_t$, ..., $^{-2}c_n\bar{B}_t\bar{\beta}_t^\nu$ intact, and is thus of a higher order in n than its companions.

Hence, as in § 8* or (8)**,

$$\frac{P_{n,\nu}(1/\bar{\beta}_t)}{Q_{n,\nu}(1/\bar{\beta}_t)} = n\bar{B}_t\Big[1+O\Big(\frac{1}{n}\Big)\Big],$$

giving

$$\lim_{n\to\infty} \frac{R_{n,\nu}(1/\bar{\beta}_t)}{n} = \bar{B}_t. \tag{5}$$

This result holds for $t = 0, 1, 2, \ldots, l$.

6. *Value on the boundary.* The value of $R_{n,\nu}(z)$ at a point on the circle $|z\bar{\beta}| = 1$ is determined as in § 9**. There are three different

contributions, obtained similarly, and giving at points on the circle, other than poles:

$$(a)\qquad \lim_{n\to\infty}\frac{P'_{n,\nu}(z)}{Q_{n,\nu}(z)}=\lim_{n\to\infty}\frac{\chi(z)}{\Pi(1-\beta_s z)\left[\sum_{t=1}^{l}C_t\Big/(z\bar{\beta}_t)^n\right]},$$

where $\chi(z)$ is a rational function of z, having singularities only at the poles of $f(z)$, the product is taken over the enclosed and boundary poles, and the C_t $(t=1, 2, \ldots, l)$ are determinate constants.

$$(b)\qquad \lim_{n\to\infty}\frac{P''_{n,\nu}(z)}{Q_{n,\nu}(z)}=\sum_s\frac{B_s}{(1-\beta_s z)^m}+\sum_{t=1}^{l}\frac{\bar{B}_t}{(1-\bar{\beta}_t z)},$$

$$(c)\qquad \lim_{n\to\infty}\frac{P'''_{n,\nu}(z)}{Q_{n,\nu}(z)}=\phi(z).$$

Hence

$$\lim_{n\to\infty}R_{n,\nu}(z)=\lim_{n\to\infty}\frac{P'_{n,\nu}(z)}{Q_{n,\nu}(z)}+\lim_{n\to\infty}\frac{P''_{n,\nu}(z)}{Q_{n,\nu}(z)}+\lim_{n\to\infty}\frac{P'''_{n,\nu}(z)}{Q_{n,\nu}(z)}$$

does not exist at any regular point of the circle $|z\bar{\beta}|=1$. (6)

7. *Multiple boundary poles. Value at an enclosed pole.* The method indicated in § 3, combined with the analysis of §§ 3**–6**, is used to discuss the case in which some or all of the boundary poles are multiple. The boundary poles at $(1/\bar{\beta}_1)$, $(1/\bar{\beta}_2)$, ..., $(1/\bar{\beta}_l)$ will be taken to have multiplicities k_1, k_2, ..., k_l, as in § 2. Of these, any k may be unity, but one at least is not.

The determinants Δ_{1Q}, Δ_{2Q} of $Q_{n,\nu}(1/\beta_r)$, corresponding to those of (2), now have the end columns

$$\begin{array}{cccc}
\ldots, 0, & \ldots, 0, & \ldots, 0, & \ldots, 0\\
\ldots, {}^{-1}c_0\bar{\beta}_1^{n-\nu+1}, & \ldots, {}^{-1}c_{k_1-1}\bar{\beta}_1^{n-\nu+1}, & \ldots, {}^{-1}c_0\bar{\beta}_l^{n-\nu+1}, & \ldots, {}^{-1}c_{k_l-1}\bar{\beta}_l^{n-\nu+1}\\
\ldots\quad\ldots & \ldots\quad\ldots & \ldots\quad\ldots & \ldots\quad\ldots\\
\ldots\quad\ldots & \ldots\quad\ldots & \ldots\quad\ldots & \ldots\quad\ldots\\
\ldots, {}^{-\nu}c_0\bar{\beta}_1^{n}, & \ldots, {}^{-\nu}c_{k_1-1}\bar{\beta}_1^{n}, & \ldots, {}^{-\nu}c_0\bar{\beta}_l^{n}, & \ldots, {}^{-\nu}c_{k_l-1}\bar{\beta}_l^{n}
\end{array},$$

$$\left|\begin{array}{cccc}
\ldots, {}^{-k_1}c_{n-\nu}\bar{B}_1, & \ldots, {}^{-1}c_{n-\nu}\bar{B}_1, & \ldots, {}^{-k_l}c_{n-\nu}\bar{B}_l, & \ldots, {}^{-1}c_{n-\nu}\bar{B}_l\\
\ldots, {}^{-k_1}c_{n-\nu+1}\bar{B}_1\bar{\beta}_1, & \ldots, {}^{-1}c_{n-\nu+1}\bar{B}_1\bar{\beta}_1, & \ldots, {}^{-k_l}c_{n-\nu+1}\bar{B}_l\bar{\beta}_l, & \ldots, {}^{-1}c_{n-\nu+1}\bar{B}_l\bar{\beta}_l\\
\ldots\quad\ldots & \ldots\quad\ldots & \ldots\quad\ldots & \ldots\quad\ldots\\
\ldots\quad\ldots & \ldots\quad\ldots & \ldots\quad\ldots & \ldots\quad\ldots\\
\ldots, {}^{-k_1}c_n\bar{B}_1\bar{\beta}_1^{\nu}, & \ldots, {}^{-1}c_n\bar{B}_1\bar{\beta}_1^{\nu}, & \ldots, {}^{-k_l}c_n\bar{B}_l\bar{\beta}_l^{\nu}, & \ldots, {}^{-1}c_n\bar{B}_l\bar{\beta}_l^{\nu}
\end{array}\right|.$$

The corresponding modifications in the β columns of Δ_{1P}, Δ_{2P} are

$$\begin{vmatrix} \ldots, 0, & \ldots, 0, & \ldots, 0, & \ldots, 0, & 1 \\ \ldots, {}^{-1}c_0\bar{\beta}_1^{n-\nu+1}, & \ldots, {}^{-1}c_{k_1-1}\bar{\beta}_1^{n-\nu+1}, & \ldots, {}^{-1}c_0\bar{\beta}_l^{n-\nu+1}, & \ldots, {}^{-1}c_{k_l-1}\bar{\beta}_l^{n-\nu+1}, & 0 \\ \ldots \quad \ldots & \ldots \quad \ldots & \ldots \quad \ldots & \ldots \quad \ldots & \ldots \\ \ldots \quad \ldots & \ldots \quad \ldots & \ldots \quad \ldots & \ldots \quad \ldots & \ldots \\ \ldots, {}^{-\nu}c_0\bar{\beta}_1^{n}, & \ldots, {}^{-\nu}c_{k_1-1}\bar{\beta}_1^{n}, & \ldots, {}^{-\nu}c_0\bar{\beta}_l^{n}, & \ldots, {}^{-\nu}c_{k_l-1}\bar{\beta}_l^{n}, & 0 \end{vmatrix},$$

$$\ldots, {}^{-k_1}c_{n-\nu}\bar{B}_1, \quad \ldots, {}^{-1}c_{n-\nu}\bar{B}_1,$$

$$\begin{matrix} \ldots, {}^{-k_1}c_{n-\nu+1}\bar{B}_1\bar{\beta}_1, & \ldots, {}^{-1}c_{n-\nu+1}\bar{B}_1\bar{\beta}_1, \\ \ldots \quad \ldots & \ldots \quad \ldots \\ \ldots \quad \ldots & \ldots \quad \ldots \end{matrix}$$

$$\ldots, {}^{-k_1}c_n\bar{B}_1\bar{\beta}_1^{\nu}, \quad \ldots, {}^{-1}c_n\bar{B}_1\bar{\beta}_1^{\nu},$$

$$\begin{vmatrix} \ldots, {}^{-k_l}c_{n-\nu}\bar{B}_l, & \ldots, {}^{-1}c_{n-\nu}\bar{B}_l, & {}^{-p-1}c_{n-\nu}B_r \\ \ldots, {}^{-k_l}c_{n-\nu+1}\bar{B}_l\bar{\beta}_l, & \ldots, {}^{-1}c_{n-\nu+1}\bar{B}_l\bar{\beta}_l, & {}^{-p-1}c_{n-\nu+1}B_r\beta_r \\ \ldots \quad \ldots & \ldots \quad \ldots & \ldots \quad \ldots \\ \ldots \quad \ldots & \ldots \quad \ldots & \ldots \quad \ldots \\ \ldots, {}^{-k_l}c_n\bar{B}_l\bar{\beta}_l^{\nu}, & \ldots, {}^{-1}c_n\bar{B}_l\bar{\beta}_l^{\nu}, & {}^{-p-1}c_nB_r\beta_r^{\nu} \end{vmatrix}$$

Proceeding as in § 5**,

$$\frac{P_{n,\nu}(1/\beta_r)}{Q_{n,\nu}(1/\beta_r)} = \Big[\sum_{t=1}^{l} n^{k_t-1}\, C_t\Big(\frac{\beta_r}{\bar{\beta}_t}\Big)\Big]\Big[1+O\Big(\frac{1}{n}\Big)\Big],$$

where
$$C_t = \frac{B_r^2}{\bar{B}_t}\,\frac{\beta_r^{2p-\nu}}{\bar{\beta}^{2k_t-\nu-1}}\,\frac{\overset{s}{\Pi}{}'(\beta_r-\beta_s)^{2m}}{\overset{s}{\Pi}{}'(\bar{\beta}_t-\beta_s)^{2m}}\,\frac{(\beta_r-\bar{\beta}_t)^{2k_t-2p-1}}{(k_t-1)\,!},$$

the factor $(\beta_r \sim \bar{\beta}_t)$ being omitted from Π', and the product extends to all the poles, including those on the boundary.

Two cases arise in the discussion of the limiting value of the sequence of approximants $R_{n,\nu}(1/\beta_r)$ as $n \to \infty$.

If one boundary pole is of higher order than the others, say that at $(1/\beta_u)$, then

$$\lim_{n\to\infty} \frac{R_{n,\nu}(1/\beta_r)}{n^{k_u-1}(\beta_r/\beta_u)^n} = C_u. \tag{7}$$

If, on the other hand, there are two or more poles of highest order, k_u, there is no unique limit, as in (7), but

$$\overline{\lim_{n\to\infty}} \frac{|R_{n,\nu}(1/\beta_r)|}{n^{k_u-1}|\beta_r/\bar{\beta}_u)|^n} \quad \text{is finite and non-zero.} \tag{8}$$

8. *Value at a boundary pole.* To obtain the values of $P_{n,\nu}(z)$, $Q_{n,\nu}(z)$ at a boundary pole at $(1/\beta_u)$, the methods of § 6**, with modifications as in § 6, are used, giving

$$\frac{P_{n,\nu}(1/\bar{\beta}_u)}{Q_{n,\nu}(1/\bar{\beta}_u)} = \left[\frac{n^{k_u}}{k_u!}\bar{B}_u + \sum_{t=1}^{t}{}' n^{k_t-1} C_t \left(\frac{\bar{\beta}_u}{\bar{\beta}_t}\right)^n\right]\left[1+O\left(\frac{1}{n}\right)\right],$$

where C_t is defined, as in § 7, but with $\bar{\beta}_u$ in place of $\bar{\beta}_r$, and the term $t = u$ is excluded from Σ'.

Different cases arise as follows :

If the pole at $(1/\bar{\beta}_u)$ is one of the highest order

$$\lim_{n\to\infty} \frac{R_{n,\nu}(1/\bar{\beta}_u)}{n^{k_u}} = \frac{\bar{B}_u}{k_u!}. \tag{9}$$

If the pole at $(1/\bar{\beta}_u)$ is of order one less than the highest, there is no unique limit, as in (9), but

$$\overline{\lim_{n\to\infty}} \frac{|R_{n,\nu}(1/\bar{\beta}_u)|}{n^{k_u}} \text{ is finite and non-zero.} \tag{10}\dagger$$

If the pole at $(1/\bar{\beta}_u)$ is any other boundary pole, and there is :

(*a*) a single boundary pole of highest order at $(1/\bar{\beta}_v)$, then

$$\lim \frac{R_{n,\nu}(1/\bar{\beta}_u)}{n^{k_v-1}(\bar{\beta}_v/\bar{\beta}_u)^n} \text{ is finite and non-zero } (k_u < k_v-1); \tag{11}\dagger$$

(*b*) more than one boundary pole of highest order, k_v, then there is no unique limit, as in (11), but

$$\overline{\lim_{n\to\infty}} \frac{|R_{n,\nu}(1/\bar{\beta}_u)|}{n^{k_v-1}} \text{ is finite and non-zero } (k_u < k_v-1). \tag{12}\dagger$$

9. *Value on the boundary.* It is required to evaluate $P_{n,\nu}(z)$, $Q_{n,\nu}(z)$, formed as in § 7, at regular points on the bounding circle $|z\bar{\beta}| = 1$. The method is similar to that of § 9**, and three contributions arise, as follows :

$$(a) \quad \lim_{n\to\infty} \frac{P'_{n,\nu}(z)}{Q_{n,\nu}(z)} = \lim_{n\to\infty} \frac{\chi(z)}{\Pi(1-\beta_s z)^m \left[\sum_{t=1}^{t} n^{k_t-1} C_t \Big/ (z\bar{\beta}_t)^n\right]},$$

† Though not affecting the nature of the result, contributions arise also from terms as in (5)**. These can be ignored in other cases as in § 4**.

where $\chi(z)$ is a rational function of z, having singularities only at the poles of $f(z)$, the product is taken over the enclosed and boundary poles, and the C_t $(t = 1, 2, \ldots, l)$ are determinate constants.

$$(b) \qquad \lim_{n\to\infty} \frac{P''_{n,\nu}(z)}{Q_{n,\nu}(z)} = \sum_s \frac{B_s}{(1-\beta_s z)^m} + \sum_{t=1}^{l} \frac{\bar{B}_t}{(1-\bar{\beta}_t z)^{k_t}}.$$

$$(c) \qquad \lim_{n\to\infty} \frac{P'''_{n,\nu}(z)}{Q_{n,\nu}(z)} = \phi(z).$$

Hence

$$\lim_{n\to\infty} R_{n,\nu}(z) = f(z) + \lim_{n\to\infty} \frac{\chi(z)}{\Pi(1-\beta_s z)^m \left[\sum_{t=1}^{\infty} n^{k_t-1} C_t \Big/ (z\bar{\beta}_t)^n\right]}.$$

Two cases arise according as all the boundary poles at $(1/\bar{\beta}_t)$ are or are not simple, $t = 1, 2, \ldots, l$.

In the first case, as $n \to \infty$, $R_{n,\nu}(z)$ does not tend to a limit at any point on the boundary, except at the singularities already discussed. This case has been dealt with in § 6.

In the second case, in *general*†,

$$\lim_{n\to\infty} R_{n,\nu}(z) = f(z). \qquad (14)$$

It should be noted that the existence of a single *multiple* pole on the boundary is sufficient to ensure convergence at non-singular points on the bounding circle.

10. *Continued fractions with the same bounding circle.* When there are l simple poles on the bounding circle, as in § 4, the bounding circle for the continued fraction of the $(\nu+1)$-th row is also the boundary for the continued fractions of the $l-1$ rows antecedent, as in the case of a single multiple boundary pole of order l (§ 8**).

Using the method of § 8**, the values of Δ_{1P}, Δ_{2P}, Δ_{1Q}, Δ_{2Q} of $P_{n,\nu-\tau}$, $Q_{n,\nu-\tau}$ $(\tau < l)$ are the $(\nu+1-\tau)$-rowed matrices obtained by suppressing the last τ row in the expressions (3) and (2). The evaluation gives rise

† As is shown in § 11, functions possessing a cyclic distribution of equi-multiple polar singularities, such as give rise to the abnormal case, form exceptions to (14).

to the following results :

At an enclosed pole at $(1/\beta_\tau)$,

$$\lim_{n\to\infty} \frac{|R_{n,\nu-\tau}(1/\beta_\tau)|}{|(\beta_\tau/\bar{\beta})|^n} \qquad [\tau = 0, 1, 2, \ldots, (l-1)] \quad (15)$$

does not exist.

At a boundary pole at $(1/\bar{\beta}_t)$,

$$\lim_{n\to\infty} \frac{R_{n,\nu-\tau}(1/\bar{\beta}_t)}{n} = \bar{B}_t \quad [\tau = 0, 1, 2, \ldots, (l-1)], \quad (16)$$

while at any regular point on the boundary $|z\bar{\beta}| = 1$, $R_{n,\nu-\tau}(z)$ does not tend to a limiting value as $n \to \infty$. (17)

When one (or more) of the several boundary poles is multiple, the evaluation of $P_{n,\nu-\tau}$, $Q_{n,\nu-\tau}$ involves $(\nu+1-\tau)$-rowed matrices, which are too unwieldy to be dealt with in detail as a general case. The above methods of operation are, however, sufficient to determine exactly the result in any given case. The process of selection of the surviving determinant bi-products proceeds as in §§ 4**–8**; but it must be noted that, in general, there will be not merely one surviving bi-product, as in § 8**, but several, in both $P_{n,\nu-\tau}$ and $Q_{n,\nu-\tau}$, so that, except in special cases, $\lim_{n\to\infty} R_{n,\nu-\tau}$ will not exist, nor will it be possible to discuss the upper or lower limits.

One result, however, stands out as exceptional in its preciseness. At a boundary pole of highest order at $(1/\bar{\beta}_u)$,

$$\lim_{n\to\infty} \frac{R_{n,\nu-\tau}(1/\bar{\beta}_u)}{n^{k_u}} = \frac{\bar{B}_u}{k_u!}. \quad (18)$$

11. *The abnormal case.* Consider, first, a function for which there are l simple poles on the circle $|z\bar{\beta}| = 1$.

As explained in § 2, the first row of the Padé table, for which the circle $|z\bar{\beta}| = 1$ is the boundary, has a special significance in reference to the abnormal case. For this row $\tau = l-1$, and in it occur all the approximants $R_{n',\nu-l+1}(z)$ which are representative of each square of abnormality, when the nature of the function gives rise to abnormal cases.

For this reason the approximants $R_{n,\nu-l+1}(z)$ are discussed separately, and it must be remembered that the results hold whether abnormal cases exist or not. With the former, however, the sequence of increasing n's is not through successive integers. Although a special case of the preceding paragraph, the results can be stated with more exactness.

At an enclosed pole at $(1/\beta_r)$ there is no limit, but

$$\lim_{n\to\infty} \frac{|R_{n,\nu-l+1}(1/\beta_r)|}{|(\beta_r/\bar{\beta})|^n} \text{ is finite and non-zero.} \tag{19}$$

Also at a boundary pole at $(1/\bar{\beta}_t)$,

$$\lim_{n\to\infty} \frac{R_{n,\nu-l+1}(1/\bar{\beta}_t)}{n} = \bar{B}_t, \tag{20}$$

while at any regular point on the boundary $|z\beta| = 1$, $R_{n,\nu-l+1}(z)$ does not tend to a limiting value as $n \to \infty$. (21)

When one (or more) of the boundary poles is multiple, and their total multiplicity is l', the behaviour of the approximant $R_{n,\nu-l'+1}(z)$ through increasing values of n is of similar interest.

Two cases arise in the evaluation at an enclosed pole at $(1/\beta_r)$:

If there is a boundary pole at $(1/\bar{\beta}_u)$ of higher order than the others,

$$\lim_{n\to\infty} \frac{R_{n,\nu-l'+1}(1/\beta_r)}{n^{1-k_u}(\beta_r/\bar{\beta}_u)^n} \text{ is finite and non-zero.} \tag{22}$$†

If there is more than one boundary pole of higher order than the others there is no unique limit as in (22), but

$$\lim_{n\to\infty} \frac{|R_{n,\nu-l'+1}(1/\beta_r)|}{n^{1-k_u}|(\beta_r/\bar{\beta}_u)|^n} \text{ exists and is finite and non-zero.} \tag{23}$$†

At a boundary pole different cases arise, as follows:

If the pole at $(1/\bar{\beta}_u)$ is one of the highest order,

$$\lim_{n\to\infty} \frac{R_{n,\nu-l'+1}(1/\bar{\beta}_u)}{n^{k_u}} = \frac{\bar{B}_u}{k_u!}. \tag{24}$$

† Compare (9)** in the case for which $\tau = k-1$.

If the pole at $(1/\bar{\beta}_u)$ is of order one less than the highest, there is no unique limit as in (24), but

$$\overline{\lim_{n\to\infty}} \frac{|R_{n,\nu-l'+1}(1/\bar{\beta}_u)|}{n^{k_u}} \text{ is finite and non-zero.} \qquad (25)\dagger$$

If the pole at $(1/\bar{\beta}_u)$ is any other boundary pole, and there is :

(*a*) a single boundary pole of highest order at $(1/\bar{\beta}_v)$, then

$$\lim_{n\to\infty} \frac{R_{n,\nu-l+1}(1/\bar{\beta}_u)}{n^{k_v-1}(\bar{\beta}_v\bar{\beta}_u)^n} \text{ is finite and non-zero } (k_u < k_v-1); \quad (26)\dagger$$

(*b*) more than one boundary pole of highest order k_v, then there is no unique limit as in (26), but

$$\overline{\lim_{n\to\infty}} \frac{|R_{n,\nu-l+1}(1/\bar{\beta}_u)|}{n^{k_v-1}} \quad (k_u < k_v-1) \qquad (27)\dagger$$

is finite and non-zero.

At any regular point on the boundary $|z\beta|=1$, $R_{n,\nu-l+1}(z)$ does not tend to a limiting value as $n\to\infty$, nor is it bounded. (28)†

This last result displays the reason for the cases of failure to (14), which has reference to the continued fraction of the $(\nu+1)$-th row; whereas, with those functions possessing a cyclic distribution of equimultiple polar singularities and thus giving rise to the abnormal cases, the continued fraction for the $(\nu+1)$-th row necessarily reduces to that for the $(\nu-l+1)$-th row, and hence satisfies (28).

12. *Summary of results.* As on previous occasions, the behaviour of the sum-function of the Taylor expansion forms a basis of comparison for those results which relate to values of the continued fractions at the different boundary poles and at regular points on the boundary.

Consider, then, the Taylor expansion of a function for which the circle of convergence, $|z\bar{\beta}|=1$, passes through polar singularities of multiplicities $k_1, k_2, \ldots, k_l$ at $1/\bar{\beta}_1, 1/\bar{\beta}_2, \ldots, 1/\bar{\beta}_l$ respectively, of total multiplicity l', there being no other singularities on the circle. The behaviour, as $n\to\infty$, of

$$S_n(z) \equiv c_0+c_1z+c_2z^2+\ldots+c_nz^n,$$

† In the evaluation, contributions from terms as in (5)** are included, since their effect cannot be ignored in these results.

will now be discussed at the different boundary poles and at regular points on the boundary.

The following results can be obtained independently, but they are here quoted as special cases of (24)–(28).

(*a*) If the pole at $(1/\bar{\beta}_u)$ is one of the highest order,

$$\lim_{n\to\infty} \frac{S_n(1/\bar{\beta}_u)}{n^{k_u}} = \frac{\bar{B}_u}{k_u!}.$$

(*b*) If the pole at $(1/\bar{\beta}_u)$ is of order one less than the highest, $|S_n(1/\bar{\beta}_u)|/n^{k_u}$ does not tend to a limit as $n\to\infty$, but its upper limit exists and is finite and non-zero.

At any other boundary pole at $(1/\bar{\beta}_u)$:

(*c*) If the pole at $(1/\bar{\beta}_v)$ is of higher order than the others,

$$\lim_{n\to\infty} \frac{S_n(1/\bar{\beta}_u)}{n^{k_v-1}(\bar{\beta}_v/\bar{\beta}_u)^n} = \text{constant} \quad (k_u < k_v - 1).$$

(*d*) If there is more than one boundary pole of highest order k_v, $|S_n(1/\bar{\beta}_u)|/n^{k_v-1}$ does not tend to a limit as $n\to\infty$, but its upper limit exists and is finite and non-zero.

(*e*) At any other point of $|z\bar{\beta}| = 1$, $S_n(z)$ does not tend to a limit as $n\to\infty$, and if one or more poles are multiple it is not even bounded.

The source of these results shows that they are generalized without loss of character in the case of the continued fraction of the first row of the sequence of l' rows for which $|z\bar{\beta}| = 1$ is the bounding circle for the corresponding continued fractions, (24)–(27). These last are more general than results (*a*)–(*d*), since the circle $|z\bar{\beta}| = 1$ may now contain any finite number of polar singularities of the function.

Passing to the values obtained for the continued fractions of the last of the l' rows, we again establish a complete parallel between (*a*)–(*d*) and (9)–(12).

The most general case, viz. that of the continued fraction of any of the l' rows, is dealt with in § 10, and we state but one definite result: the value of a boundary pole of the highest order is given by

$$\lim_{n\to\infty} \frac{R_{n,\nu-\tau}(1/\bar{\beta}_u)}{n^{k_u}} = \frac{B_u}{k_u!} \quad (\tau = 1, 2, \ldots, l').$$

This result is fundamental, and, as in the second paper, it persists in the most general cases, not only in character, but also in detail.

In considering the behaviour of the continued fractions at an enclosed pole, results are obtained which have no counterpart in those arising from the Taylor expansion of a function.

The nature of the divergence of the continued fraction of the last of the l' rows at an enclosed pole at $(1/\beta_r)$ is given by the following :

If the boundary pole at $(1/\beta_u)$ is of higher order than the others

$$(f) \qquad \lim_{n\to\infty} \frac{R_{n,\nu}(1/\beta_r)}{n^{k_u-1}(\beta_r/\bar{\beta}_u)^n} = C_u, \text{ as in (7).}$$

If there are two or more poles of highest order k_u, no unique limit exists, but

$$(g) \qquad \lim_{n\to\infty} \frac{|R_{n,\nu}(1/\beta_r)|}{n^{k_u-1}|(\beta_r/\bar{\beta}_u)|} \text{ is finite and non-zero, as in (8).}$$

For the continued fraction of the first of the l' rows, the corresponding result to (f) is

$$\lim_{n\to\infty} \frac{R_{n,\nu-l'+1}(1/\beta_r)}{n^{1-k_u}(\beta_r/\bar{\beta}_u)^n} \text{ is finite and non-zero, as in (22),}$$

and corresponding to (g) no unique limit exists, but

$$\lim_{n\to\infty} \frac{|R_{n,\nu-l'+1}(1/\beta_r)|}{n^{1-k_u}|(\beta_r/\bar{\beta}_u)|^n} \text{ is finite and non-zero, as in (23).}$$

In comparison with the results obtained in the case of a single boundary pole, whether simple or multiple, those arising from several boundary poles are characterized by a variety in the nature of the divergence depending on the relative multiplicities of the different boundary poles. This is further manifested in the fact that the order of divergence has frequently to be expressed in terms of an upper limit, whereas the preciseness of the former results gives rise to unique limits.

In the previous papers the dominating effect of the boundary pole was largely in evidence. As the multiplicity of this pole increases, so does the order of divergence of the continued fraction at the enclosed and boundary poles. The same feature characterizes the effect of several boundary poles, and in an even more remarkable manner, since it is the *dominant* boundary pole or poles which determine the order of divergence in every case, the effects of the boundary poles of lower multiplicity being

negligible in comparison. This is particularly noticeable when the order of divergence of the continued fraction is considered at the boundary poles themselves. As is seen from (11), (12) and (26), (27), at a boundary pole of lower multiplicity, the order of divergence is completely determined by the multiplicity of the dominant boundary pole or poles.

The behaviour of the continued fractions at regular points on the boundary gains interest from the fact that, when all the boundary poles are simple, the continued fraction is indeterminate at such points, whereas, when the boundary poles are multiple, the continued fraction for the last of the l' rows converges to the value of the function at regular points on the boundary. The existence of a single multiple boundary pole is sufficient to ensure this last result. On the other hand, the continued fraction for the first of the l' rows is indeterminate at all regular points on the boundary, and is even unbounded unless all the boundary poles are simple. These results suggest a parallel with those given at the end of § 9** in the case of the continued fractions of the successive rows for which the bounding circle was determined by a single multiple pole.

ON THE CONVERSE OF CAUCHY'S THEOREM

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1. The following converse of Cauchy's theorem is due to Morera :

If, in a domain D, the function $f(z)$ is single-valued, continuous in z and such that

$$\int_C f(z)\,dz = 0 \tag{1.1}$$

for all closed rectifiable paths of integration C in D, then $f(z)$ is regular in D.

A domain is, as usual, a connected open set. Any two points of D can be joined by a path consisting of a finite number of segments lying entirely in D, and, if ζ is any point of D, then there is a corresponding positive number σ such that every z satisfying

$$|z-\zeta| < \sigma \tag{1.2}$$

also belongs to D.

If required, the path joining any two points of D can be made up of segments parallel to one or other of the axes, so that it is what is described as a "stepped" path.

By saying that $f(z)$ is regular in D is meant here that, if ζ is any point of D and σ is as above, then there are coefficients γ_n such that

$$f(z) = \sum_{n=0}^{\infty} \gamma_n (z-\zeta)^n \tag{1.3}$$

for all the points z given by (1.2).

2. The theorem of § 1 is an immediate consequence of Cauchy's theorem. For, if c is any point of D and

$$F(z) = \int_c^z f(u)\,du,$$

then (1 . 1) shows that $F(z)$ is single-valued. Since $f(z)$ is continuous in z,

$$F'(z) = f(z)$$

and, by Cauchy's theorem and its immediate consequences, $F(z)$, and therefore its differential coefficient $f(z)$, is regular.

3. It is the object of this paper to consider the converse of Cauchy's theorem in a somewhat different way. It will be seen that the theorem and its converse form a single direct theorem with a hypothesis that falls into two parts.

If $$x = x+iy, \quad f(z) = p(x, y)+iq(x, y),$$

where x, y, $p(x, y)$, $q(x, y)$ are real, then this theorem states that $f(z)$ is regular in D if certain conditions relating to the continuity of $f(z)$ are satisfied and either

$$\frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x}, \quad \frac{\partial p}{\partial x} = \frac{\partial q}{\partial y}, \tag{3 . 1}$$

or

$$\int_C p dx - q dy = 0, \quad \int_C q dx + p dy = 0 \tag{3 . 2}$$

for all closed rectifiable paths of integration C in D.

Thus Cauchy's theorem states in effect that $f(z)$ is regular if certain differential equations are satisfied, and the converse states that $f(z)$ is regular if certain integral equations are satisfied.

4. It is known that in some cases integral equations are more readily solved than differential equations, and advantage has been taken of this to facilitate the solution of certain differential equations by replacing them by a corresponding set of integral equations. From what follows it will be seen that the equations (3 . 1) can be discussed very conveniently by reducing them to a set of integral equations of the form (3 . 2). In other words, the converse of Cauchy's theorem can be substituted for the direct theorem and used as an alternative approach to the subsequent theory.

5. It is convenient to begin with a special case in which not all paths of integration in D are considered, but only those which are composed of radial lines through a given point, together with an arc of a circle with the point as centre. In this case the method so often employed of solving equations by means of Fourier series leads very quickly to a solution.

Let ζ be the given point and let r, θ be respectively the modulus and amplitude of $z-\zeta$. Further, let

$$f(z) = \lambda(r, \theta)+i\mu(r, \theta), \tag{5.1}$$

where $\lambda(r, \theta)$ and $\mu(r, \theta)$ are real functions.

Take any positive number ρ less than σ, and denote by A, P, Q the respective points ζ, $\zeta+\rho$, $\zeta+\rho e^{i\omega}$, where $0<\omega\leqslant 2\pi$. Then, if C is the path $APQA$, (1.1) gives

$$\int_{AQ} f(z)\,dz-\int_{AP} f(z)\,dz = \int_{PQ} f(z)\,dz. \tag{5.2}$$

Write now

$$\lambda(r, \theta)\cos\theta-\mu(r, \theta)\sin\theta = \phi(r, \theta),$$

$$\lambda(r, \theta)\sin\theta+\mu(r, \theta)\cos\theta = \psi(r, \theta).$$

Then (5.2) becomes, on separation of its real and imaginary parts,

$$\begin{cases} \int_0^\rho \{\phi(r, \omega)-\phi(r, 0)\}\,dr = -\rho\int_0^\omega \psi(\rho, \theta)\,d\theta, \\ \int_0^\rho \{\psi(r, \omega)-\psi(r, 0)\}\,dr = \rho\int_0^\omega \phi(\rho, \theta)\,d\theta. \end{cases} \tag{5.3}$$

6. To make matters definite let the sign of integration be interpreted in the general Denjoy sense. Then, for the paths just described, (1.1) requires $\phi(r, \theta)$ and $\psi(r, \theta)$ to be integrable in this sense, within D, along the radial lines through ζ, and round the circles with ζ as centre.

In particular, for $0<r<\sigma$, $\phi(r, \theta)$ and $\psi(r, \theta)$ must be integrable functions of θ in $(0, 2\pi)$, and so they have Fourier series:

$$\begin{cases} \phi(r, \theta) \sim \sum_{n=1}^{\infty} \{a_n(r)\cos n\theta+b_n(r)\sin n\theta\}, \\ \psi(r, \theta) \sim \sum_{n=1}^{\infty} \{c_n(r)\cos n\theta+d_n(r)\sin n\theta\}. \end{cases} \tag{6.1}$$

In these series there are no constant terms. For, if C is the circle with centre ζ and radius r, then (1.1) gives

$$-r\int_0^{2\pi} \psi(r, \theta)\,d\theta = 0, \quad r\int_0^{2\pi} \phi(r, \theta)\,d\theta = 0,$$

and these show that the constant terms vanish.

Since Fourier series are integrable term by term*,

$$(6.2)\qquad \begin{cases} \int_0^\omega \phi(r,\theta)\,d\theta \sim \sum_{n=1}^{\infty} \frac{1}{n}\Big[-b_n(r)\cos n\theta + a_n(r)\sin n\theta\Big]_0^\omega, \\ \int_0^\omega \psi(r,\theta)\,d\theta \sim \sum_{n=1}^{\infty} \frac{1}{n}\Big[-d_n(r)\cos n\theta + c_n(r)\sin n\theta\Big]_0^\omega. \end{cases}$$

On the other hand, a calculation of Fourier coefficients shows that

$$\int_0^\rho \phi(r,\theta)\,dr \sim \sum_{n=1}^{\infty} \{A_n(\rho)\cos n\theta + B_n(\rho)\sin n\theta\},$$

$$\int_0^\rho \psi(r,\theta)\,dr \sim \sum_{n=1}^{\infty} \{C_n(\rho)\cos n\theta + D_n(\rho)\sin n\theta\},$$

where

$$A_n(\rho) = \int_0^\rho a_n(r)\,dr, \quad B_n(\rho) = \int_0^\rho b_n(r)\,dr,$$

$$C_n(\rho) = \int_0^\rho c_n(r)\,dr, \quad D_n(\rho) = \int_0^\rho d_n(r)\,dr,$$

provided that these integrals exist and that

$$(6.3)\qquad \begin{cases} \int_0^{2\pi} d\theta \cos n\theta \int_0^\rho \phi(r,\theta)\,dr = \int_0^\rho dr \int_0^{2\pi} \phi(r,\theta)\cos n\theta\,d\theta, \\ \int_0^{2\pi} d\theta \sin n\theta \int_0^\rho \phi(r,\theta)\,dr = \int_0^\rho dr \int_0^{2\pi} \phi(r,\theta)\sin n\theta\,d\theta, \\ \int_0^{2\pi} d\theta \cos n\theta \int_0^\rho \psi(r,\theta)\,dr = \int_0^\rho dr \int_0^{2\pi} \psi(r,\theta)\cos n\theta\,d\theta, \\ \int_0^{2\pi} d\theta \sin n\theta \int_0^\rho \psi(r,\theta)\,dr = \int_0^\rho dr \int_0^{2\pi} \psi(r,\theta)\sin n\theta\,d\theta. \end{cases}$$

If (6.3) holds, since a function cannot have two distinct Fourier series, it follows from (5.3) that

$$(6.4)\qquad \begin{cases} A_n(\rho) = \frac{\rho d_n(\rho)}{n}, \quad B_n(\rho) = -\frac{\rho c_n(\rho)}{n}, \\ C_n(\rho) = -\frac{\rho b_n(\rho)}{n}, \quad D_n(\rho) = \frac{\rho a_n(\rho)}{n}, \end{cases}$$

for $n \geqq 1$ and $0 < \rho < \sigma$. These plainly also hold for $\rho = 0$.

* The series in (6.2) are summable $(C, 1)$ and the constant terms are the sums $(C, 1)$ of

$$\sum_{n=1}^{\infty} \frac{b_n(r)}{n}, \quad \sum_{n=1}^{\infty} \frac{d_n(r)}{n}$$

respectively.

7. Equations (6 . 4) give, for $0 < \rho < \sigma$,

$$(7\,.\,1)\qquad \begin{cases} a_n(\rho) = \dfrac{n}{\rho}\displaystyle\int_0^\rho d_n(r)\,dr, \quad d_n(\rho) = \dfrac{n}{\rho}\int_0^\rho a_n(r)\,dr, \\ b_n(\rho) = -\dfrac{n}{\rho}\displaystyle\int_0^\rho c_n(r)\,dr, \quad c_n(\rho) = -\dfrac{n}{\rho}\int_0^\rho b_n(r)\,dr, \end{cases}$$

and show that the functions $a_n(r)$, $b_n(r)$, $c_n(r)$, $d_n(r)$ are all continuous save possibly at $r = 0$.

It follows from (7 . 1) that each of the four functions satisfies the equation

$$h(\rho) = \frac{n}{\rho}\int_0^\rho dr\,\frac{n}{r}\int_0^r h(s)\,ds,$$

and, therefore, since each is continuous,

$$\left(\frac{d}{dr}\,\frac{r}{n}\right)\left(\frac{d}{dr}\,\frac{r}{n}\right)h(r) = h(r),$$

the complete primitive of which, in the linear domain $r > 0$, is

$$h(r) = Kr^{n-1} + Lr^{-(n+1)},$$

where K and L are constants.

To determine the constants substitute in the original integral equations (7 . 1). It appears that in all cases $L = 0$, since otherwise the integrals concerned would not exist. Further, K is the same for $a_n(r)$ and $d_n(r)$, while the values of K for $b_n(r)$ and $c_n(r)$ are equal but opposite in sign. Thus

$$a_n(r) = a_{n-1}\,r^{n-1}, \qquad b_n(r) = b_{n-1}\,r^{n-1},$$
$$c_n(r) = -b_{n-1}\,r^{n-1}, \quad d_n(r) = a_{n-1}\,r^{n-1},$$

where a_{n-1}, b_{n-1} do not depend on r; and

$$\phi(r, \theta) \sim \sum_{n=1}^{\infty} (a_{n-1}\,r^{n-1}\cos n\theta + b_{n-1}\,r^{n-1}\sin n\theta),$$

$$\psi(r, \theta) \sim \sum_{n=1}^{\infty} (-b_{n-1}\,r^{n-1}\cos n\theta + a_{n-1}\,r^{n-1}\sin n\theta)$$

for $0 < r < \sigma$.

From these it follows that

$$(7\,.\,2)\qquad \begin{cases} \lambda(r, \theta) \sim \displaystyle\sum_{n=0}^{\infty} (a_n r^n \cos n\theta + b_n r^n \sin n\theta), \\ \mu(r, \theta) \sim \displaystyle\sum_{n=0}^{\infty} (-b_n r^n \cos n\theta + a_n r^n \sin n\theta), \end{cases}$$

for $0 < r < \sigma$.

If ρ is as before, by a known property of Fourier series of functions which are integrable in Denjoy's sense,

$$a_{n-1}\rho^{n-1}, \quad b_{n-1}\rho^{n-1} = o(n). \tag{7.3}$$

Thus the series in (7.2) are absolutely convergent for $r < \rho$ and so for $r < \sigma$.

A term by term integration gives

$$\int_0^\theta \lambda(r, t)\,dt = \sum_{n=0}^{\infty} \frac{r^n}{n}\Big[a_n \sin nt - b_n \cos nt\Big]_0^\theta,$$

$$\int_0^\theta \mu(r, t)\,dt = \sum_{n=0}^{\infty} \frac{r^n}{n}\Big[-b_n \sin nt - a_n \cos nt\Big]_0^\theta,$$

the qualifying $(C, 1)$ being omitted, since the series are known to be convergent. Further, in view of the uniform convergence given by (7.3), the series just obtained can be differentiated term by term. Thus

$$\frac{d}{d\theta}\int_0^\theta \lambda(r, t)\,dt = \sum_{n=0}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta),$$

$$\frac{d}{d\theta}\int_0^\theta \mu(r, t)\,dt = \sum_{n=0}^{\infty} r^n(-b_n \cos n\theta + a_n \sin n\theta),$$

and the correspondences (7.2) have been replaced by equalities.

Thus, provided that $\lambda(r, \theta)$ and $\mu(r, \theta)$ are θ-differential coefficients,

$$\lambda(r, \theta) = \sum_{n=0}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta),$$

$$\mu(r, \theta) = \sum_{n=0}^{\infty} r^n(-b_n \cos n\theta + a_n \sin n\theta),$$

for $0 < r < \sigma$, *i.e.*

$$f(z) = \sum_{n=0}^{\infty} (a_n - ib_n)(z-\zeta)^n,$$

for $0 < |z-\zeta| < \sigma$.

If, further, $f(z)$ is continuous in r at $r = 0$, then the above expansion plainly holds also at $z = \zeta$, and so throughout the domain (1.2).

8. The following result has now been obtained:

If, in the domain (1.2), $f(z)$ *is single-valued and integrable in the general Denjoy sense along the radial lines through* ζ *and round the circles with* ζ *as centre, and* (1.1) *holds for every closed path* C *in* (1.2) *composed of parts of the radial lines and arcs of the circles, and the re-*

peated integrals in (6 . 3) *exist and are subject to inversion of the order of integration, then, provided that* $f(z)$ *is a* θ*-differential coefficient for* $0<|z-\zeta|<\sigma$ *and is continuous in* $|z|$ *at* ζ, *there is an expansion* (1 . 3) *valid in the domain considered.*

It may be noted that the special condition relating to $f(z)$ and θ is satisfied if $f(z)$ is continuous in θ in $0<|z-\zeta|<\sigma$.

9. An elementary sufficient condition for (6 . 3) is that $\phi(r, \theta)$ and $\psi(r, \theta)$ are absolutely integrable with respect to (r, θ) over the range

$$0 \leqslant r \leqslant \rho, \quad 0 \leqslant \theta \leqslant 2\pi,$$

for all positive values of ρ less than σ. A special case of this condition is that in which $f(z)$ is bounded in the range considered.

10. A sufficient condition which has a bearing on the direct theorem is that in which $\phi(r, \theta)$ and $\psi(r, \theta)$ are finite and each both r- and θ-differential coefficients, or, what comes to the same thing, $f(z)$ is finite and both an r- and a θ-differential coefficient, and

$$F(z) = \int_{\zeta}^{z} f(u)\, du,$$

the path of integration being the radial line from ζ to z, is continuous in z.

Write $$F(z) = \Lambda(r, \theta) + i\mathrm{M}(r, \theta),$$

so that

$$\Lambda(r, \theta) = \int_0^r \phi(u, \theta)\, du, \quad \mathrm{M}(r, \theta) = \int_0^r \psi(u, \theta)\, du. \tag{10 . 1}$$

In virtue of (5 . 3),

$$\begin{cases} \Lambda(r, \theta) = \int_0^r \phi(u, 0)\, du - r\int_0^\theta \psi(r, t)\, dt, \\ \mathrm{M}(r, \theta) = \int_0^r \psi(u, 0)\, du + r\int_0^\theta \phi(r, t)\, dt. \end{cases} \tag{10 . 2}$$

From (10 . 1) and (10 . 2), since $\phi(r, \theta)$ and $\psi(r, \theta)$ are each both r- and θ-differential coefficients,

$$\begin{cases} \dfrac{\partial \Lambda}{\partial r} = \phi, & \dfrac{\partial \Lambda}{\partial \theta} = -r\psi, \\ \dfrac{\partial \mathrm{M}}{\partial r} = \psi, & \dfrac{\partial \mathrm{M}}{\partial \theta} = +r\phi. \end{cases} \tag{10 . 3}$$

These show that $\phi(r, \theta)$ is the r-differential coefficient of a function which is continuous in (r, θ) and also, when $r \neq 0$, the θ-differential coefficient of another such function. It is easily verified that the same is true of $\phi(r, \theta)\cos n\theta$, so that, by a fundamental result due to Looman*,

$$\int_\epsilon^\rho dr \int_0^{2\pi} \phi(r, \theta)\cos n\theta\, d\theta = \int_0^{2\pi} d\theta \int_\epsilon^\rho \phi(r, \theta)\cos n\theta\, dr,$$

for $0 < \epsilon \leqslant \rho$.

Now, by (10 . 3),

$$\int_\epsilon^\rho \phi(r, \theta)\cos n\theta\, dr = \{\Lambda(\rho, \theta) - \Lambda(\epsilon, \theta)\}\cos n\theta$$

and is thus continuous in (ϵ, θ). It follows that

$$\lim_{\epsilon \to 0} \int_0^{2\pi} \{\Lambda(\rho, \theta) - \Lambda(\epsilon, \theta)\}\cos n\theta\, d\theta = \int_0^{2\pi} \lim_{\epsilon \to 0} \{\Lambda(\rho, \theta) - \Lambda(\epsilon, \theta)\}\cos n\theta,$$

and (10 . 4) yields

$$\int_0^\rho dr \int_0^{2\pi} \phi(r, \theta)\cos n\theta\, d\theta = \int_0^{2\pi} d\theta \int_0^\rho \phi(r, \theta)\cos n\theta\, dr.$$

A similar argument yields the remaining three results of (6 . 3), and this is all that is required.

A particular case is that in which $f(z)$ is continuous in the separate variables r and θ, and is such that $F(z)$ is continuous in z.

11. If, in addition to (1 . 1),

$$\int_C zf(z)\, dz = 0 \tag{11 . 1}$$

for the particular paths considered, then the condition relating to the continuity in z of $F(z)$ can be replaced by the condition that

$$F_1(z) = \int_\zeta^z F(u)\, du$$

is continuous in z.

For (11 . 1) yields, after integration by parts,

$$[zF(z)]_C - \int_C F(z)\, dz = 0,$$

* 2, 99.

i.e. since $F(z)$ is single-valued,

$$\int_C F(z)\,dz = 0.$$

Now $F(z)$, as an integral, is continuous in r and θ separately. By the above it must be regular, and the same is true of $f(z)$, its r-differential coefficient.

12. In terms of rectangular coordinates x, y there is the following converse of Cauchy's theorem:

If, in a domain D, $f(z)$ is single-valued, finite, and is both an x- and a y-differential coefficient,

$$\int_R f(z)\,dz = 0, \tag{12.1}$$

for all rectangles R in D, and

$$F(z) = \int_c^z f(u)\,du, \tag{12.2}$$

where c is a point of D, is continuous in z, then $f(z)$ is regular in D.

It is to be understood that the path of integration from c to z is stepped. The existence of the integral (12.2) is then a consequence of Denjoy's theorem to the effect that every finite differential coefficient is integrable.

In virtue of (12.1), $F(z)$ is single-valued.

If
$$F(z) = P(x, y)+iQ(x, y),$$

then, by hypothesis, $P(x, y)$ and $Q(x, y)$ are continuous in (x, y).

Now, if $c = a+ib$,

$$P(x, y) = \int_{(a, b)}^{(x, y)} p\,dx - q\,dy, \quad Q(x, y) = \int_{(a, b)}^{(x, y)} q\,dx + p\,dy, \tag{12.3}$$

the path of integration being in each case a stepped path.

Since $f(z)$ is both an x- and a y-differential coefficient,

$$\frac{\partial P}{\partial x} = p, \quad \frac{\partial P}{\partial y} = -q,$$

$$\frac{\partial Q}{\partial x} = q, \quad \frac{\partial Q}{\partial y} = p,$$

which show that $P(x, y)$ and $Q(x, y)$ satisfy the Cauchy-Riemann differ-

ential equations

$$\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}.$$

By Looman's theorem it now follows that

$$\int_R P dx - Q dy = 0, \quad \int_R Q dx + P dy = 0,$$

for every rectangle R in D, so that

$$\int_R F(z)\, dz = 0$$

for every rectangle R in D.

But $F(z)$ is continuous in z, and therefore, since an arbitrary path can be approximated to by a stepped path,

$$\int_C F(z)\, dz = 0$$

for every closed rectifiable path in D.

Take any point ζ of D and identify C with one of the paths considered in § 5. It follows from § 9 that $F(z)$ is regular in D.

Since $$F'(z) = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} = p + iq = f(z),$$

$f(z)$ is regular in D.

13. As in § 11, the condition of continuity of $F(z)$ may be replaced by an alternative, namely

$$\int_R z f(z)\, dz = 0,$$

together with the continuity in Z of

$$F_1(z) = \int_c^z dv \int_c^v f(u)\, du, \tag{13.1}$$

the paths of integration being in all cases stepped.

14. Let ξ, η be curvilinear coordinates such that there is a (1, 1) correspondence with non-vanishing Jacobian between a domain D in the (x, y)-plane and a domain Δ in the (ξ, η)-plane. Further, let x and y be twice differentiable functions of (ξ, η).

This makes

$$\frac{\partial x}{\partial \xi}, \quad \frac{\partial x}{\partial \eta}, \quad \frac{\partial y}{\partial \xi}, \quad \frac{\partial y}{\partial \eta} \tag{14.1}$$

all continuous, so that to each rectifiable path which, together with its end points, lies in Δ there corresponds a rectifiable path in D. Also, since the Jacobian does not vanish, the same is true with D and Δ interchanged. Finally, since the differential coefficients (14.1) are differentiable,

$$\frac{\partial^2 x}{\partial\xi\,\partial\eta} = \frac{\partial^2 x}{\partial\eta\,\partial\xi}, \quad \frac{\partial^2 y}{\partial\xi\,\partial\eta} = \frac{\partial^2 y}{\partial\eta\,\partial\xi}. \tag{14.2}$$

15. Write

$$f(z) = p(x, y)+iq(x, y) = \lambda(\xi, \eta)+i\mu(\xi, \eta),$$

where $p(x, y)$, $q(x, y)$, $\lambda(\xi, \eta)$, $\mu(\xi, \eta)$ are all real, and suppose that $\lambda(\xi, \eta)$ and $\mu(\xi, \eta)$ are continuous in the separate variables ξ and η.

Consider now paths of integration in D composed of a finite number of arcs $\xi = \text{const.}$, $\eta = \text{const.}$, and assume that (1.1) holds when C is made up in this way, or, what is equivalent, that (1.1) holds whenever C is the image of a rectangle R in Δ.

Define $F(z)$ by (12.2), the path of integration being as above, so that, in virtue of (1.1), $F(z)$ is single-valued.

Write

$$F(z) = P(x, y)+iQ(x, y) = \Lambda(\xi, \eta)+i\mathrm{M}(\xi, \eta),$$

and let $\gamma = a+i\beta$ be the point in Δ whose image in D is $c = a+ib$.

In the equations (12.3) the variables can be changed. For the paths considered can be divided into a finite number of arcs, in each of which either ξ alone varies or η alone varies, and for each of these arcs the required change is then justified as a change of variables in an ordinary integral. Thus

$$\Lambda(\xi, \eta) = \int_{(a, \beta)}^{(\xi, \eta)} \left(\lambda \frac{\partial x}{\partial\xi} - \mu \frac{\partial y}{\partial\xi}\right) d\xi + \left(\lambda \frac{\partial x}{\partial\eta} - \mu \frac{\partial y}{\partial\eta}\right) d\eta,$$

$$\mathrm{M}(\xi, \eta) = \int_{(a, \beta)}^{(\xi, \eta)} \left(\mu \frac{\partial x}{\partial\xi} + \lambda \frac{\partial y}{\partial\xi}\right) d\xi + \left(\mu \frac{\partial x}{\partial\eta} + \lambda \frac{\partial y}{\partial\eta}\right) d\eta.$$

Since $\lambda(\xi, \eta)$, $\mu(\xi, \eta)$, and the differential coefficients (14.1) are continuous in the separate variables ξ and η, it follows, by the usual formula for differentiating the integral of a continuous function, that

$$\begin{cases} \dfrac{\partial\Lambda}{\partial\xi} = \lambda \dfrac{\partial x}{\partial\xi} - \mu \dfrac{\partial y}{\partial\xi}, & \dfrac{\partial\Lambda}{\partial\eta} = \lambda \dfrac{\partial x}{\partial\eta} - \mu \dfrac{\partial y}{\partial\eta}, \\ \dfrac{\partial\mathrm{M}}{\partial\xi} = \mu \dfrac{\partial x}{\partial\xi} + \lambda \dfrac{\partial y}{\partial\xi}, & \dfrac{\partial\mathrm{M}}{\partial\eta} = \mu \dfrac{\partial x}{\partial\eta} + \lambda \dfrac{\partial y}{\partial\eta}. \end{cases} \tag{15.1}$$

Now the real and imaginary parts of

$$\int_C F(z)\,dz, \tag{15.2}$$

where C is the image of a rectangle, are respectively

$$\int_R \left(\Lambda\frac{\partial x}{\partial\xi}-\mathrm{M}\frac{\partial y}{\partial\xi}\right)d\xi+\left(\Lambda\frac{\partial x}{\partial\eta}-\mathrm{M}\frac{\partial y}{\partial\eta}\right)d\eta \tag{15.3}$$

$$\int_R \left(\mathrm{M}\frac{\partial x}{\partial\xi}+\Lambda\frac{\partial y}{\partial\xi}\right)d\xi+\left(\mathrm{M}\frac{\partial x}{\partial\eta}+\Lambda\frac{\partial y}{\partial\eta}\right)d\eta. \tag{15.4}$$

These can both be shown to vanish.

For, as is easily verified, by (15.1),

$$\frac{\partial}{\partial\eta}\left(\Lambda\frac{\partial x}{\partial\xi}-\mathrm{M}\frac{\partial y}{\partial\xi}\right)=\Lambda\frac{\partial^2 x}{\partial\eta\,\partial\xi}-\mathrm{M}\frac{\partial^2 y}{\partial\eta\,\partial\xi}+\lambda\left(\frac{\partial x}{\partial\xi}\frac{\partial x}{\partial\eta}-\frac{\partial y}{\partial\xi}\frac{\partial y}{\partial\eta}\right)$$
$$-\mu\left(\frac{\partial x}{\partial\xi}\frac{\partial y}{\partial\eta}+\frac{\partial x}{\partial\eta}\frac{\partial y}{\partial\xi}\right),$$

$$\frac{\partial}{\partial\xi}\left(\Lambda\frac{\partial x}{\partial\eta}-\mathrm{M}\frac{\partial y}{\partial\eta}\right)=\Lambda\frac{\partial^2 x}{\partial\xi\,\partial\eta}-\mathrm{M}\frac{\partial^2 y}{\partial\xi\,\partial\eta}+\lambda\left(\frac{\partial x}{\partial\xi}\frac{\partial x}{\partial\eta}-\frac{\partial y}{\partial\xi}\frac{\partial y}{\partial\eta}\right)$$
$$-\mu\left(\frac{\partial x}{\partial\xi}\frac{\partial y}{\partial\eta}+\frac{\partial x}{\partial\eta}\frac{\partial y}{\partial\xi}\right)$$

so that, in virtue of (14.2),

$$\frac{\partial}{\partial\xi}\left(\Lambda\frac{\partial x}{\partial\eta}-\mathrm{M}\frac{\partial y}{\partial\eta}\right)=\frac{\partial}{\partial\eta}\left(\Lambda\frac{\partial x}{\partial\xi}-\mathrm{M}\frac{\partial y}{\partial\xi}\right).$$

If, now, $F(z)$ is continuous in z, so that

$$\Lambda\frac{\partial x}{\partial\xi}-\mathrm{M}\frac{\partial y}{\partial\xi},\quad \Lambda\frac{\partial x}{\partial\eta}-\mathrm{M}\frac{\partial y}{\partial\eta} \tag{15.5}$$

are continuous in (ξ, η), it follows, by Looman's theorem, that (15.3) vanishes. A similar argument shows that the same is true of (15.4).

Since the functions (15.5) are continuous in (ξ, η), their integrals along any rectifiable path in Δ can be approximated to within any standard by integrals along stepped paths made up of lines parallel to one or other of the axes. Thus the vanishing of the integrals (15.3) and (15.4) implies the vanishing of the integrals round any closed rectifiable curve in Δ, and, in virtue of the remark at the beginning of the

section, the integral (15 . 2) must vanish for all closed rectifiable paths C in D. By Morera's theorem $F(z)$ is regular in D.

Now $f(z)$ is continuous in ξ, so that it is one of the derivatives of $F(z)$. But $F(z)$ is regular, and therefore it has only one derivative. Thus

$$f(z) = F'(z),$$

and this shows that $f(z)$ is regular in D.

16. As in § 11, the conditions relating to the continuity of $F(z)$ can be replaced by an alternative. If, in addition to (1 . 1), (11 . 1) holds for the particular paths considered, then the continuity in z of $F_1(z)$, as defined in (13 . 1), is sufficient.

17. The Cauchy-Riemann equations are a particular case of the following :

$$\frac{\partial(\lambda, x)}{\partial(\xi, \eta)} = \frac{\partial(\mu, y)}{\partial(\xi, \eta)}, \quad \frac{\partial(\lambda, y)}{\partial(\xi, \eta)} = -\frac{\partial(\mu, x)}{\partial(\xi, \eta)}, \tag{17 . 1}$$

which reduce to the Cauchy-Riemann equations (3 . 1) when $x = \xi$, $y = \eta$.

When $p(x, y)$ and $q(x, y)$ are differentiable functions of (x, y) and ξ, η are as described in § 14, it is possible to transform (17 . 1) into the Cauchy-Riemann equations by the ordinary methods for change of variables. If, however, $p(x, y)$ and $q(x, y)$ are not differentiable, this may be no longer the case. At a given point (x, y) the Cauchy-Riemann equations may hold, but not (17 . 1); or (17 . 1) may hold, but not the Cauchy-Riemann equations.

18. If ξ, η are as described, and the partial derivatives which occur in (17 . 1) exist and are finite, then (17 . 1), when supplemented in the usual way by continuity, is a sufficient condition for Cauchy's theorem.

For let Γ be a simple closed curve of bounded variation in the (x, y)-plane, let D be the domain which it encloses, and let Δ be the domain in the (ξ, η)-plane which corresponds to D. Suppose that $f(z)$ is continuous in z in D and on Γ relative to $\Gamma + D$, and that the equations (17 . 1) are satisfied in Δ, the partial derivatives mentioned being existent and finite. Then it will be shown that

$$\int_\Gamma f(z)\, dz = 0. \tag{18 . 1}$$

Let C be the image of any rectangle R in Δ. Then

$$(18.2)\qquad \int_C f(z)\,dz = \int_R \left(\lambda \frac{\partial x}{\partial \xi} - \mu \frac{\partial y}{\partial \xi}\right) d\xi + \left(\lambda \frac{\partial x}{\partial \eta} - \mu \frac{\partial y}{\partial \eta}\right) d\eta$$
$$+ i \int_R \left(\mu \frac{\partial x}{\partial \xi} + \lambda \frac{\partial y}{\partial \xi}\right) d\xi + \left(\mu \frac{\partial x}{\partial \eta} + \lambda \frac{\partial y}{\partial \eta}\right) d\eta.$$

Now

$$\frac{\partial}{\partial \xi}\left(\lambda \frac{\partial x}{\partial \eta} - \mu \frac{\partial y}{\partial \eta}\right) = \lambda \frac{\partial^2 x}{\partial \xi\, \partial \eta} - \mu \frac{\partial^2 y}{\partial \xi\, \partial \eta} + \frac{\partial \lambda}{\partial \xi}\frac{\partial x}{\partial \eta} - \frac{\partial \mu}{\partial \xi}\frac{\partial y}{\partial \eta},$$

$$\frac{\partial}{\partial \eta}\left(\lambda \frac{\partial x}{\partial \xi} - \mu \frac{\partial y}{\partial \xi}\right) = \lambda \frac{\partial^2 x}{\partial \eta\, \partial \xi} - \mu \frac{\partial^2 y}{\partial \eta\, \partial \xi} + \frac{\partial \lambda}{\partial \eta}\frac{\partial x}{\partial \xi} - \frac{\partial \mu}{\partial \eta}\frac{\partial y}{\partial \xi},$$

and it is easily seen that, in virtue of (14.2) and (17.1), these two expressions are the same.

Since $f(z)$ is continuous in z, $\lambda(\xi, \eta)$ and $\mu(\xi, \eta)$ are continuous in (ξ, η). The same is true of

$$\lambda \frac{\partial x}{\partial \eta} - \mu \frac{\partial y}{\partial \eta}, \quad \lambda \frac{\partial x}{\partial \xi} - \mu \frac{\partial y}{\partial \xi},$$

and it follows, by Looman's theorem, that

$$\int_R \left(\lambda \frac{\partial x}{\partial \xi} - \mu \frac{\partial y}{\partial \xi}\right) d\xi + \left(\lambda \frac{\partial x}{\partial \eta} - \mu \frac{\partial y}{\partial \eta}\right) d\eta = 0.$$

Similarly

$$\int_R \left(\mu \frac{\partial x}{\partial \xi} + \lambda \frac{\partial y}{\partial \xi}\right) d\xi + \left(\mu \frac{\partial x}{\partial \eta} + \lambda \frac{\partial y}{\partial \eta}\right) d\eta = 0.$$

These two equations, in virtue of (18.2), give (1.1) for the particular paths C considered.

Since $f(z)$ is continuous in z, it follows that (1.1) holds for all closed rectifiable paths in D. Since $f(z)$ is continuous on Γ, (17.2) now follows because Γ can be approximated to by a path C in D in such a way that the integral round C approximates to that round Γ.

Thus Cauchy's theorem is proved under the hypothesis (17.1) without any assumption of differentiability. It is assumed only that $f(z)$ is continuous in z, and has partial derivatives with respect to ξ and η which are everywhere finite and have real and imaginary parts which satisfy (17.1).

19. The distinction, when $p(x, y)$ and $q(x, y)$ are not differentiable, between the Cauchy-Riemann equations and (17.1) can be illustrated by

means of the following analysis. Take

$$p(x, y) = \frac{x^3-y^3}{x^2+y^2} \quad (x^2+y^2 \neq 0),$$

$$= 0 \qquad (x^2+y^2 = 0),$$

$$q(x, y) = \frac{x^3+y^3}{x^2+y^2} \quad (x^2+y^2 \neq 0),$$

$$= 0 \qquad (x^2+y^2 = 0).$$

Then it is easily verified that $p(x, y)$ and $q(x, y)$ are both continuous in (x, y), and have the origin as a point of non-differentiability.

At the origin

$$\frac{\partial p}{\partial x} = +1, \quad \frac{\partial p}{\partial y} = -1,$$

$$\frac{\partial q}{\partial x} = +1, \quad \frac{\partial q}{\partial y} = +1,$$

so that the Cauchy-Riemann equations are satisfied.

Now make the substitutions

$$x = a\xi+b\eta, \quad y = c\xi+d\eta,$$

where

(19.1) $\quad a^2+c^2 = 1, \quad b^2+d^2 = 1, \quad a^2 \neq d^2, \quad b^2 \neq c^2, \quad ad-bc \neq 0.$

Then

$$p(x, y) = \lambda(\xi, \eta) = \frac{(a\xi+b\eta)^3-(c\xi+d\eta)^3}{(a\xi+b\eta)^2+(c\xi+d\eta)^2},$$

$$q(x, y) = \mu(\xi, \eta) = \frac{(a\xi+b\eta)^3+(c\xi+d\eta)^3}{(a\xi+b\eta)^2+(c\xi+d\eta)^2},$$

and it is easily verified that, at the origin,

$$\frac{\partial \lambda}{\partial \xi} = a^3-c^3, \quad \frac{\partial \lambda}{\partial \eta} = b^3-d^3,$$

$$\frac{\partial \mu}{\partial \xi} = a^3+c^3, \quad \frac{\partial \mu}{\partial \eta} = b^3+d^3,$$

so that

$$\frac{\partial(\lambda, x)}{\partial(\xi, \eta)} = ab(a^2-b^2)+ad^3-bc^3, \quad \frac{\partial(\mu, y)}{\partial(\xi, \eta)} = cd(c^2-d^2)+a^3d-b^3c,$$

$$\frac{\partial(\lambda, y)}{\partial(\xi, \eta)} = a^3d-b^3c-cd(c^2-d^2), \quad \frac{\partial(\mu, x)}{\partial(\xi, \eta)} = ab(a^2-b^2)-ad^3+bc^3,$$

and (17 . 1) hold only if

$$ab(a^2-b^2)-cd(c^2-d^2)+\{bc(b^2-c^2)-ad(a^2-d^2)\} = 0,$$

$$ab(a^2-b^2)-cd(c^2-d^2)-\{bc(b^2-c^2)-ad(a^2-d^2)\} = 0,$$

which require $$bc(b^2-c^2)-ad(a^2-d^2) = 0.$$

Now, from (18 . 1),

$$b^2-c^2 = a^2-d^2 \neq 0,$$

so that $$bc = ad,$$

which contradicts (19 . 1). Thus (17 . 1) do not hold.

20. The example just given shows the advantage of considering the integral equations (3 . 2) rather than the differential equations (3 . 1). Under the hypothesis that $f(z)$ is continuous in z there is no simple means of proceeding from the differential equations referred to one set of variables to the corresponding equations referred to another set of variables. On the other hand, the corresponding change for the integral equations presents no difficulty.

21. An indefinite complex integral, unlike the corresponding real integral, is not necessarily continuous. For take

$$f(z) = \exp(-z^{-4}) \quad (z \neq 0),$$

$$= 0 \quad (z = 0),$$

and consider $$F(z) = \int_a^z f(u)\,du,$$

where a is an arbitrarily chosen finite number and the path of integration is any stepped path.

The function $f(z)$ has no residue at the origin, so that $F(z)$ is single-valued and (1 . 1) holds for all paths of the kind considered. Also, as is easily verified, $f(z)$ is continuous in the separate variables x and y at all finite points, the origin included.

If now $F(z)$ were continuous in z, then, by § 12, $f(z)$ would be regular, which is not the case.

References.

1. W. H. Young, *Proc. London Math. Soc.* (2), 10 (1912), 1–6.
2. H. Looman, *Göttinger Nachrichten*, 1923, 97–108.

SOME RESULTS FOR THE GEOMETRY OF LINE SYSTEMS IN SPACE OF FOUR DIMENSIONS†

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IN a recent paper‡ I have considered the problem of the geometry of line systems in higher space of arbitrary dimension, confining myself to defining a system of characteristics and to determining formulae for the characteristics of the system of intersection of two given systems of suitable dimensions. In the present note we consider some further points in connexion with line systems in space of four dimensions and with their systems of intersections.

1. The system of all lines in this space is of six dimensions, and we therefore have to deal with the following systems of lower dimension :

(i) the hypercomplex (∞^5),

(ii) the hypercongruence (∞^4),

(iii) the complex (∞^3),

(iv) the congruence (∞^2),

and finally (v) the scroll.

The first three of these are termed unrestricted systems, since in general at least a finite number of their lines respectively pass through an arbitrary point of space, though in the case of the complex this is not invariably the case. Complexes are in some respect analogous to congruences of ordinary space, and each is such that its lines (in general)

† This paper was communicated to the London Mathematical Society on the advice of Dr. J. W. L. Glaisher, Editor of the *Quarterly Journal of Mathematics*. Mr. James had communicated it to the *Quarterly Journal* before his death.—F. Bath.

‡ "Formulae for the intersection of line systems in higher space", *Proc. London Math. Soc.* (2), 24 (1925), 359.

are tritangent † to a certain form, the *focal form* ‡ ; while hypercomplexes are analogous to complexes of ordinary space.

The characteristics of these systems, according to the scheme in the paper cited (which we subsequently refer to as F), are as follows. The hypercomplex has a single characteristic : its degree k, or the degree of its equation in line coordinates, defined as the order of the three-dimensional cone of its lines through a fixed point. The lines in a space§ form a complex of the same degree, and so also its lines in a plane envelop a curve of the k-th class.

A *hypercongruence* has two main characteristics : its order m, namely the order of the two-dimensional cone of its lines through a point, and its class n, or the number of its lines in a plane. The system of its lines in a space is a congruence (m, n). We define, in addition, its rank as the rank of this congruence.

For *a complex* we have similarly (i) the order m, or the number of its lines through an arbitrary point ; (ii) the class n, or the order of the scroll of its lines in a space ; and (iii) the rank R, or the number of lines of the complex concurrent on a line and cospacial with a fixed plane through it ‖. We use ψ to denote the order of the double curve of the n-tic scroll of lines in a space, and other numbers are defined later (§ 7). *The complex projects into a complex of degree* n¶.

Of restricted systems the congruence only need be considered. The lines of such a system generate a form, and in general only one line passes through each point of the form. Suppose, however, that κ lines

† Segre, "Un' osservazione sui sistemi ...", *Rend. Circ. Mat. Palermo*, 2 (1888), 148.

‡ The term "form" is used in this paper to denote a construct of dimension $r-1$ in space $[r]$.

§ Used in this paper to denote a space of three dimensions.

‖ The first two of these are established definitions used in connexion with the several systems which have been previously considered by various authors.

¶ The theorem that the quadric is the only doubly ruled surface of ordinary space is essentially identical with the theorem that *the only complexes of space of four dimensions lying in a form are the systems of all lines on a quadric form or on a plane-scroll.* The theorem therefore does not extend to higher space. In fact, consider in a space $[r]$, of r dimensions, an m-tic form V^m_{r-1} (by hypothesis not generated by linear spaces), and its V_{r-i} of $(i+1)$-point-contact tangents at any point. It is seen at once that if V^m_{r-1} have a V^r_{r-2}, which is $(m-i)$-fold, then ∞^{r-2-i} of these tangents have $m+1$ intersections with the form and therefore lie in it, forming in all a system of dimension $2r-i-4$. The greatest system is then of dimension $2r-6$ (for $i=2$), and the values of i range from 2 to $4r-2$. Such forms exist for all values of m for $\nu=1$, and for some of the earlier values of m for higher values of ν. Generally, if the form have a V_{r-j}, which is $(m-i)$-fold $(j=2, 3, \ldots, r-2)$, where i ranges from unity to $r-j$ $(j>2)$, it contains a system of lines of dimension $2r-i-j-2$, of which at least a finite number pass through each point of the form.

In particular, there are an infinite number of forms in space of five dimensions which contain systems of four dimensions (therein analogous to congruences of ordinary space).

pass through each of its points, and let m/κ be the order of the form. Then m is, by definition, the order of the congruence, namely the number of its lines meeting a fixed line. Its class is defined as the number of its lines in a given space†.

The congruence is projected from a fixed point into a congruence of type (m, n), whose rank equals the class of the envelope of lines in a plane of the four-dimensional space which join intersections with the plane of rays cospacial with it, together with the number of rays concurrent on and cospacial with it (if any).

1.1. It is perhaps worth while pointing out that there are an unlimited number of congruences for which κ is greater than unity, contrary to analogy with scrolls in ordinary space. This is seen by considering (as in a previous note) a V_3^μ with a $(\mu-2)$-fold surface of order ν, for which we must have

$$\mu-1 \geqslant \nu(\mu-3).$$

Through each point of the form passes a quadric cone of principal tangents (3-point tangents), 2ν of which meet the multiple surface. Each meets V_3^μ in $\mu+1$ points and therefore lies in it, giving a congruence for which $\kappa = 2\nu$. For a quartic form we can have $\nu = 1$, 2, or 3 (though in the last case the form is plane-ruled), but for higher forms $\nu = 1$ only. Notice also that in the case of a multiple plane the number of lines through each of its points is $2(\mu-2)$, falling into $\mu-2$ pairs each corresponding to one of the sheets of the form. This is the normal (or general) configuration at the multiple surface in all cases, though overlapping between the various sets of lines is possible.

The value of κ cannot exceed six (being the number of lines having four-point contact at a general point of a form) except for systems lying in a space, quadric form, or plane-ruled form.

2. *On the system of intersection of two given systems.*

The dimension of the system common to two systems of dimensions i and j respectively is $i+j-2(r-2)$, and its characteristics are determined in terms of those of the original systems by means of the formulae

† The only congruences previously considered are those lying on cubic forms. Notice in this connexion that a given form may contain several congruences; for the general cubic form $\kappa = 6$, and we have a single congruence of order 18, but in particular cases the form contains several congruences with various values of m and κ. (Suitably counted $\Sigma m = 18$, $\Sigma\kappa = 6$.) The number κ has previously been taken as the order of the system. Cf. Segre, "Sulle varieta cubiche ...", *Mem. Torino* (2), 39 (1889).

of the paper cited. We quote only those which are really essential and from which the remainder are deducible by repetition, combined with simple degeneration arguments.

I. *The number of lines common to a hypercomplex of degree k and a scroll of order n is kn.*

II. *The number of lines common to a hypercongruence (M, N)† and a congruence (m, n) is $mM+nN$.*

III. *The number of lines common to two complexes of characteristics (m, n)* and *(m', n')* is $mm'+nn'$.

The results, though elementary, are given in tabular form for the sake of reference. In Table I we give the characteristics for the sub-systems of a given system meeting prescribed linear spaces. In the first column are stated the types of the space or spaces in question; p, p', etc., denoting planes, and l, l', etc., lines. The subsequent columns give the characteristics for (i) a hypercomplex of degree k, (ii) a hypercongruence (m, n), (iii) a complex (m, n), and finally (iv) a congruence (m, n). Inside the brackets in the symbols are given the order and class of the sub-system preceding the colon, or for a scroll the order only; while the symbols $(p, p', \ldots)^{\alpha}$, $(l, l', \ldots)^{\alpha}$ have the following meanings. The former denotes that the planes entering have, as the case may be, multiplicity α on a form, or cut a scroll in a curve of order α; while the second denotes that the lines concerned are of multiplicity α on the form or scroll in question. The exterior suffixes give the dimensions of the systems. For all the hypercongruences and complexes the rank is zero, and for the congruences $\kappa = 1$ (in general), though this is not assumed for the original systems.

TABLE I.

Linear space	(i)	(ii)	(iii)	(iv)
p	$\{k, k:\}_4$	$\{m, m+n:\}_3$	$\{m+n, n: p^m\}_2$	$\{m+n: p^m\}_1$
l	$\{k, k:\}_3$	$\{m+n, m: l^m\}_2$‡	$\{m+n: l^m\}_1$	—
p, p'	$\{k, 2k\}_3$	$\{2m+n, m+n: (p, p')^m\}_2$	$\{2m+n: (p, p')^{m+n}\}_1$	—

	(i)	(ii)
p, l	$\{2k, k: p^k, l^k\}_2$	$\{2m+n: p^{m+n}, l^m\}_1$
l, l'	$\{2k: (l, l')^k\}_1$	—
p, p', p''	$\{3k, 2k: (p, p', p'')^k\}_2$	$\{3m+2n: (p, p', p'')^{2m+n}\}_1$
$p, p'\ l$	$\{3k: (p, p')^{2k}, l^k\}_1$	—

† In all such symbols m, M, etc., refer to the orders, and n, N, etc., to the classes of the systems under consideration.

‡ This form has a residual double surface on which the line is h-fold, where h denotes the number of apparent generators of the cone of rays of the original system passing through a point. By considering an arbitrary space section the order of this surface is seen to be $R+h+\frac{1}{2}n(n-1)$.

In the second table we give formulae for the characteristics of the systems of intersection of two line systems, the results being so arranged that above each symbol is stated one of the systems, and on the extreme left of its own row the second system from which it is obtained by section. For convenience we have written the system $\begin{bmatrix} \mu \\ \nu \end{bmatrix}_i$ to denote a system of order μ, class ν, and dimension i.

TABLE II.

First system	$\{5\}$ (k')	$\{4\}$ (M', N')	$\{3\}$ (m', n')	$\{2\}$ (m, n)	$\{1\}$ (n)
$\{5\}$ (k)	$\begin{bmatrix} kk' \\ kk' \end{bmatrix}_4$	$\begin{bmatrix} kM' \\ k(M'+N') \end{bmatrix}_3$	$\begin{bmatrix} k(m'+n') \\ kn' \end{bmatrix}_2$	$[k(m+n)]_1$	$[kn]_0$
$\{4\}$ (M, N)	—	$\begin{bmatrix} MM'+MN'+M'N \\ MM'+NN' \end{bmatrix}_2$	$[m'M+n'M+n'N]_1$	$[mM+nN]_0$	—
$\{3\}$ (m, n)	—	—	$[mm'+nn']_0$	—	—

The characteristics of the system of intersection of three or more systems can be written down by repeated use of this Table.

3. Except for a hypercomplex of general character there are in all cases essential characteristics other than those already considered. Of these the most important are the rank of a complex, as already defined, and the orders of the loci of double points on the constructs generated by the restricted systems, it being noted that in the case when the form generated by a congruence contains other such systems we associate to each the order of the double surface (two-dimensionality) generated by the points of intersection of pairs of its own rays.

4. After these preliminaries we pass on to give some results for congruences of the four-dimensional space, of which the most important is

THEOREM I. *An arbitrary line of a congruence of general character of order m and class n meets $m-3$ other lines, or, in other words, meets the double surface of the form filled in $m-3$ points* †.

Through each point of the form F generated by the congruence passes a single line of the system (in virtue of the restriction to the general case). Before passing on we note the following theorem of Schubert ‡ required during the course of the proof.

LEMMA. *Given in a plane a system of ∞^2 point-pairs (A, B) such that an arbitrary point determines, as a point A, β points B, and, as a*

† This theorem corresponds to a well known theorem on scrolls in ordinary space, but the usual proof of this latter theorem does not readily generalize to the present case. On the other hand, it may also be proved by a method precisely analogous to that about to be given.

‡ "Die n-dimensionalen Verallgemeinungen der Anzahlen für die vielpunktig berührenden Tangenten einer punktallgemeiner Fläche m-ten Grades", *Math. Ann.*, 26 (1886), 52.

point B, α points A, while γ pairs lie on a given line, then the number of pairs for which A and B coincide, while their join passes through a given fixed point, is equal to $\alpha+\beta+\gamma$.

Let a fixed ray g of the congruence be taken and two spaces π, π' cutting in a plane ϖ; let π' cut g in Q. Let η be any one of the ∞^2 planes through g, cutting π' in a line a passing through Q and π in a line b, meeting the section of the form F in $m-1$ points L. The rays l of the congruence through these points L cut π' in $m-1$ points A which we associate with the lines a through Q. It is seen that if A lie on a, then either l meets g, or a meets the section of ϖ and F. Let QA meet ϖ in A', and let η (or a) meet the same plane in B'. If A lie on a, A' and B' coincide. There are ∞^2 pairs (A', B'), such that to B' correspond $m-1$ points A' (arising from the plane $\eta \equiv B'g$); while, given A', there are $m-1$ corresponding points B' arising from the intersections of $A'Q$ and the form F. Finally the number of pairs $A'B'$ lying on a line of ϖ equals the number of pairs (A, a) in a space through g, and there is one such for each of the $n-1$ rays of the congruence other than g in this space.

Among the coincidences are those which arise from the rays of the congruence meeting g (the number of which is sought), and the limiting joining line is unrestricted in direction so that each point counts once in the theorem of the lemma. The remainder are the points of the curve in which ϖ meets F. The class of the envelope of limiting joining lines is equal to the class of the envelope of spaces defined by g and the rays of the congruence meeting ϖ, the plane envelope being, in fact, the envelope of the sections of ϖ by these spaces. Its class is therefore equal to $m+n$.

By substituting these numbers in the theorem of the lemma, Theorem I is immediately proved.

4.1. *Extension of the theorem.* Suppose that g is an i-fold ray of the congruence and therefore an i-fold line of the form F. Then by an analogous method it is proved that the *i-fold ray meets $m-3i$ ordinary rays*. In fact, the correspondence (A', B') is now of indices $m-i$, $m-i$, while $n-i$ pairs lie on a given line of ϖ. *These points are i-ple points of the double surface*, which must be met by g in $3i-3$ other points (through which pass no other rays of the congruence)†.

† In the same way it can be formally proved that an i-ple generator of a scroll of ordinary space is met by $m-2i$ ordinary generators in i-ple points of the double curve, which is met again by the multiple generator in $2i-2$ simple points.

4.2. *Corollaries of Theorem* I.

(i) The principal (or three-point-contact) tangents to F at a point on a generator describe a quadric cone V_2^2 passing through g. Consider the section f of F by an arbitrary space through g. Its Hessian is of order $4(m-2)$ and is touched by g in those points where g meets it, which are simple on F (or f)†. Now a point on the double curve of f is triple on the Hessian, and its tangent cone contains the principal tangents in each sheet of f at the point. Thus g meets the Hessian four times in each of the $m-3$ points where it meets the double curve of f. Hence it is elsewhere bitangent to the Hessian, so that there exist two points at which g coincides with the other principal tangent to f. Now principal tangents of f are principal tangents of F, and hence, since two such planes lie in a space through g,

The principal tangents of F meeting g are such that the planes defined by g and neighbouring principal tangents form a quadric [1]*-cone*‡.

The following analogous result is the dual of a well known theorem§ in connexion with systems of ∞^2 planes in space of four dimensions.

The spaces defined by the ray g and neighbouring rays of the congruence envelop a quadric [1]*-cone.*

The spaces in question are evidently the tangent spaces to F along g, so that, in other words, *the class of immersion*|| *of any ray in the form F is equal to two*, a result which can be readily verified algebraically for the congruence on a cubic form. Further, the quadric cone enveloped is seen to be identical with the previous cone by considering the plane common to two neighbouring spaces of the envelope, since its residual section with F touches the line g.

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† Salmon-Rogers, *A treatise on the analytical geometry of three dimensions*, 2 (5th ed., London, 1915), 288, § 287.

‡ This symbol denotes a cone with a line vertex.

§ Cf. Segre, "Sui fochi di 2° ordine dei sistemi infiniti di piani", *Rend. Accad. Lincei*, 5 (1921), 30.

The theorem in its original form may be applied to the system of planes defined by the rays of the congruence which intersect. Its characteristics, dual to m, n respectively, are y, σ defined later; and these are also the characteristics of the congruence formed by cutting the system by a fixed space. *Each of the planes is met in a conic by the consecutive planes of the system*, and the planes are thus conic-tropes on a form of order $2y(\sigma-1)-2\rho$, where ρ is the rank of the last named congruence of lines of ordinary space. Each line of the original system is therefore $2(m-3)$-tangent to this form in the general case.

|| F. Severi, "Sulli intersezione delle varietà algebriche . . .", *Mem. Torino* (2), 52 (1908), 61.

(ii) Consider, further, an arbitrary plane ϖ through g, cutting F in a residual curve of order $m-1$. The intersections of g with the double surface account for $m-3$ of the intersections of this curve and g, leaving two to be accounted for by contacts of the plane and F. *Each plane through the generator g is therefore bitangent to the form F*, or, in other words, *an arbitrary plane through g contains two principal tangents having their points of contact on g.*

These principal tangents meet g in two points, which give the foci of g when the congruence is projected into a space congruence from any point of ϖ. Now any congruence in [3] can be derived by projection from one in [4], for which κ is equal to unity. For we obtain such a congruence by joining each of the lines of the system in the [3] to a fixed point in [4], and selecting in each plane so derived a line chosen rationally (say so as to meet two fixed planes). A proof is therefore deduced of the theorem that an arbitrary line of a congruence of ordinary space is met by two consecutive lines. More generally from the extended theorem *it follows that an i-ple ray of a space congruence is met by $2i$ consecutive ordinary rays.*

Conversely, assuming this result we have a new proof† of Theorem I and its extension, though a direct proof appears initially preferable.

(iii) The theorems (i) and (ii) suggest that *the system of ∞^2 principal tangents meeting the generator g generate a form which is of the same nature for all possible congruences*, a result strictly analogous to the well known theorem that the principal tangents to a scroll of ordinary space in the points of a generator describe a scroll which is of the same nature for all scrolls (namely a regulus). That this theorem is true is seen by a degeneration argument. Let the congruence degenerate into two congruences on distinct forms F_1, F_2, and suppose g to lie on F_1. Then all intersections of g and F_2 are required to complete the $m-3$ intersections of g and other rays of the congruence. Now the tangent lines to F_1 at such an intersection are indeed principal lines of the combined form F_1+F_2; but they correspond to the tangent lines to one of the sheets of F at one of the double points (binodes) of F on g (namely that sheet containing g), and are not derived from principal lines of F

† And analogously, since each ray of a system ∞^{r-2} in a space of dimension $r-1$ is $(r-2)$-tangent to a certain focal form (Segre, "Un' osservazione . . .", *loc. cit.*), we have the general result: *Each ray of a system of lines of dimension $r-2$ in a space of dimension r meets the double construct of the form generated in $m-r+1$ points, m being the order of the form question, etc.*

at simple points on g. It follows that to the form of principal lines for F corresponds in the degeneration that for F_1. The result stated follows†.

The order of the form which we are considering can be found from any special case, say the case $m = 3$, it being noted that the last two corollaries are applicable to any congruence, including those for which κ, the number of generators through an arbitrary point of the locus, is greater than unity. The order is equal to the order of the scroll of principal tangents along a line g on a cubic surface in ordinary space. A plane through g contains two such tangents, and, in addition, the line g is a simple directrix and two-fold generator on the scroll, the last statement following from the fact that g is touched by the residual section of two planes through it, or from Theorem (i) of the present section. *The form of principal lines is therefore of order five in all cases, the line itself being triple on the form*, counting once as a directrix and twice as a generator. One property of this locus of principal tangents may be noted. It is known that in a system of ∞^2 planes in space of four dimensions‡ there exist in each plane six points (lying on the focal conic) in which two consecutive planes are concurrent. Dually, *through our generator g, pass six spaces, which contain in all three neighbouring lines*. In each such space the three lines define as directrices a regulus whose generators are principal tangents. This is only compatible if the section of F by the space has an extra binode (at least), and conversely this is sufficient, since each binode, as we have seen, reduces the order of the scroll of principal tangents by three. (In fact, there is removed from the scroll the cubic cone of lines lying in the sheet containing g, which meet F in four consecutive points.)

† The theorem is verified algebraically as follows. In the first place the determination of the order is equivalent to the same problem for the section by an arbitrary space through g. Consider in the first place a surface of order m passing through g but otherwise arbitrary, and let g be the line $x_1 = x_3 = 0$, so that one of its points can be taken as $(0, \mu, 0, 1)$. Then the surface of principal lines is given by elimination between relations of the form $(x\,\overline{|}\,1, \mu)^{m-1} = 0$, $(x^2\,\overline{|}\,1, \mu)^{m-2} = 0$, the expressions entering being respectively linear and quadratic in (x), and of degrees in μ given by the indices. It is thus of order $3m-4$. Now suppose that the point $\mu = 0$, or $(0, 0, 0, 1)$, is a binode of f, the tangent spaces being $x_1 = 0$, $x_2 = 0$. Then the equation of f is of the form

$$x_1 x_2 x_4^{m-2} + x_3 u_{m-1} = 0,$$

where u_{m-1} is an expression of degree $m-1$. The expression in the first equation contains μ as a factor and reduces to $x_1 + x_3(1, \mu)^{m-2} = 0$, while that in the second is such that the highest power of μ is always multiplied by x_3, and hence can be eliminated without increasing the degree in (x), leaving a relation $(x^2\,\overline{|}\,1, \mu)^{m-3} = 0$. The new eliminant is of order $3m-7$, so that the binode reduces by three the order of the eliminant. As there are $m-3$ such, the final order equals five, as determined below.

‡ Segre, "Sui fochi di 2° ordine . . .", *Rend. Accad. Lincei* (5), 30 (1921), 67.

In other words, the theorem implies that *there exist six tangent spaces to F passing through g which cut the form in surfaces having a binode at the point of contact*†.

5. *Some other results for congruences of general character.*

The most important of these are connected with the scroll of lines of the congruence meeting a fixed plane ϖ, a scroll S^{m+n} of order $m+n$, cutting the plane in a curve, C^m, of order m, the section of F. Suppose, in the first place, that the congruence be projected from a fixed point Q in ϖ on to an arbitrary space π cutting ϖ in a line h. Then the points where h meets the focal surface of the projected congruence are the points of intersection of h and the lines through Q tangent to C^m. Hence, if r denote the rank of the projected congruence‡, and ψ denotes the order of the double surface on F, we have $2n(m-1)-2r = m(m-1)-2\psi$ by a well known theorem. Thus

$$r = (n-\tfrac{1}{2}m)(m-1)+\psi. \tag{1}$$

Consider now the spaces through ϖ. Each contains n rays; and hence, apart from the plane, the cone of apparent self-intersection of S^{m+n} from Q is of order $r+\frac{1}{2}n(n-1)$, and its curve of intersection with S^{m+n} is of order exactly twice this number. Associating the points of intersection in pairs gives y, *the number of improper double points of the scroll, excluding the ψ points on ϖ*, thus

$$4\{r+\tfrac{1}{2}n(n-1)\}-2\{r+\tfrac{1}{2}n(n-1)\} = 2y+\phi,$$

where ϕ is the class of the focal surface of the projected congruence. This gives on reduction§

$$y = 2r+(\tfrac{1}{2}n-m)(n-1) = \tfrac{1}{2}n^2-m^2+mn+2m-\tfrac{5}{2}n+2\psi. \tag{2}$$

The total number of improper double points is $y+\psi$, and the order of the cone of self-intersection from an arbitrary point is

$$r+\tfrac{1}{2}n(n-1)+\tfrac{1}{2}m(m-1).$$

† Notice in the case of the section by an ordinary tangent space that when the section acquires an ordinary node there is removed from the quintic scroll a quadric cone, leaving a (Cayley) cubic scroll.

In the case of a cubic form it can be readily verified algebraically that if a cubic surface possess three neighbouring lines, it has a binode on their common position, and that there are six and only six tangent spaces to the form passing through g and possessing extra binodes.

‡ Namely the number of pairs of its rays concurrent on and coplanar with a fixed line.

§ This result may also be found from general researches on surfaces in four dimensions (noting that the total number of improper double points is $y+\psi$) given by Severi, "Intorno ai punti doppi impropri di una superficie . . .", *Rend. di Palermo*, 15 (1901), 33.

The class† *of* S^{m+n}, *or the number of tangents passing through a point*, is equal to the sum of the orders and classes of the focal surface, or

$$4mn-2n-2m+4r,$$

since the tangents passing through Q in ϖ are the tangents to the curve C^m and the lines through Q meeting two coincident rays cospacial with ϖ.

On the scroll of intersection of the congruence (m, n) *and an arbitrary* k*-tic hypercomplex.*

This scroll S is of order $k(m+n)$ by § 2, or by the argument about to be given. To determine its characteristics allow the hypercomplex to degenerate into k special linear hypercomplexes, each consisting of the system of all lines meeting a fixed plane. Then the scroll is replaced by k scrolls of the type just discussed, sharing in pairs $m+n$ generators. Now the absorption by each such common generator of the points of intersection of the two scrolls is the same, and equals $1+2z$, where z is the so-called *class of immersion* of the generator in either scroll, namely the order of the form of tangent planes to the scroll in points of the line‡. This form in the present case is the space defined by the generator and a consecutive generator, so that z is equal to unity. It follows that the number of distinct intersections of two of the scrolls is

$$(m+n)(m+n-3).$$

Since these arise in the degeneration from improper double points of S, *the total number of double points on this scroll* is

$$d_k = k(\gamma+\psi)+\binom{k}{2}(m+n)(m+n-3). \tag{3}$$

Alternatively we can calculate from the same degenerate figure the genus of cross section of S, using the well known formula for the genus of a degenerate curve. The result is (in the general case)

$$p_k = kp+\tfrac{1}{2}k(k-1)(m+n)-k+1, \tag{4}$$

where p is the genus of cross section in the case $k=1$, *i.e.* the genus

† Severi, *loc. cit.*

‡ For this definition and formula see Severi, "Sulle intersezioni delle varietà algebriche . . .", *Mem. Torino* (2), 52 (1903), 61, chap. 4, n. 11.

of a plane section of the form F. It can be verified that these two values satisfy the equation

$$d_k = \binom{k(m+n-2)}{2} - 3p_k,$$

as they should do†. The value of p_k is easily modified for the case in which F possesses multiple curves.

6. *Focal properties of congruences.* There are in general ∞^1 lines of the system meeting consecutive lines, and these will be termed *focal rays*, and the scroll they describe the *focal scroll*. We shall use N to denote the order of the scroll, ν that of *the focal curve* C^ν, the locus of the intersections of three consecutive rays, and the cuspidal curve of the form F, and finally M to denote the order of the form described by the focal planes, namely the planes defined by the same pairs of rays. We obtain formulæ for these three numbers in terms of the previously defined characteristics, together with

σ, the number of pairs of intersecting rays whose plane passes through a fixed point external to F‡, and

ζ, the order of the form of planes defined by the pairs of rays of the congruence which meet in points of a fixed space.

The first of the relations is obtained from a correspondence in the star of ∞^2 spaces through a fixed line q. Rewording Theorem I of § 4 to apply to this case we require, in addition to the indices, the number γ of pairs of corresponding spaces passing through a fixed plane through q, and for each type of self-corresponding space the number of spaces whose plane of limiting intersection lies in a fixed space. The correspondence in question is obtained by associating spaces through q, which contain intersecting rays of the congruence. Both indices are $n(m-3)$, $\gamma = \psi$, and there are two types of coincidences, the first arising from the focal rays, and counting as ν, and the second arising from pairs of intersecting rays in spaces through q, and counting as 2ζ. Thus

$$2n(m-3)+2\psi = \nu+2\zeta, \tag{5}$$

giving the value of ν. The formula may also be obtained by considering the correspondence between the pairs of points in a space π' which lie on rays of the congruence concurrent on a second space π. The rays in

† Severi, "Intorno ai punti doppi . . .", *loc. cit.* The remaining characteristics can be calculated from the formulae of this paper.

‡ This number equals the number of double rays of the projected congruence. For the theory of general congruences in ordinary space, see C. M. Jessop, *A treatise on the line complex* (Cambridge, 1903), chap. 14, § 341, *et seq.*, though the formulae are restricted to the case when the number of double rays is zero.

question describe a scroll of order† $n(m-3)+2\psi$, and hence the corresponding pairs lie on a curve of this order; while their joining lines describe a scroll of order ζ. The proper united points are τ, and the improper ones (each counting twice) ζ in number. The relation between these numbers gives (5).

To obtain a second relation we proceed as follows. Let g be a fixed external line, and let us consider the pairs of rays which intersect and are cospacial with g. From the intersection with a fixed plane through g it is seen that the scroll generated by these lines is of order $m(m-3)+2y$; and from the same consideration the planes which they define generate a form of order $y+\sigma$, it being noted that the multiplicity of g thereon is σ. Taking the intersection of the whole figure by a space π there is a (1, 1)-correspondence on the section of the scroll, whose proper coincidences are M in number (arising from the M intersections of g and the form of focal planes), and whose improper coincidences (counting twice) are similarly ζ in number. We therefore have the equation

$$2m(m-3)+4y-2(y+\sigma) = M+2\zeta,$$

or

$$2m(m-3)+2y = M+2\zeta+2\sigma, \tag{6}$$

from which the value of M is obtained. An analogous method is to consider the pairs of points in the section of F by a fixed space π through which pass concurrent rays. Projecting these pairs on to a fixed plane ϖ from a fixed point Q in the space we have a plane correspondence of involutory character whose indices are $m(m-3)$ and for which $\gamma = y$. The united points arise from the points where π meets the scroll of focal lines the class of the envelope of limiting joins being M, from the intersections with the double surface for which the class similarly equals ζ, and from the pairs of intersecting rays coplanar with Q, which give rise to complete coincidences, each counting twice. Substituting in the lemma of § 4 gives the result precisely in the form (6).

The value of N, *the order of the scroll of focal rays*, is very easily obtained, by considering the number of points of intersection of two of the scrolls S^{m+n} of rays of the congruence meeting fixed planes, which we have already seen to be equal to $(m+n)(m+n-3)$. It is further equal to the order of the scroll S' obtained by taking those rays which meet the rays of one of the S^{m+n}, and this scroll S' cuts the plane ϖ associated with S^{m+n}

† For the order of this scroll is equal to the order of the curve of intersections of the rays of the congruence meeting a fixed plane with other rays. Taking a space through the plane, the result is found at once.

in $2\psi+2y+N$ points. Hence

$$N=(m+n)(m+n-3)-2\psi-2y, \tag{7}$$

and N is expressible in terms of the original characteristics only, thus

$$N=3m^2-7m+2n-6\psi=6p+2(m+n)-6, \tag{8}$$

where p is the genus of a plane section of F†.

7. *Further results in connexion with complexes of general character.* In addition to the characteristics m, n, R, and ψ (§ 1), we use in the discussion the following numbers, connected by a variety of relations :

s, the number of triple points on the double curve of the scroll S^n in an arbitrary space (triple on the curve and on the scroll), a number equal to $\frac{1}{3}(n-4)\{2n-n^2+3\psi\}$‡. By duality this number is also equal to the number of tritangent planes of the scroll, or the number of triads of coplanar rays of Γ (as we shall call the complex in future) in a fixed space.

T_1, the number of triads of rays of Γ intersecting on a plane ϖ and cospacial with it; T_2, the number of triads of rays concurrent on a fixed line and cospacial with it; χ, the number of points on the plane section of the scroll S^n in a space through which pass pairs of rays lying in some other space through the plane; δ, the number of points in a plane ϖ through which pass two pairs of rays respectively cospacial with the plane; and, finally, θ, the order of the focal form of the complex (touched three times by every ray of Γ). By a method analogous to that used to prove the corresponding result for congruences in ordinary space§, this number is found to be equal to $2n(m-1)-2R$‖.

7.1. *The number T_2 can also be expressed in terms of the previous characteristics.* Let g be the fixed line and let us associate spaces through g, each of which contains two rays of Γ meeting g, one of which is common to the two spaces. The correspondence is between the spaces of a system ∞^1 in the [1]-star of axis g, the system being of class R.

† This result may be compared with the formula for the number of singular generators of a scroll of order n in ordinary space, namely $4p+2n-4$ (Lurôth's formula).

‡ See Wiman, *Acta Math.*, 19 (1894), 63.

§ For details of the proof, see James, "Formulae for the intersection of line systems in higher space", *loc. cit.*

‖ In the general case there are only a finite number of multiple rays (double). If, however, there exist a congruence of k-ple rays of order μ this expression is reduced by $\mu k(k-1)$.

The indices are each $2(m-2)$, the correspondence being involutory, and the self-corresponding spaces are

(i) the T_2 spaces each containing a triad of rays concurrent on g, each space counting six times; and

(ii) the spaces containing a ray, one of whose foci† lies on g, and a second ray through the same point of g, each counting once. Finally the planes of intersection of corresponding spaces generate a form of order n (multiply counted), so that, applying the well known general correspondence principle in dual form

$$4R(m-2)-2n\binom{m-1}{2} = 6T_2+\theta(m-2),$$

whence
$$T_2 = (m-2)\{R-\tfrac{1}{2}n(m-1)\}. \qquad (9)$$

A second proof can be based on the correspondence of pairs of points (A, B) on g, such that through A passes a ray which meets the plane of a pair of rays through B.

7.2. *Some elementary formulae.*

(i) *On the locus of planes of pairs of rays concurrent on a line g.* Its order is seen to be $\frac{1}{2}m(m-1)+R$ from the section by a plane through g, and its genus can be written down by a well known theorem of Zeuthen‡. Its multiple constructs are g, which is $\frac{1}{2}m(m-1)$-fold, the scroll of rays of Γ meeting g, which is $(m-1)$-fold, and a double surface the locus of intersections of pairs of planes, passing $\frac{1}{2}\binom{m}{3}\binom{m-2}{2}$ times through g, and meeting a plane through g in $\frac{1}{2}R(R-1)$ points. To find the order of this double surface, consider an arbitrary space through g cutting the surface in g counted $3\binom{m}{4}$ times and a residual curve X. X contains $\frac{1}{2}R(R-1)$ points in any plane through g in addition to those on g. It has triple points in the T_2 points on g and cuts it again in those points where the spaces defined by two pairs of rays of Γ (not sharing a ray) taken separately with g cut in a plane of a fixed space. They are thus in number equal to the order of the cone of these planes. Let us consider the association of these pairs of spaces, an involutory correspondence of indices $\frac{1}{2}(m-2)(m-3)$, with no united spaces. The class of the envelope of spaces is R. Hence by the correspondence principle in dual form the order of the form of planes of intersection is $R\binom{m-2}{2}$

X is therefore of order $$R\binom{m-2}{2}+3T_2+\binom{R}{2},$$

and the double surface of order $3\binom{m}{4}+R\binom{m-2}{2}+3T_2+\binom{R}{2}$.

† This term denotes the points of intersection of two consecutive rays.

‡ *Math. Ann.*, 3 (1871), 150-156. The theorem is that if two series of ∞^1 elements, of genera p_1, p_2 respectively, are in (x_1, x_2) correspondence, and if there are t_1, t_2 coincidences of the involutions defined thereby in each, then

$$t_1-t_2 = 2x_2(p_1-1)-2x_1(p_2-1).$$

It contains a number of lines easily found by associating pairs of points on g through which pass rays cospacial with one of the planes under discussion.

(ii) *On the system of planes of pairs of rays concurrent on a fixed plane.* We define as the order (class) of an ∞^i of planes in space of four dimensions the number dual to the class (order) of an ∞^i of lines. In the present case the order is $\binom{m}{2} + \omega$, where ω denotes the class of the envelope of lines in a plane which are concurrent and coplanar with a pair of rays of Γ.

The class of the system is $R + \psi$.

(iii) *The system of planes of pairs of rays concurrent on a fixed space* is of order ψ and class $\psi + R$.

By considering the numbers of spaces through a suitable linear space in a line, plane, or space, respectively, *the classes of the developables of the* $(\infty^1, \infty^2, \infty^3)$ *spaces of triads of rays concurrent on these respective constructs are seen to be* $\binom{m}{3} + T_2$, $T_1 + T_2$, $T_1 + s$ respectively.

Similarly—

(iv) *The locus of points of intersection of pairs of rays in a plane passing through a fixed point* A is of order $\binom{m}{2} + R$, and passes $\binom{m}{2}$ times through A. The system of planes of pairs of rays (∞^4) is of class ψ and order† y (say), y being the number of such planes passing through a line.

(v) *The locus of intersections of pairs of rays in a space which passes through a fixed line* is a form of order $\binom{m}{2} + R$, while, if it passes through a plane, the locus is a surface of order $R + \psi$.

(vi) *The loci of points of intersection of triads of concurrent rays of* Γ in spaces which pass respectively through a fixed point A, a line g, or a plane, are respectively—

a form of order $\binom{m}{3} + T_2$, having A as an $\binom{m}{3}$-fold point;

a surface of order $T_1 + T_2$, meeting l in T_2 points;

and a curve of order $T_1 + s$, meeting the plane in T_1 points.

8. *On the scroll S of lines of the complex Γ meeting a fixed line g.* The scroll S is of order $m+n$, and passes m times through g. We denote its genus by p, and the order of its apparent double curve (or the order of the cone of lines through a fixed point A meeting two generators of the scroll) by h. This cone passes through the points R on the line g in association with the plane gA, and through the y points not on g in which generators of the scroll intersect.

Consider now a pair of generators $R'Q'$, $R''Q''$ meeting g in R', R'' respectively, and such that $Q'Q''$ passes through the fixed point A. The cone described by $AQ'Q''$ is the cone of order h, and the curve described by the points Q', Q'' is of order $2h$, since it does not pass through A, while a space through A contains $2h$ of its points. The curve has nodes

† The notation used in § 7, *et seq.*, is independent of that previously employed in the discussion of the congruence.

at the R and the y points previously mentioned. Consider now the (1, 1) correspondence between the points Q' and Q''. The scroll of joining lines is of order h, so that, by the correspondence principle, the total number of united points is $2h$. These include the R and the y points, all counting twice, and the points (say x in number) through which pass tangents to S passing through A, so that as a first relation between the characteristics of the scroll S we have

$$2h = x+2y+2R. \tag{10}$$

Further, the points R', R'' are in $\{m(n-1), m(n-1)\}$ correspondence, since each ray of the scroll contains $n-1$ points Q arising from the $n-1$ remaining generators in the space defined by the ray in question and the plane Ag. The united points are the points on g corresponding to the x united points of the previous correspondence, and the points (R) counting twice, since the correspondents of such a point distinct from it are $(m-2)(n-1)+2(n-2)$ in number. Thus

$$2m(n-1) = x+2R. \tag{11}$$

Thirdly, consider the section of the scroll by an arbitrary space π, and let P', P'' be the points where π cuts $R'Q'$ and $R''Q''$. These points are in $(n-1, n-1)$ correspondence, and the correspondents of a given point P are cut out on the curve (P) by the space defined by g and AP, so that the valency is unity, while the united points are x in number.

Hence
$$x = 2n+2p-2. \tag{12}$$

Since $h = \frac{1}{2}(m+n-1)(m+n-2)-\frac{1}{2}m(m-1)-p$, the equations (10) ... (12) give the values

$$\left.\begin{aligned} p &= (n-1)(m-1)-R, \quad & h &= \tfrac{1}{2}n(n-1)+R, \\ x &= 2m(n-1)-2R, & y &= \tfrac{1}{2}n(n-1)-m(n-1)+R \end{aligned}\right\}. \tag{13}$$

The value of h may also be read off directly by considering the section of the cone (AQ) by a space through g and A, or from the correspondence between corresponding points R' and Q', etc. Similarly, the value of p can be found from Lurōth's formula for the number of singular generators of a scroll in ordinary space (see Equation 8, note), for the projection of S from A is a scroll S' of the same order and genus, having a nodal h-tic curve meeting g in R points and containing x cuspidal points. The singular generators of S' arise from the θ generators of S, one of whose

respective foci lies on g, and the generators on which lie the points x. Hence $\theta+x=2(m+n)+4p-4$, from which (12) again follows.

The results (10)–(12) are readily modified to allow for the case when S has double or multiple generators (in particular, when Γ has a multiple congruence), and verifications can then be obtained from the complexes defined by conditions of incidence with fixed surfaces and curves.

9. We can also determine *the number of apparent generators of the scroll S, i.e.* the number of lines passing through an external point A meeting a triad of generators of S. For this purpose consider the correspondence of points Q', Q_1' on the same generator of S and through which pass chords of S which also pass through A. It is of indices $n-2$, $n-2$, and the scroll of joining lines is S itself, counted $(n-1)(n-2)$ times. The united points are the points on the t lines sought and $n-2$ points for each of the x generators of the figure. There are three points for each of the t generators, and each is a double point on the curve C^{2h}, and therefore counts twice, but the latter points count singly. Hence

$$2.2h.(n-2)-(m+n)(n-1)(n-2)=6t+x(n-2),$$

so that
$$t=\tfrac{1}{6}(n-2)\{6R+(n-3m)(n-1)\}. \tag{14}$$

9.1. *The genus of the apparent nodal curve of S*, etc. We consider for this purpose the correspondence between g and the curve C^{2h} when corresponding points lie on the same generator of S, applying to it the theorem of Zeuthen, previously quoted.

We are to write [see the note in § 7.2, (i)]

$$x_1=1,\quad x_2=m(n-1),\quad t_1=0,\quad t_2=\theta(n-1)+x(n-2),$$

and hence the genus of C^{2h} is

$$q_1=1+(n-1)(2mn-n-3m)-R(2n-3). \tag{15}$$

Similarly from the correspondence between g and the rays of the cone (AQ), for which

$$x_1=2,\quad x_2=m(n-1),\quad t_1=x,\quad t_2=\theta(n-1)+x(n-2),$$

we deduce that the genus q_2 of the cone in question is

$$q_2=1+\tfrac{1}{2}(n-1)(2mn-4m-n)-R(n-2). \tag{16}$$

From these (or directly from the correspondence of the points Q and the generators of the cone on which they lie) we have $q_1=2q_2+\frac{1}{2}x-1$.

10. It may be noted that the formulae of §§ 8, 9 *are applicable to any scroll in* [4] *with a multi-directrix line*, provided that m, n, and R are interpreted in terms of properties of the scroll only, namely

m is the multiplicity of the directrix;

$n+m$ is the order of the scroll (and n equals the class of immersion of the scroll therein); and

R is the number of intersections of the directrix and the residual double curve on the scroll of projection in ordinary space.

The converse of the alternative proof of the value of x shows that in the present case there are $\theta = 2n(m-1)-2R$ points on the directrix at which two of the m rays coincide.

In particular, *for a scroll of order d with a simple directrix*,

$$\left.\begin{gathered} p=0, \quad R=\theta=0, \quad h=\tfrac{1}{2}(d-1)(d-2), \\ x=2(d-2), \quad y=\tfrac{1}{2}(d-2)(d-3), \quad t=\tfrac{1}{6}(d-2)(d-3)(d-4), \\ q_1=(d-3)^2, \quad q_2=\tfrac{1}{2}(d-3)(d-4), \end{gathered}\right\} \tag{17}$$

and verifications are given by the cubic scroll and by the quartic scroll which possesses a directrix.

11. *On the rank of the complex common to a hypercomplex Λ of degree k and a hypercongruence λ of order m and class n*, the complex in question being of order km and class $k(m+n)$. It will be convenient to denote these numbers by μ and ν respectively. From the last few sections it appears that the rank sought is known when the number of isolated double points or the scroll S of its lines meeting a fixed arbitrary line g is known, and it is this latter number which we determine directly.

Let us associate points A and B on g, when a line of the hypercongruence through A and a line of the hypercongruence through B meet on S, which is of order $k(2m+n)$, but not on g. The points B corresponding to a point A are found by taking the $m\nu-\mu$ intersections† of the m-tic cone of A and S, which are neither at A nor on the generators through it. Through each such point pass $k-1$ rays of Λ besides the generator of the scroll. The first index is therefore $(m\nu-\mu)(k-1)$.

To find the second index, consider an auxiliary correspondence associating points P and Q on a generator of the scroll, when the plane of this line and g contains a ray of λ through Q and PB belongs to Λ.

† To read off this number allow the m-tic cone to degenerate into m planes, μ of which contain generators of S. Each of these is met by $\nu-1$ other generators, while the remainder are each met by ν generators. Taking these together gives the value written.

Then

(i) the indices are k, $n-1$;

(ii) the order of the curve (P) is $k(\mu+\nu)-\mu$;

(iii) the joining scroll is the scroll S counting $k(n-1)$ times;

(iv) the order of the curve (Q) is equal to the number of its points in a space through g and not on g, *i.e.* $\nu(n-1)$, together with the number of its points on g, which we obtain as follows. Consider the scroll of lines of λ meeting in pairs on g and coplanar with it. Through a point of g pass H pairs of lines, where H is the number of apparent double generators on an arbitrary cone of λ, so that the present cone has H cuspidal branches through g. Further, a space through g contains r pairs of lines, where r is the rank of λ. The present scroll is therefore of order $2(H+r)$, and $2k(H+r)$ of its lines belong to Λ, and their intersection with g are the points sought.

Thus the order of the curve (Q) is $2k(H+r)+\nu(n-1)$. By the general correspondence theorem the number of united points of the correspondence (P, Q) is

$$\{k(\mu+\nu)-\mu\}(n-1)+\{2k(H+r)+\nu(n-1)\}k-k(n-1)(\mu+r),$$

which is the second index.

Finally the united points of (A, B) are

$$2k^2(H+r)+k\nu(m+n-1)-k\mu-\mu(n-1)-(m\nu-\mu)$$

or
$$k^2\{2H+2r+(m+n)^2-2m-n\}-km(m+2n-2) \qquad (18)$$

in number. This is, therefore, twice the number of the double points on the scroll in question. For a hypercongruence which is the complete intersection of two hypercomplexes of degrees k', k'' we have

$$m=n=k'k'', \quad H=\tfrac{1}{2}k'k''(k'-1)(k''-1), \quad r=k'k''(k'-1)(k''-1)^{\dagger},$$

and

The number of improper double points on the scroll of rays of the complex common to three hypercomplexes meeting a fixed line is

$$\tfrac{1}{2}kk'k''(7kk'k''-3kk'-3kk''-3k'k''+2), \qquad (19)$$

and by the equations (13) *this is also equal to the rank of the complex*‡.

† For the value of H see Salmon-Rogers, *op. cit.*, 1, 356, and for the value of r Jessop, *op. cit.*, 274.

‡ This formula may be verified by noting that if two of the hypercomplexes are linear the formula must give $\frac{1}{2}k(k-1)$, k being the degree of the third system. This formula is given at once by considering the case of two *special* systems, each of which is defined by the condition that its lines meet a fixed plane.

The formula (18) is modified by the addition of $\frac{1}{2}(km+kn-1)(m-n)$ to give the corresponding rank.

12. *On the form F^{m+n} of lines meeting a plane ϖ*†.

The discussion of this form enables us to express the numbers χ and ω_1^* in terms of m, n, R, ψ, δ and T_1, and gives a number of other results of varying importance. We begin with an important lemma.

On the intersection of certain curves and the focal form F. Consider a scroll of rays of Γ with a directrix plane ϖ meeting it in a curve C possibly multiple and of any order. Then the scroll is tangent to F along a curve met three times by each generator, this curve meeting the plane in certain points which it is sufficient to suppose simple on F. They are of two kinds. At the first only one sheet of the scroll touches the form along a curve through the point. The generators of the remaining sheets are distinct from that of this sheet, and C must touch F, for, considering (for example) a real point Q of F, otherwise the generator in question would pass from real to imaginary or *vice versa* at the passage through Q of its intersection with ϖ. There may, however, be two sheets whose curves of contact pass through Q. Since C is simple on F, the sheets must share the same generator at this point, which is therefore a cuspidal point§ on the multiple curve C, but the direction of the curve is in no way restricted‖.

There are, in general, other intersections at which none of the generators touches F, and these are simple intersections.

Thus the curve C touches, or simply intersects, F according as one, or either none or a coinciding pair of the generators of the scroll at Q belongs to sheets whose branches of the curve of contact pass through Q.

12.1. We now introduce the following numbers whose values, in terms of the above characteristic numbers, are obtained in the course of the discussion:

f_1, the number of the pairs of coinciding rays concurrent on and cospacial with a plane ϖ;

† It is to be remarked that many of these results are inapplicable as they stand to quadratic complexes ($m = 2$), the discrepancies being accounted for by the presence of singular points and multiple rays in more than finite number. There are thus no quadratic complexes of "general" character.

‡ See § 7–§ 7.2.

§ Or pinch point.

‖ We shall not need to consider the case when there are more than two such sheets.

f_2, the number of points on ϖ through which pass two rays cospacial with ϖ and such that a focus of one of them lies at the point of intersection;

f_3, the number of points Q common to ϖ and the focal form F through which pass two rays cospacial with ϖ and neither of whose foci lies at Q;

$(n\theta)_1$, the number† of points Q in ϖ such that through Q passes a ray in a fixed space through ϖ and one of whose foci lies at Q; and

$(n\theta)_0$, the number of points Q in ϖ and on F such that one of the rays non-tangent to F thereat lies in a fixed space through ϖ.

In virtue of the lemma,

$$f_1+2f_2+f_3 = R\theta, \quad 2(n\theta)_1+(n\theta)_0 = n\theta, \tag{20}$$

since the points $(f_1) \dots (f_3)$ are common to the curve C^θ, the section of F, and the curve C^R, of order R, the locus of points through which pass two rays concurrent on and cospacial with ϖ; and since the points $(n\theta)_1$ and $(n\theta)_0$ are common to C^θ and the curve C^n of intersection of ϖ and the scroll of lines of Γ in a space π through ϖ. To find the latter pair of numbers, notice that through each point of C^n pass $m-1$ other rays of Γ defining with ϖ the same number, $m-1$, of spaces π' which are in $n^2(m-1)$ involutory correspondence, since there are n^2 points on the curve C^n, with the property that a second ray through such a point lies in a second space π', and each such point gives $m-2$ spaces π' corresponding to the first. This correspondence gives

$$2n^2(m-2) = (n\theta)_0+2\chi,$$

whence, since evidently $Rn = 2\psi+\chi$, we obtain

$$(n\theta)_0 = 2n^2(m-2)-2Rn+4\psi, \tag{21}‡$$

and therefore, using (20), $\quad (n\theta)_1 = n^2-2\psi. \tag{22}§$

12.2. *Preliminary results*‖. The form of lines of Γ meeting ϖ is of order $m+n$, with a double surface of order $R+\psi$, cutting ϖ in C^R.

† In a notation to be explained later.

‡ Or by considering the correspondence between the points in which the various ray involved meet an arbitrary fixed space A.

§ Or from the singular correspondence π, π'.

‖ We omit the detailed proofs of the results in this sub-section.

Notice that C^R has triple points at the points, T_1 in number, in which concur triads of rays of Γ cospacial with ϖ, and has nodes at the points, δ in number, in which concur two pairs of rays separately cospacial with ϖ. It is also the locus of double points of the curve C^n. The form has a triple curve (also triple on the double surface) of order T_1+s, and the rays concurrent in the points of this curve generate a scroll of order $3s+a$, where a is the order of its curve of intersection C^a with ϖ. There is also a cuspidal curve of order $f_1+2n+4p'-4$, where p' is the genus of the scroll of rays of Γ in a given space, *i.e.* $\frac{1}{2}(n-1)(n-2)-\psi$.

The rays meeting in pairs on C^R and cospacial with ϖ describe a scroll of order $2R+2\psi$, and their planes a form of order $\psi+\omega$ on which ϖ is ω-fold, since the multiplicity of a surface S on a locus of ∞^1 surfaces, all of which meet it in curves, equals the number of these curves through a fixed point of S. It is of the same genus as C^R.

12.3. *The values of the symbols* f_1, f_2, f_3.

Let π be an arbitrary space through ϖ: it contains ψ pairs of rays concurrent in points Q of the curve C^R, and through each such point there pass $m-2$ further rays of Γ, defining with ϖ as many spaces π' which we associate with π. Conversely, to a given space π' correspond χ spaces π. π and π' coincide when Q coincides with one of the sets of points (T_1) or (f_2). Hence

$$3T_1+f_2=\psi(m-2)+\chi,$$

or
$$f_2=Rn+\psi(m-4)-3T_1. \tag{23}$$

Similarly, from the correspondence of the spaces π' among themselves,

$$f_3=2(Rn-2\psi)(m-3)-4\delta. \tag{24}$$†

From (20), (23), and (24) we deduce

$$f_1=2Rn+2\psi(m-2)+6T_1+4\delta-2R^2. \tag{25}$$

13. *Multiple surface on the focal form* F. The correspondences used in the last subsection are typical of a number of sets of which one can be formed for every curve in ϖ such that one or more of the complex rays through it satisfy some definite geometrical condition. They are considered systematically at a later stage, but it

† (23) and (24) can also be found by associating the intersections with a fixed space Λ of a ray through Q and one of the remaining $m-2$ rays (for 23), or these remaining rays in pairs (for 24). Associating the pair of rays defining Q gives $f_1+2\omega=2R+2\psi$, a result which we shall reobtain in a different connection (§ 16).

is convenient to discuss at once a further example, namely for the curve C^θ, the section of F. Through a point Q on this form there pass a pair of coincident rays g and $m-2$ residual rays g'. Let ϕ_1 denote the number of points in ϖ through which pass three coincident rays, and ϕ_2 the number of those through which pass two pairs of coincident rays. These are the orders of the surfaces generated by points with these higher focal properties; we shall call them the trifocal surface and the doubly-focal surface respectively. They are respectively triple and double on F.

Associating the spaces defined by the g and ϖ with those defined by the g' with ϖ, we have

$$(n\theta)_1(m-2)+(n\theta)_0 = f_2+3\phi_1,$$

whence
$$\phi_1 = n^2(m-2)-\psi(m-4)-Rn+T_1, \tag{26}$$

and so also, associating the spaces $g'\varpi$ in pairs, we have

$$2(n\theta)_0(m-3) = 2f_3+2\phi_2,$$

whence
$$\phi_2 = 2n^2(m-2)(m-3)-4Rn(m-3)+8\psi(m-3)+4\delta. \tag{27}$$

There are, in addition, curves on the focal form with still higher focal properties, but for the sake of brevity we shall not attempt to determine their orders.

14. *Consideration of a set of curves in association with the plane* ϖ.

In addition to the curves C^m, C^R, and C^θ, we introduce

(i) the curve C^a previously mentioned, the locus of intersections with ϖ of triads of rays cospacial with it. It has triple points at the points T_1, and two types of double points (according as the two triads defining these share or do not share a ray).

(ii) The curve C^b of order b, the locus of intersections with ϖ of rays, one of whose focal planes (defined by the rays with incident consecutive rays) has a line intersection with ϖ. It passes through the points (f_1).

(iii) There are ∞^1 generators AB tangent to the double surface of F^{m+n}, *i.e.* meeting two consecutive generators BC, $B'C'$. If A and C are the points of intersection of the rays concerned with ϖ, the locus of A is a curve C^d and that of C a curve C^c, these curves being of orders d, c respectively.

(iv) The curve C^e of order e, the locus of the intersections with ϖ of triads of coplanar rays in spaces through ϖ.

(v) The curve $C^{\bar{y}}$ of order $\bar{y} = 2y + m(n-2)$, the locus of intersections with ϖ of pairs of intersecting rays such that their join passes through a fixed point Q of ϖ. The point Q is $m(n-2)$-fold on the curve, and its order is read off from the number of intersections with a line through Q. It has nodes at the ω points on C, for which the corresponding line of the envelope (ω) (7 . 2) pass through Q, and at the three points on the lines of intersection of the planes in (iv), which pass through Q.

At each point of one of these curves there are m rays of Γ, and of these one or more will be responsible for the fact that the point lies on the curve in question. We therefore call these the *significant rays*, the remainder being the residual rays. *We denote by* $\Sigma(\alpha\beta)_i$ *the number of points common to* C^α *and* C^β, *and such that the same* i *rays are significant for each.* For $i \geqslant 1$ these points are subdivided as follows. Taking $i = 1$, each significant ray meets $n-2$ others in the space defined by it and ϖ, and of these a certain number enter into the definition for the curve. These we call the *significant-incident* rays; they may or may not be the same for each curve giving rise to the subdivision in question. For C^R there are additional intersections denoted by such symbols as $(\alpha R)^*$, and noted in detail later. *In the normal case we have the relation* $\Sigma\Sigma(\alpha\beta)_i = \alpha\beta$, *but for* C^θ *and a curve* other than C^R we have

$$2(\alpha\theta)_1 + (\alpha\theta)_0 = \alpha\theta, \tag{28}$$

and there are exceptional cases for the curve C^R, namely

$$\left.\begin{aligned} (R\theta)_0 + 2(R\theta)_1 + (R\theta)_2 &= R\theta \\ 2(nR)_2 + (nR)_0 &= nR, \text{ etc.} \end{aligned}\right\}. \tag{29}$$

If the curve dealt with is a straight line, we regard as significant all the rays through each of its points. The following symbols have known values.

$(nR)_0 = \chi$, $(nR)_2 = \psi$ [there is no $(nR)_1$], $(n\bar{y})_1 = 2\psi$,

$(n\theta)_0$, $(n\theta)_1$, (Equations 21, 22, $(na)_1 = 3s$ (§ 7), $(nb)_1 = 2n + 4p' - 4$,

$(nc)_1 = (nd)_1 = 4\psi(n-3) - n(n-2)(n-3) - 6s$†, $(ne) = 3s$.

The values of the corresponding $(n\alpha)_0$ are thus also known. Further

$$(R\theta)_0 = f_3, \quad (R\theta)_1 = f_2, \quad (R\theta)_2 = f_1.$$

† Namely, the number of generators of the scroll S'' in a space tangent to the double curve. The points $(nc)_1 (nd)_1$ are, of course, distinct.

We shall use ϵ to denote $(Rd)^*$ or $(Rc)^*$, namely, *the number of points on* ϖ *at which one of the pairs of significant rays touches the double surface.* They are points of contact of C^R and C^d, and points of simple intersection of C^R and C^c. We further use $(eR)^*$ to denote the number of triads of coplanar rays meeting ϖ, two of which concur on ϖ.

We develop later a system of relations between the intersection symbols and other important numbers, which will suffice to determine all their values in terms of the characteristics (m, n, R, ψ, δ, and T_1), (though we do not state them explicitly). The most important of these symbols† are

A_1, the number of quadruple points of F^{m+n}.

A_2, the number of the points of intersection of the triple curve of F^{m+n} and the focal form, such that the three generators are distinct, and that a fourth ray consecutive to one of them passes through the point. The same constructs cut also in the A_4 points through which pass three generators, two of which coincide.

A_5, the number of the points on the triple curve at which one of the three generators is tangent to another sheet of F^{m+n}.

A_9, the number of points on the triple curve such that the plane of two generators contains another ray of Γ.

B_1, the number of planes containing two coincident rays of Γ and a third ray, all meeting ϖ.

B_2, the number of points on the cuspidal curve at which the generator has one more intersection with F^{m+n} than for a general point of the curve.

C_1, the number of sets of three concurrent and coplanar generators of F^{m+n}.

C_2, the number of points on the double surface of F^{m+n} at which one generator is three-point tangent to the sheet containing the other.

C_3, the number of points on the same surface at which each generator is tangent to the sheet containing the other.

$D_5(F_1)$, the number of curves on sections of scrolls S^n in spaces through ϖ possessing tangents of four-point contact (tacnodes).

E_2, the number of sets of four coplanar generators of F^{m+n}.

Further, in terms of the intersection symbols, etc., can be expressed the numerical properties of curves, etc., connected with the form F^{m+n}.

† The notation appears later in § 18, for the A in formulae II, the B in III, the C in IV, the D in VIII, the E in IX, and the F in X.

Thus from the curve C^a we deduce that the order of the triple curve of F^{m+n} equals T_1+s, and that the order of the scroll of generators through its points is $3s+a$.

From the curve C^b we deduce that the order of the scroll of singular generators of the same form, and for which the focal point is not on ϖ (see § 6), is $b+2n+4p'-4$. [There is, in addition, a scroll of order $\theta+(n\theta)_1$ for which the focal point is on ϖ, namely the scroll of significant rays for the curve C^θ.] The order of the cuspidal curve, locus of these foci is $f_1+2n+4p'-4$.

From the curve C^d we deduce that the order of the scroll of generators of the same form and tangent to its double surface is $d+(nd)_1$, while the order of the curve of their points of contact is $\epsilon+(nd)_1$.

The double surface cuts the focal form in the cuspidal curve, and in a curve of order $f_2+(nw)_1$ (see § 17, equation 43), through each of whose points pass two generators of F^{m+n} and a third ray near to one of them (and in a residual curve, through whose points pass two generators of F^{m+n} and two distinct and themselves coinciding rays), and so on.

15. *The values of* b, N_b, *and* ω can be obtained directly and explicitly in terms of the characteristics, and we pass on to do so, obtaining, at the same time, the values of d, and e in terms of a, the value of which is obtained implicitly in the calculations giving the values of the symbols just stated. The value of c is obtained in the same connexion.

In the first place let A be any point in ϖ, and let the m generators through A cut the double surface of the form F^{m+n} in points B. Take a fixed line l in arbitrary position, and let it be joined to the various points (B), a correspondence being obtained between the planes of the $[1]$-star of axis l, when the planes passing through points on the same ray of Γ are associated in pairs. The indices are each $2(R+\psi)(n-3)$, since a given plane η meets the double surface in $R+\psi$ points, each of which gives $2(n-3)$ corresponding planes η'. The number of pairs of corresponding planes in a fixed space π through l equals $\frac{1}{2}n(n-2)(n-3)$, since π contains n lines of the form. The self-corresponding planes are as follows :

(i) The planes meeting the scroll of significant rays for C^a. We require the number of such planes for which the limiting joining space passes through a fixed plane of the star $\{1\}$, and these spaces are the spaces of the united planes and the generators of the scroll whose order $3s+a$ therefore is the number sought.

(ii) The planes passing through a point of contact of a ray meeting ϖ and tangent to the double surface of the form; they count, as in (1), for $d+(nd)_1$ in the formula.

(iii) The planes of l and rays meeting l, each counting $(n-2)(n-3)$ times.

Substituting these numbers in the first lemma of § 4, as applied to the $\{1\}$-star, we obtain

$$4(R+\psi)(n-3)+n(n-2)(n-3)$$
$$= 2(3s+a)+\{d+(nd)_1\}+(m+n)(n-2)(n-3), \quad (30)$$

so that d is given in terms of a and known numbers.

15 . 1. Now let g be a fixed line in ϖ, let A denote a typical point on it, and B a point in which a ray through A meets the double surface of the form F^{m+n}. Further, let (C) denote the points where the other rays through the points (B) meet ϖ. Then the points B describe a curve of order $2R+n(n-2)$, having nodes at the points where g cuts C^R; and the points (C) describe a curve of order $\mu = 2R+2y+b$, this being the number of its points on g; and finally the class ν of the envelope of lines AC is $2y+m(n-2)$, this being the order of one of the curve $C^{\bar{y}}$.

The points C, C_1, ... arising from the same ray meeting g are in $(n-3, n-3)$ involutory correspondence, and the united planes are d in number, so that, from the correspondence theorem, the envelope of joining lines is of class $(n-3)(2R+2y+b)-\frac{1}{2}d$. We now associate with all the points D cut out on g by the joins CC_1, ... of points (C) arising from A, whence (omitting details)

$$2e+d-2b(n-3) = m(n-2)(n-3)+4y(n-3), \quad (31)$$

a second relation between the orders sought.

15 . 2. *The value of b can be found explicitly* by considering the form of lines of Γ meeting the scroll of its lines meeting g, a form of order $(m+n)^2$. Each ray of the scroll is a triple generator of the form, and thus each sheet of the scroll gives $m+2$ sheets of the form in a region remote from g. At a point Q on g there are thus $3m$ simple sheets and $\frac{1}{2}m(m-1)$ sheets of the second order for the form, since, if g_1, g_2 are two rays through Q, there is a sheet arising from the rays near g_2 meeting the sheets of the scroll containing g_1, and one from the interchange of g_1 and g_2; and these sheets coincide at Q. It follows that g is $m(m+2)$-fold on the form; and the order of the curve of points (C), which is the residual section by an arbitrary plane, is thus $(m+n)^2-m(m+2)$, and

this is equal to $2R+2y+b$, whence

$$b = 4m(n-1)+n-4R. \qquad (32)$$

The same result may be obtained by equating to $2(m+2n)+6p-6$ (where p is the genus of a plane section of the form of lines F^{m+n}) the order of the total scroll of its singular generators, namely

$$(b+2n+4p'-4)+\theta+(n\theta)_1$$

(See § 6, equation 8, and § 14.) The formulae (2), (5), and (6) give verifications of other results when applied to the same form. By associating the points B, B_1, ... arising from the same point A in the previous figure (§ 15 . 1), the formula (30) is verified, and (31) is verified by associating in pairs the planes ABC arising from the same point A.

The focal planes of Γ, defined by rays and incident near rays, form a system whose order we shall denote by N_b (namely the class of the envelope which they cut out on a fixed plane), and whose class (the number of planes in a space) is $2n+4p'-4$, which we subsequently denote by P. To find N_b, consider the correspondence of pairs of points AC in ϖ when through them pass intersecting rays. The indices are each $m(n-2)$, whence, from the lemma of § 4,

$$2m(n-2)+2y = N_b+2\omega,$$

and by a later result (giving ω)

$$N_b = n(n-1)-2m-2R^2+2Rn+2\nu(m-3)+6T_1+4\delta. \qquad (33)$$

15 . 3. *A first expression for the value of a.* Taking the figure of § 15 . 1, consider the scroll generated by the $m-2$ residual rays through the points (B). Taken with the scroll (AB), counted $n-2$ times, and the scroll (BC), it forms the total scroll of rays meeting the curve (B), and hence its order is

$$\{n(n-2)+2R\}(m+n)-(m+n)(n-2)-\{(2y+2R+b)+n(n-2)\}.$$

From the intersection of this scroll and ϖ its order is also seen to be $2a+2R(m-2)+w+w'$, where $w(w')$ is the number of intersections of the curve (B) with the focal form F, which arise from rays passing through a point B, and near to the corresponding AB (BC), but not coinciding with BC (AB). Now the points of intersection of lines AB with consecutive rays of Γ lie on a curve of order $3n+\theta$ (as is seen from its intersections with a fixed space through g), meeting each generator in three points. It therefore cuts the curve (B) in

$$(\theta+3n)(n-2)+3\{2R+n(n-2)\}-3(m+n)(n-2)$$

points, and these are also $b+w$ in number. Hence

$$w = 2mn^2-11mn+n^2+10m-3n-2R(n-7) \tag{34}$$

and
$$2a = (n-m)(n^2-6n+7)+2R(2n-5)+m-w'. \tag{35}$$

The value of w' is found later.

16. *The values of ω and ϵ.* Let A be a point of the curve C^R and let the pair of significant rays cut the double surface in points B, B_1, ... There are $n-3$ points B on each of the rays, and through them pass rays of Γ meeting ϖ in points C, C_1, ... Further, let the $m-2$ residual rays at A cut the same double surface in points B^*, etc. The curve (B) has six-fold points at each point T_1 for ϖ, and passes through the ϵ points $(dR)^*$, and is thus of order $2\psi(n-3)+6T_1+\epsilon$. The curve (B^*) is similarly of order $\chi(n-2)+6T_1+4\delta$. By considering a slight variation of the curve C^R it is seen that it counts twice as the locus of points (B) derived similarly from an *arbitrary* R-tic curve, whence

$$\{2\psi(n-3)+6T_1+\epsilon\}+\{\chi(n-2)+6T_1+4\delta\}+2R = R\{n(n-2)+2R\},$$

or, on reduction,
$$\epsilon = 2R^2-2R+2\psi-12T_1-4\delta. \tag{36}$$

The same figure gives the value of ω (§ 7 . 2). Let the orders of the curves (C), (C^*) be denoted by μ_R, μ_R^* and the classes of the envelopes (AC), (AC^*) by ν_R, ν_R^*. Then, from the correspondence of the points A, C and A, C^* respectively,

$$\left.\begin{aligned} 2R(n-3)+\mu_R-\nu_R &= 6T_1+(bR)_1+\epsilon \\ R(m-2)(n-2)+\mu_R^*-\nu_R^* &= 6T_1+(bR)_0+4\delta \end{aligned}\right\}. \tag{i}$$

Reasoning as above we have, further,

$$\left.\begin{aligned} \nu_R+\nu_R^*+2\omega &= R\nu \\ \mu_R+\mu_R^*+2R &= R(2R+2y+b) \end{aligned}\right\}, \tag{ii}$$

where $\nu = 2y+m(n-2)$, and, again,

$$(bR)_0+(bR)_1+f_1 = bR. \tag{iii}$$

Adding the equations (i), and using (ii) and (iii), we have

$$2R^2-4R+(2\omega+f_1) = 12T_1+4\delta+\epsilon,$$

so that $2\omega+f_1 = 2\psi+2R$, or, finally,

$$\omega = R^2-R(n-1)-\psi(m-3)-3T_1-2\delta. \tag{37}$$

16 . 1. Before proceeding further with the calculations we propose to determine two orders, of which the first is *the order of the congruence of significant rays for the trifocal surface* (§ 13), *i.e.* the congruence of rays of Γ meeting two consecutive rays in the same point.

We take a line g in ϖ and the scroll of order $m+n$ of lines meeting g, and consider thereon the curve X of order $3n+\theta$, which is such that its points are foci of generators of the scroll. Let A be a point on g and l an arbitrary fixed line. The rays through A give $3m$ points on X, through each of which pass $m-2$ residual rays. These define, with the fixed line l, spaces cutting g in $3m(m-2)$ points B which we associate with A. The second index is equal to the order $(3n+\theta-6)(m+n)$ of the scroll of residual rays. The united points are : (i) the points where g meets F, each counting $2(m-2)$ times ; (ii) the points where g cuts the form whose order is sought, each counting twice ; and (iii) the points A on g, such that one of the m rays of Γ through A and one of the residual rays arising from this ray lie in a plane meeting l. By taking l in ϖ and meeting g in Q this last number is seen to be equal to the number of correspondents of Q as a point A, together with w. (§ 15 . 3.) It follows that the order sought is

$$\tfrac{1}{2}\{(3n+\theta-6)(m+n)-2\theta(m-2)-w_2\},$$

or

$$\tfrac{1}{2}\{2R(m-11)-2nm^2+24mn-16m-11n\}. \qquad (38)$$

This value can also be obtained by comparing two values for the order of the scroll of residual rays for the curve X†.

From the same figure there can also be deduced *the order of the congruence associated with the doubly-focal surface*, namely by associating in pairs the projections on g of the residual rays which concur in the same point of the curve X. The united points arise from the intersections of g and the form generated by the congruence in question, and from the points where l cuts the form of the planes defined by these residual rays in pairs. This form is of order

$$L = \{\tfrac{1}{2}m(m-1)+R\}(3n+\theta)-(3m+b)-2\{3m(m-2)+w\}$$

(the first term giving the order of the form described by the planes of the pairs of rays of Γ meeting on an arbitrary curve of order $(3n+\theta)$, the second the order of the form of focal planes corresponding to the foci of

† All such arguments can alternatively be expressed as correspondence arguments.

which X is the locus, and the last the order of the form described by the planes of the rays meeting g with the corresponding residual rays). The second order sought is therefore

$$2(3n+\theta-6)(m+n)(m-3)-2L. \tag{39}$$

17. We pass on to determine the more important intersection symbols $(\alpha\beta)$, the values of a, etc., and other important numbers, though to avoid systematic error we do not state their values explicitly. Returning to the notation of § 14, we denote by A a point on one of our curves, by B the significant points, if any, corresponding to A, *i.e.* the points where the significant rays of A are met by the significant-incidence rays, and by B' the points where the former rays are met by the remaining rays meeting ϖ. The latter rays cut ϖ in points (C') on a curve of order μ_α (say), and the envelope of lines AC' is taken to be of class ν_α, C^α being the curve under consideration. S^α denotes the scroll of significant rays for the same curve. In addition, we use the following symbols (left-hand column) to denote the numbers of the right-hand column.

N The class of one of the n-tic curves C^n $[N = n(n-1)-2\psi]$.

m_α The class of the envelope of the tangent at A to the curve C^n which is the section of the scroll S^n containing the significant rays for the curve C^α, as A varies on this curve. (This tangent we shall subsequently say is in the direction at A corresponding to the significant ray in question.) In particular, m_R denotes the class of the envelope of nodal tangents of the curves C^n. (Note that $m_b = N_b$.)

$(\alpha R)_1$ The number of intersections A of the curves C^α, C^R, for which the space of the significant rays for C^α contains a further ray through A.

$(\alpha\theta)_1$ The number of points on C^α through which pass rays near to one of the significant rays.

N^α The class of the envelope of joins of the intersections with ϖ of triads of rays cospacial with ϖ.

$N_{c,\,d}$ The class of the envelope of lines AA' in ϖ such that through A passes a ray which meets a ray through A' and a consecutive ray in the same space with ϖ.

N_c The class of the envelope of limiting joining lines of the sections on ω of the pairs of consecutive rays in the same figure. $(N_c = m_c)$.

N_s The class of the envelope of the sections on ϖ of tritangent planes of scrolls S^n.

The following additional curves are now introduced :—

(*a*) *The curve* $C^{\bar{x}}$. Let Q be a fixed point in ϖ. A line g through Q meets x pairs of consecutive rays in spaces through ϖ (§ 8, equation 13.) Their intersections on g as g varies describe this curve of order $x+m$. A given space contains N generators of the scroll $S^{\bar{x}}$, which is therefore of order $x+m+N$.

(*b*) *Two associated curves of order* w, w'. (§ 15 . 3.) Through a point of C^w and an associated point of $C^{w'}$ two intersecting rays pass, such that through their intersection passes a ray consecutive to the ray for C^w. An alternative value of w' is obtained by taking the curve of foci of the rays BC in the figure of § 15 . 1, a curve of order $\mu_\theta+3n(n-2)$, when, from its intersections with the curve (B), we deduce

$$3\{n(n-2)+2R\}+\{3n(n-2)+\mu_\theta\}-3\{\mu+n(n-2)\}=w'+b. \quad (40)$$

The value of $(nw)_1$ or $(nw')_1$ is obtained† from the number of intersections of the double curve of the scroll S^n in a given space through ϖ with the curve of order $3n^2+(n\theta)_1$ of foci of generators of this scroll. We have

$$6\psi+\{3n^2+(n\theta)_1\}(n-2)-3n(n-2)=(nw)_1+2P, \quad (41)$$

where P, as usual, denotes $2n+4p'-4$, the number of singular generators of S^n. The scrolls S^w, $S^{w'}$ are of orders $w+(nw)_1$, $w'+(nw)_1$ respectively.

(*c*) *Two associated curves* $C^{x'}$, $C^{x''}$. The curves C^n through any point of the plane define m directions at it. For a point of $C^{x''}$ the ray corresponding to one direction is met by a ray meeting the tangent to the direction in question (in the corresponding point on $C^{x'}$). The joining lines form the envelope $(N_{x',x''})$. To determine the orders x' and x'', let g be a fixed line in ϖ. At a point A of this line there are $m(n-2)$ directions of the envelope of lines AC (Fig., § 15 . 1), and each such line cuts the n-tic curve for the space from which it arises in $n-2$ points on a curve, whose order, by a simple correspondence

† The value of $(nw)_1$ can also be obtained from the order of the scroll of residual rays for the double curve of S^n, this gives

$$\psi(m+n)-n(n-2)=(nw)_1+\psi(m-2)+3s. \quad (41a)$$

argument (Cf. § 18, VIII) is found to be

$$\mu' \equiv \nu n + mn(n-2) - m(n-2) - \mu. \tag{42}$$

Associating respectively the points (A, P) and (C, P), we obtain

$$m(n-2)^2 + \mu' - \nu(n-2) = 2R(n-3) + x'', \tag{43}$$

$$\mu(n-2) + \mu' - \nu(n-2) = x' + \mu_R, \tag{44}$$

so that x'' is known. Alternatively we can find it by associating the m sets of $n-2$ directions of the envelope (ν) with the corresponding directions of the curves C^n through A (as A varies on g). To find the value of $(nx')_1 = (nx'')_1$, the scroll S^n in a space through ϖ is taken, and at each point A on the section C^n we associate the tangent with the $n-2$ directions joining A to points P in which C^n is cut by rays which meet the generator through A, obtaining

$$N(n-2) + 2\psi - m(n-2) = P + (nx'')_1. \tag{45}$$

The scrolls $S^{x'}$, $S^{x''}$ are respectively of orders $x' + (nx'')_1$, $x'' + (nx'')_1$.

(*d*) *The curve C^T of order T, the locus of inflexions of the curves C^n.* If, again, we take a line g in ϖ there are m directions at a point A on it (those previously considered), and these cut the corresponding C^n in points P on a curve of order (Cf. again § 18, VIII) $(x+m)n + mn - m$, $x+m$ being the class of the envelope of these sets of m lines. Associating A, P,

$$m(n-2) + \{(x+m)n + mn - m\} - (x+m)(n-2) = T + 2R, \tag{46}$$

so that T is known. Note that $(nT)_1 = 3(N-n)$, the number of inflexions of a given curve C^n. The class of the envelope of inflexional tangents is n_T, which we conventionally put equal to $\frac{1}{2}m_T$.

18. We pass on to certain systematic sets of formulae.

Formulae I. By associating the spaces defined by ϖ with the significant rays and those defined by the same plane with the residual rays at the points (A) of a curve C^α we have (omitting details, for which cf. § 12 . 2),

$$(an)_1(m-1) + (an)_0 = (aR)_1 + (a\theta)_1 + (aR)^*, \tag{Iα}$$

where a stands for a, b, c, d, e, $\bar{x}$, w, w', x', x'', T in turn, and in all these cases $(an)_0 = an - (an)_1$; and the symbols $(aR)^*$, when not already defined (as they are for c, d, e), are as follows:

$$(aR)^* = 6T_1, \quad (bR)^* = f_1, \quad (wR)^* = (w'R)^* = f_2, \quad (\bar{x}R) \equiv 0, \quad (TR)^* = 0,$$

and finally $(x'R)^*$ or $(x''R)^*$ is equal to the number of lines of the envelope (ω) (§ 7 . 2, § 16) tangent to one of the branches of the curve C^n, which has the corresponding point on C^R as node. For $C^{\bar{y}}$ we have, in detail,

$$2\psi(m-2)+(n\bar{y}-2\psi) = \nu_R+\nu_\theta+2\omega. \qquad (\mathrm{I}\bar{y})$$

Notice that $(xR)_1 = m_R$, $(x\theta)_1 = m_\theta$, etc., and that $(TR)_1$ *is the number of curves* C^n *with flecnodes.*

Similar formulae are obtained by associating in pairs the residual spaces of the figure (see § 20).

Formulae II. Taking the curves of significant points (B) in the various figures, the order of the total scroll of rays of Γ meeting such a curve is equal to its order multiplied by $m+n$. The order of the scroll of residual† rays meeting this curve can therefore be written down and equated to the number of such rays meeting ϖ. This gives for the curves mentioned on the left of each result :

$(C^{\bar{y}})$ $\{w+\psi\}(m+n)-\{\bar{y}+m(n-2)\} = w(m-2)+N_a+N_{w,w'}$ $(\mathrm{II}\bar{y}A)$,

(C^a) $\{s+T_1\}(m+n)-(3s+a) = 4A_1+A_2+T_1(m-3)$ $(\mathrm{II}a)$,

(C^b) $\{P+f_1\}(m+n)-2(P+b) = A_3+A_4+f_1(m-2)$ $(\mathrm{II}b)$,

$(C^c,\ C^d)$ $\{(nd)_1+\epsilon\}(m+n)-\{c+(nd)_1\}-\{d+(nd)_1\} = A_5+A_6+A_7+\epsilon(m-2)$ $(\mathrm{II}d)$,

(C^e) $\{3s+(eR)^*\}(m+n)-2(3s+e) = A_8+A_9+(eR)^*(m-2)$ $(\mathrm{II}e)$,

$(C^w,\ C^{w'})$ $\{(nw)_1+f_2\}(m+n)-2\{w+(nw)_1\}-\{w'+(nw)_1\}$
$= 2(A_{10}+A_{11}+A_2)+f_2(m-3)$ $(\mathrm{II}w)$,

$(C^{x'},\ C^{x''})$ $\{(nx')_1+(x'R)^*\}(m+n)-\{x'+(nx')_1\}-\{x''+(nx')_1\}$
$= A_{12}+A_{13}+A_{14}+(x'R)^*(m-2)$ $(\mathrm{II}x')$.

The new symbols in this list denote the numbers of the following sets of points, and the sets of letters in brackets following them give the sets of curves which have common intersections arising from the figure in question.

A_1 Quadruple points of the form F^{m+n}.

A_2 Points on its triple curve through which pass rays consecutive to one of the three generators at the point (w, a), (w', a).

A_3 Points on the cuspidal curve with the same property (w, w', b).

A_4 Points on the triple curve at which two of the three generators of F^{m+n} coincide. (Intersections with the cuspidal curve), (a, b, c, w); (a, d, w').

† That is to say those rays not (in general) meeting ϖ.

A_5 Points on the triple curve at which one of the three generators is tangent to another sheet of F^{m+n} (a, c), (a, d).

$A_6(A_7)$ Points on the significant curve of C^c and C^d, through which pass rays of Γ consecutive to the generators of S^d (S^c), (c, w'), (d, w), $\{(c, w), (d, w')\}$.

A_8 The analogous number for C^e (e, w), (e, w').

A_9 Points on the triple curve such that the plane of two generators contains another (fourth) ray (a, e).

$A_{10}(A_{11})$ Points on the double surface of F^{m+n} through which pass two rays consecutive to one (respectively to each) of the two generators of F^{m+n}. The latter is an intersection (w, w').

$A_{12}(A_{13})$ The numbers analogous to A_7 (A_6) for the curves $C^{x'}$, $C^{x''}$.

A_{14} Points on the triple curve such that the plane of two generators of F^{m+n} cuts ϖ in a line tangent to the associated curve C^n at one extremity (a, x'), (a, x'').

For the curves C^R, etc., there are no significant points, but the corresponding argument for the residual points gives

$$(C^{\bar{y}}) \quad \{2\psi(n-3)+\nu_R\}(m+n)-(2\psi+\bar{y})(n-3)-\{2\psi(n-3)+\mu_{\bar{y}}\} = \nu_w+2\nu_a \qquad (\text{II}\bar{y}B),$$

$$C^{(R)} \quad \{2\psi(n-3)+6T_1+\epsilon\}(m+n)-(2R+2\psi)(n-3)-\{\mu_R+2\psi(n-3)\} = 2(aR)_1+(wR)_1+(w'R)_1+3T_1(m-3) \qquad (\text{II}R),$$

$$C^{(\theta)} \quad \{f_1+f_2+(n\theta)_1(n-2)\}(m+n)-\{\theta+(n\theta)_1\}(n-2)-\{\mu_\theta+(n\theta)_1(n-2)\} = 2(a^\theta)_1+(w\theta)_1+(w'\theta)_1 \qquad (\text{II}\theta),$$

$$(C^{\bar{x}}) \quad \{N(n-2+m_R\}-(x+m+N)(n-2)-\{\mu_{\bar{x}}+N(n-2)\} = 2m_a+m_w+m_{w'} \qquad (\text{II}\bar{x}),$$

$$(C^T) \quad \{3(N-n)(n-2)+(TR)_1\}(m+n)-\{T+3(N-n)\}(n-2)-\{\mu_T+3(N-n)(n-2)\} = 2(aT)_1+(wT)_1+(w'T)_1 \qquad (\text{II}T),$$

where $(\alpha T)_1$ denotes the number of points on C^α which are inflexions on the curves C^n corresponding to the significant rays thereat. The corresponding formulae for the remaining curves are required in § 20.

Formulae III. By associating corresponding points on the pairs of curves C^c, C^d, etc., or the single curves C^a, etc., we have :

$$4a-2N_a = A_4+6T_1 \quad (\text{III}a): \qquad w+w'-N_{w,w'} = 2A_3+f_2 \quad (\text{III}w),$$
$$c+d-N_{c,d} = B_2+\epsilon \quad (\text{III}c): \qquad x'+x''-N_{x',x''} = B_3+(x'R)^* \quad (\text{III}x'),$$
$$4e-6N_e = 2(eR)^*+B_1 \quad (\text{III}e):$$

where

B_1 denotes the number of planes containing two coinciding rays of Γ and a third distinct ray, all meeting ϖ (e, b, x'', c), (e, d, x').

B_2 denotes the number of points on the cuspidal curve at which the generator (*i.e.* the limit of two coinciding generators) *has one further intersection with* $F^{m+n}(b, c, d)$.

B_3 denotes the number of points on C^b for which the corresponding line of the envelope N_b has three-point contact with the curve C^n corresponding to the significant ray (b, x', x'').

Formulae IV. Associating in pairs the significant points (B) for the curve C^e which lie on the same ray, we have

$$4\{3s+(eR)^*\}-2(e+3s) = B_1+6C_1; \qquad \text{(IV}e, A\text{)}$$

and associating the points (B) with the points (B') for the various figures, we deduce the formulae

$$\begin{aligned}
(C^{\bar{y}}) \quad & 3\{\psi+w\}(n-3)+\{2\psi(n-3)+\nu_R\}-(2\psi+\bar{y})(n-3) && = N_a+N_{c,d} && \text{(IV}\bar{y}A\text{)},\\
(C^a) \quad & 3\{s+T_1\}(n-4)+\{3s(n-4)+(aR)_1\}-(3s+a)(n-4) && = 12A_1+A_5 && \text{(IV}a\text{)},\\
(C^b) \quad & \{f_1+P\}(n-3)+\{P(n-3)+(bR)_1\}-(b+P)(n-3) && = B_2+A_4 && \text{(IV}b\text{)},\\
(C^c) \quad & \{\epsilon+(nd)_1\}(n-3)+\{(nd)_1(n-3)+(cR)_1\}-\{c+(nd)_1\}(n-3) && = A_5+C_3 && \text{(IV}c\text{)},\\
(C^d) \quad & \{\epsilon+(nd_1)\}(n-4)+\{(nd)_1(n-4)+(dR)_1\}-\{d+(nd)_1\}(n-4) && = A_5+C_2 && \text{(IV}d\text{)},\\
(C^e) \quad & \{3s+(eR)^*\}(n-4)+\{3s(n-4)+(eR)_1\}-(3s+e)(n-4) && = A_9+C_4 && \text{IV}eB\text{)},\\
(C^w) \quad & \{f_2+(nw)_1\}(n-3)+\{(nw')_1(n-3)+(wR)_1\}-\{w+(nw)_1\}(n-3) && = 2A_2+A_6 && \text{(IV}w\text{)},\\
(C^{w'}) \quad & \{f_2+(nw)_1\}(n-3)+\{(nw)_1(n-3)+(w'R)_1\}-\{w'+(nw)_1\}(n-3) && = 2A_2+A_7 && \text{(IV}w'\text{)},\\
(C^{x'}) \quad & \{(x'R)^*+(nx')_1\}(n-3)+\{(nx')_1(n-3)+(x'R)_1\}-\{x'+(nx')_1\}(n-3) && = A_{14}+C_5 && \text{(IV}x'\text{)},\\
(C^{x''}) \quad & \{(x'R)^*+(nx')_1\}(n-3)+\{(nx')_1(n-3)+(x''R)_1\}-\{x''+(nx')_1\}\ n-3) && = A_{14}+C_6 && \text{(IV}x''\text{)},
\end{aligned}$$

where the new symbols denote the numbers of the following entities :—

C_1 Sets of three coplanar and concurrent generators of $F^{m+n}(a, e)$.

C_2 Points on the double surface at which one generator is three-point tangent to the sheet containing the other.

C_3 Points on the double surface at which each generator is tangent to the sheet containing the other (c, d).

C_4 Rays in tritangent planes of scrolls S^n which have a contact with the other sheet of F^{m+n} at one of the points where they cut its double surface (e, c), (e, d).

$C_5(C_6)$ Lines of the envelope $N_{c,d}$, which are tangent to the curve C^n, of which the corresponding S^n contains the significant ray of the point where the line meets C^c, (C^d), $(x''c)(x', d)\{(x'c), x'', d)\}$.

For the remaining curves we write down the formulae obtained by associating the residual points (B') in pairs. The corresponding formulae for the curves just considered are required in § 20.

$$(C^R) \quad 2\{6T_1+2\psi(n-3)+\epsilon\}(n-4)-(2R+2\psi_1)(n-3)(n-4) = (dR)_1+2(aR)_1 \quad (\text{IV}R),$$

$$(C^\theta) \quad 2\{f_1+f_2+(n\theta)_1\}(n-3)-\{\theta+(n\theta)_1\}(n-2)(n-3) = (d\theta)_1+2(a\theta)_1 \quad (\text{IV}\theta),$$

$$(C^{\bar{y}}) \quad 2\{2\psi(n-3)+\nu_R\}(n-4)-\{\bar{y}+2\psi\}(n-3)(n-4) = \nu_d+2\nu_a \quad (\text{IV}\bar{y}B),$$

$$(C^{\bar{x}}) \quad 2\{N(n-2)+m_R\}(n-3)-(x+m+N)(n-2)(n-3) = m_d+2m_a \quad (\text{IV}\bar{x}),$$

$$(C^T) \quad 2\{3(N-n)+(TR)_1\}(n-3)-\{T+3(N-n)\}(n-2)(n-3) = (dT)_1+2(aT)_1 \quad (\text{IV}T).$$

Formulae V. On each scroll S^α there is a curve, which we may call the *focal curve*, through each of whose points a ray of Γ passes near to the generator of the scroll, though in the cases of S^b and S^w the loci of residual foci (met twice by each generator) is thereby referred to. The orders of these curves are readily written down, and, by the usual formula, the number of intersections with the significant curve (B) or the residual curve (B') is obtained. (The cases correspond to those for *Formulae* I.) This gives the results:

$$(C^{\bar{y}}) \quad 3(w+\psi)+(6\psi+\nu_\theta)-3(\bar{y}+2\psi) = 2N_b+N_{w,w} \quad (\text{V}\bar{y}A),$$

$$(C^a) \quad 3(s+T_1)+\{9s+(a\theta)_1\}-3(a+3s) = 2A_4+A_2 \quad (\text{V}a),$$

$$(C^b) \quad 2(f_1+P)+\{2P+(b\theta)_1\}-2(b+P) = A_3 \quad (\text{V}b),$$

$$(C^c) \quad 3\{\epsilon+(nd)_1\}+\{3(nd)_1+(c\theta)_1\}-3\{c+(nd)_1\} = A_4+A_7+B_2 \quad (\text{V}c),$$

$$(C^d) \quad 3\{\epsilon+(nd)_1\}+\{3(nd)_1+(d\theta)_1\}-3\{d+(nd)_1\} = B_2+A_6 \quad (\text{V}d),$$

$$(C^e) \quad 3\{3s+(eR)^*\}+2\{9e+(e\theta)_1\}-6\{e+3s\} = A_8+2B_1 \quad (\text{V}e),$$

$$(C^w) \quad 2\{f_2+(nw)_1\}+\{2(nw)_1+(w\theta)_1\}-2\{w+(nx)_1\} = 2A_3+A_{10} \quad (\text{V}w),$$

$$(C^{w'}) \quad 3\{f_2+(nw)_1\}+\{3(nw)_1+(w'\theta)_1\}-3\{w'+(nw)_1\} = 2A_{11}+2A_3 \quad (\text{V}w'),$$

$$(C_{x'}) \quad 3\{(x'R)^*+(nx')_1\}+\{3(nx')_1+(x'\theta)_1\}-3\{x'+(nx')_1\} = B_3+A_{12} \quad (\text{V}x'),$$

$$(C^{x''}) \quad 3\{(x'R)^*+(nx')_1\}+\{3(nx')_1+(x''\theta)_1\}-3\{x''+(nx')_1\} = B_3+A_{13}+B_1 \quad (\text{V}x''),$$

$$(C^{\bar{y}}) \quad 3\{\nu_R+2\psi(n-3)\}+\{6\psi+\nu_\theta\}(n-3)-3(n-3)(\bar{y}+2\psi) = \nu_w+\nu_b \quad (\text{V}\bar{y}B),$$

$$(C^R) \quad 3\{2\psi(n-3)+6T_1+\epsilon\}+\{6\psi+f_1+f_2\}(n-3)-3(2R+2\psi)(n-3) = (bR)_1+(wR)_1 \quad (\text{V}R),$$

$$(C^\theta) \quad 2\{(n\theta)_1(n-2)+f_1+f_2\}+\{3\phi_1+2(n\theta)_1\}(n-2)-2\{\theta+(n\theta)_1\}(n-2) = (b\theta)_1+(w\theta)_1 \quad (\text{V}\theta),$$

$$(C^{\bar{x}}) \quad 3\{N(n-2)+m_R\}+\{m+3N\}(n-2)-3(x+m+N)(n-2) = N_b+m_w \quad (\text{V}\bar{x}),$$

$$(C^T) \quad 3\{3(N-n)(n-2)+(TR)_1\}+\{9(N-n)+(T\theta)_1\}(n-2)-3\{T+3(N-n)\}(n-2) = (wT)_1+B_3 \quad (\text{V}T).$$

Formulae VI. Let Q be a fixed point in ϖ and let g be a line through it, cutting C^α in points (P). The significant ray (or rays) for each such

point defines with ϖ a space, and the S^n in this space is cut by g in points (P'), other than P, and which we associate with P as g varies. They lie on a curve of order $\alpha(n-1)+m(n\alpha)_1$ (as is seen from its intersections with any line g through Q). The correspondences P, P' give† for $\alpha = \bar{y}, a, b, c, d, e, w, w', x', x'', T$ and θ,

$$\alpha(n-1)+\{\alpha(n-1)+m(n\alpha)_1\}-\alpha(n-1) = m_\alpha+(\alpha R)_1, \qquad \text{(VI}\alpha\text{)}$$

while, for the curve C^R,

$$R(n-2)+m\psi = 3T_1+m_R. \qquad \text{(VI}R\text{)}$$

Analogous arguments for the curves $C^{\bar{y}}$ and $C^{\bar{x}}$ taking as Q the respective fixed points used to construct these curves, give

$$\bar{y}(n-2)+2m\psi = N_{x',x''}+\nu_R, \qquad \text{(VI}\bar{y}\beta\text{)}$$

$$(x+m)(n-2)+mN = N_T+m_R. \qquad \text{(VI}\bar{x}\text{)}$$

Formulae VII. Taking the same figure we may associate the planes defined by the lines g and the significant rays through the points (P) on it with those defined by each g and the rays through the points (P') on the same line and in the same space. Using the correspondence principle on a plane of cross section, the curves $\alpha = \bar{y}, a, b, c, d, e, \theta, w, w', x', x''$ and T give

$$\alpha(n-1)+(\alpha n)_1(n-1) = m_\alpha+\nu_\alpha. \qquad \text{(VII}\alpha\text{)}$$

Further (as in VI),

$$(2R+2\psi)(n-2) = m_R+\nu_R, \qquad \text{(VII}R\text{)}$$

$$(y+\psi)(n-2) = N_{x',x''}+3N_e, \qquad \text{(VII}\bar{y}B\text{)}$$

$$(N+x)(n-2) = N_{x',x''}+N_T. \qquad \text{(VII}\bar{x}\text{)}$$

Formulae VIII. Analogous relations can be obtained, using, in place of the pencil (Q, ϖ), one of the envelopes with special properties. If (N_α) be the envelope chosen, one of its lines cuts C^α in a set of k (say) points, and the significant rays at such a set (P) define a space, the S^n of which cuts the line chosen in points (P') other than the points of the set (P). Associating the (P) and (P') as g varies gives the relation sought. The order n_α of the locus of the (P') is found as follows. Let l be a fixed line in ϖ. The N_α lines through a point A on l define N_α spaces through ϖ, and the C^n for these spaces cut l in $N_\alpha . n$ points B

† For $\alpha = \bar{y}$ the point Q is here distinct from the fixed point used in the construction of the curve. Note that $m_{\bar{y}} = \nu_{\bar{x}}$, $m_b = N^b$, $m_e = N_e$, $m_T = 2N_T$.

which we associate with A. Conversely, through a given point B on l, there pass m rays of Γ, and hence we get $m(n\alpha)_1 k'$ points A, where k' is a numerical factor varying from case to case. The coincidences A, B arise from the points where l cuts C^α and where it cuts the locus whose order is sought. Thus for $\alpha = a$, b, c, e and T,

$$n_\alpha = N_\alpha n + k' m(\alpha n)_1 - \alpha,$$

where $k' = 1$, except for $\alpha = e$ when $k' = \frac{1}{3}$. For the envelope (ω) in connexion with the curve C^R we have, however, $n_\omega = \omega n + m\psi - 2R$.

When, however, the envelope arises from two curves, say, the case of $(N_{\alpha,\beta})$, the points (P') are the intersections residual to the points on both curves, and thus for α, $\beta = c$, d; w, w'; x', x'' we have, for the order of their locus,

$$n_{\alpha,\beta} = N_{\alpha,\beta} n + m(\alpha n)_1 - \alpha - \beta.$$

Each case gives rise to two relations according as we associate the points (P') with points on C^α or on C^β. The results are :

(N_a)	$a(n-2) + 2n_a - 2N_a(n-2)$	$= A_{14} + (aR)_1$	(VIIIa),
(N_b)	$b(n-2) + n_b - N_b(n-2)$	$= B_8 + (bR)_1$	(VIIIb),
$(N_{c,d})$	$c(n-2) + n_{c,d} - N_{c,d}(n-2)$	$= C_5 + (cR)_1$	(VIIIc, A),
	$d(n-2) + n_{c,d} - N_{c,d}(n-2)$	$= C_6 + (dR)_1$	(VIIId),
(N_c)	$c(n-2) + n_c - N_c(n-2)$	$= D_1 + (cR)_1$	(VIIIc, B),
(N_e)	$e(n-3) + 3n_e - 3N_e(n-3)$	$= D_2 + (eR)_1$	(VIIIe),
$(N_{w,w'})$	$w(n-2) + n_{w,w'} - N_{w,w'}(n-2)$	$= A_{13} + (wR)_1$	(VIIIw),
	$w'(n-2) + n_{w,w'} - N_{w,w'}(n-2)$	$= A_{12} + (w'R)_1$	(VIIIw'),
(ω)	$R(n-2) + n_\omega - \omega(n-2)$	$= 3T_1 + (x'R)^*$	(VIIIω),
$(N_{x',x''})$	$x'(n-3) + n_{x',x''} - N_{x',x''}(n-3)$	$= D_3 + (x'R)_1$	(VIIIx'),
	$x''(n-3) + n_{x',x''} - N_{x',x''}(n-3)$	$= D_4 + (x''R)_1$	(VIIIx''),
(N_T)	$T(n-3) + n_T - N_T(n-3)$	$= D_5 + (TR)_1$	(VIIIT).

In these formulae D_1, ..., D_5 denote the numbers of the following entities :

D_1 Points on the curve C^c at which the tangent to the C^n associated with the significant ray has inflexional contact $[=(cT)_1]$.

D_2 Points on the curve C^e at which the line of the envelope N_e connected with the significant ray is tangent to the corresponding $C^n.(e, x')$, (e, x'').

$D_3(D_4)$ Bitangent planes of scrolls S^n whose sections on ϖ are respectively inflexional tangent to C^n at one of (or tangent at both of) the points in which ϖ is cut by the two generators in the plane.

D_5 Curves C^n with tangents of four-point contact.

Formulae IX. Taking the same figure the arguments analogous to those giving the *Formulae* VII give

$$(N_a+3s)(n-2) = A_9+A_{14} \quad (\text{IX}a), \qquad \{N_{w,w'}+(nw)_1\}(n-2) = A_{12}+A_{13}+A_8 \quad (\text{IX}w),$$
$$(N_b+P)(n-2) = B_1+B_3 \quad (\text{IX}b), \qquad \{\omega+\psi\}(n-2) = (x'R)^*+(eR)^* \quad (\text{IX}\omega),$$
$$\{N_{c,d}+(nd)_1\}(n-2) = C_4+C_5+C_6 \quad (\text{IX}d), \qquad \{N_{x',x''}+(nx')_1\}(n-3) = D_2+D_3+D_4 \quad (\text{IX}x'),$$
$$\{N_c+(nd)_1\}(n-2) = D_1+E_1 \quad (\text{IX}c), \qquad \{N_T+3N-3n\}(n-3) = D_8+D_5 \quad (\text{IX}T),$$
$$\{N_e+s\}(n-2) = D_2+E_2 \quad (\text{IX}e),$$

E_1 denotes the number of points on C^c whose significant rays are met by rays of Γ also meeting the corresponding line of the envelope N_T (but which are not significant rays for the respective corresponding points on C^d); and E_2 denotes *the number of sets of four coplanar rays meeting* ϖ.

Formulae X. Corresponding formulae may be developed using the envelope of nodal tangents of class m_R. The order (say n_R) of the curve of points (P') is $m_R n+2m\psi-2R$, and thus

$$2R(n-3)+n_R-m_R(n-3) = (TR)_1+6T_1. \qquad (\text{X } 1)$$

In the analogue of *Formulae* IX there are three sets of planes which may be made to correspond in pairs, giving

$$m_R+2\psi = f_1+(x'R)^*, \qquad (\text{X } 2)$$

$$(m_R+2\psi)(n-3) = (TR)_1+(x''R)_1 = 2F_1+F_2, \qquad (\text{X } 3, 4)$$

where F_1 is *the number of curves* C^n *with tacnodes*, and F_2 is the number of lines of the envelope ν_R which coincide with lines of m_R at the same points of C^R, excluding the points (ϵ).

Analogous arguments using the envelopes (m_a) for $a = a,\ c,\ d,\ w,\ w',\ x'$ and θ give, first, the values of the $(aT)_1$ in terms of the corresponding $(aR)_1$ and known numbers; and then in terms of the $(aT)_1$ the numbers of points on each curve C^a at which a line of the envelope (ν_a) is tangent to the curve C^n corresponding to the significant ray. Both sets of numbers are thereby eventually known. We can apply the method yet again to the envelope (ν_R) for which the curve of points (P') is of order (say) $\mu'_R = \nu_R n+2\psi m(m-3)-2R-\mu_R$. Omitting details, the correspondences give

$$2R(n-3)+\mu'_R-\nu_R(n-3) = (x''R)_1+3T_1(n-4), \qquad (\text{X } 5)$$

$$\mu_R(n-3)+\mu'_R-\nu_R(n-3) = (x'R)_1+2F_3, \qquad (\text{X } 6)$$

$$\nu_R+2\psi(n-3) = f_1(n-3)+2(eR)^*+\epsilon, \qquad (\text{X } 7)$$

$$\{\nu_R+2\psi(n-3)\}(n-3) = 2(eR)_1+(x'R)_1+(x''R)_1 = F_2+2F_4, \qquad (\text{X } 8, 9)$$

where F_3 is *the number of double points (not on ϖ) of the scroll of order $2(R+\psi)$ of pairs of rays concurrent on and cospacial with ϖ*; and F_4 is the number of scrolls S^n having two rays in respective bitangent planes concurrent on the plane ϖ. The analogous formulae for the remaining curves supply verifications and unimportant numbers only. (Cf. § 20.) In each of the arguments of VII–X we may consider, in place of the points (P') or the corresponding planes, the points where the lines of the envelope under consideration cut an associated curve other than in the points (P), or the corresponding planes, giving results required in § 20.

XI. *Special formulae.* (i) Returning, in the first place, to the figure of § 15 . 1, let the rays (other than AB) meeting the lines BC cut it in points B'', and ϖ in points C'' on a curve of order γ (say). The points B'' are on a curve of order $\mu_R+n(n-2)(n-3)$. The class of the envelope CC'' is $\mu_{\bar{y}}$. If z denote the number of pairs of rays meeting a line g in ϖ and both met by a third ray meeting ϖ, we have $\gamma = 2z+c$. Associating the points (B, B'') and (C, C'') respectively give

$$\{2R+n(n-2)\}(n-3)+\{\mu_R+n(n-2)(n-3)\}-\{\mu+n(n-2)\}(n-3) = c+2a, \qquad \text{(XI 1)}$$

$$\mu(n-3)+(2z+c)-2\mu_{\bar{y}} = \mu_b+\mu_R. \qquad \text{(XI 2)}$$

(ii) Now consider the curve $C^{\bar{y}}$ associated with a point Q, and notice that from the lines of the envelope $\nu_{\bar{y}}$ passing through Q we have

$$\nu_{\bar{y}} = 6N_e+m(n-2)(n-3). \qquad \text{(XI 3)}$$

(iii) Again, let A be any point in ϖ, and let the generators of F^{m+n} through it cut the double surface in points (B), and let the other generators through these points cut ϖ in points (C), and let us associate in pairs the points (C) arising from the same point A, as A varies over the whole plane. We find

$$2m(n-2)(n-3)+z = 2(eR)^*+N_e. \qquad \text{(XI 4)}$$

(iv) Associating the points (A) and (C') for the curves, C^{ν}, C^{b}, $C^{\bar{x}}$, and C^{θ} give

$$\bar{y}(n-3)+\mu_{\bar{y}}-\nu_{\bar{y}} = \nu_R+\nu_b, \qquad \text{(XI}\bar{y}\text{)}$$

$$b(n-3)+\mu_b-\nu_b = (bR)_1+B_2+2G, \qquad \text{(XI}b\text{)}$$

$$(x+m)(n-2)+\mu_{\bar{x}}-\nu_x = m_R+N_b, \qquad \text{(XI}\bar{x}\text{)}$$

$$\theta(n-3)+\mu_\theta-\nu_\theta = f_1+f_2+(bR)_1. \qquad \text{(XI}\theta\text{)}$$

G is *the number of doubly-focal generators of* F^{m+n}, *i.e. generators met by two consecutive generators.* (See § 6.) The remaining formulae of this type are required in § 20. Taking, again, the figure for C^a, and associating in pairs A, B, C for the same point B, we obtain

$$4(3s+a)-6s = 6C_1+A_4+2(3s+a). \qquad \text{(XI 5)}$$

For the other curves we may associate the planes ABC and $AB'C$ for the points B, B' on the same significant ray, but these give verifications only. The association of planes $AB'C'$ (in pairs) is required in § 20. Other methods giving, for the most part, verifications and minor numbers only are the associations of lines of the various envelopes m_a, ν_a, and N_a (or analogous envelopes) in pairs, or of pairs of lines of one of them, arising from the same points of C^a, and the use of the correspondence theorem in dual form. The only formulae obtained in this way and required for the purpose of the present calculations are given by associating the lines of N_c and $N_{c,d}$ at the points of C^c, from which

$$N_c+N_{c,d}-c = B_1, \qquad \text{(XI 6)}$$

and by associating those of ν_R and m_R at points of C^R, from which

$$\nu_R+m_R(n-3)-2R(n-3) = F_2 = (bR)_1+(x''R)_1 \qquad \text{(XI 7, 8)}$$

(v) We can also consider the scrolls (BC) or $(B'C')$ of the various figures and the points B'' (say) where their generators are met by other generators of F^{m+n}, and construct the formulae strictly analogous to those of I, ..., X (Cf. § 20).

19. From the Formulae I to XI we can read off, in turn, the values of the unknown symbols, which, however, we do not propose to record explicitly both for brevity and to avoid systematic error. The following is, however, the order in which they are obtainable. Since Formulae I always connect $(a\theta)_1$ and $(aR)_1$ and VI, VII in turn connect m_a and ν_a with $(aR)_1$, we name in the list only the first occurring of each of these sets of four members :

m_R(VIR), ν_R(VIIR), ν_θ(I$\bar{y}$), $(bR)_1$(VIb), μ_θ(XIθ), $w'(40)$, $a(35)$, $d(30)$, $e(31)$, μ_R[§ 16 (i)], c(XI 1), $(x'R)^*$(VIIIω), $(eR)^*$(IXω, X 7), N_T(VI$\bar{x}$), $N_{x',x''}$(VII$\bar{x}$), m_θ(I$\bar{x}$, VIθ), $N_{w,w'}$(V$\bar{y}A$), N_a(II$\bar{y}A$), N_e(VII$\bar{y}B$), B_1(IIIe), C_1(IVeA), A_4(IIIa, XI 5), A_3(IIb), $(wR)_1$(VR), B_2(IVb), $\nu_{\bar{x}} = m_{\bar{y}}$(VI$\bar{y}$), $\mu_{\bar{x}}$(XI$\bar{x}$), $\nu_{\bar{y}}$(XI 3), $\mu_{\bar{y}}$(XI$\bar{y}$), $N_{c,d}$(IIId, IV$\bar{y}A$), N_c(XI 6), z and γ(XI 4), μ_b(XI 2), ν_a(IIyB), ν_c(VIIc), $(cR)_1$(VIc), ν_c(IV$\bar{y}B$, IVR), A_2(Va), A_1(IIa), A_5(IVa), C_2(IVd), C_3(IVc), A_6(IVw, Vd), A_7(Vc, IIc), $(w'R)_1$(IIR, IVw'),

A_{14}(VIIIa), A_9(IXa), A_8(IIe), $(e\theta)_1$(Ve), A_{11}(Vw'), C_4(IVeB), A_{10}(Vw), C_5(VIIIc), C_6(VIIId), D_1(VIIIc, B), x'(44), x''(43), B_3(IIIx', IXb, VIIIb), A_{13}(VIIIw), A_{12}(VIIIw', II$\bar{x}$, IXw), E_1(IXe), D_2(VIIIe), E_2(IXe), $(x'\theta)_1$(Vx'), $(x''\theta)_1$(Vx''), D_3(VIIIx'), D_4(VIIIx''), D_5(IXT), $(TR)_1$(VIIIT, X3), ν_T(VIIIT), F_3(X6), F_2(XI7), F_1(X9), F_1'(X4), C(XIb).

In each case we have named in the bracket following each symbol the equation, or equations, from which it is immediately derived.

20. The arguments which have been mentioned but not discussed in detail give (as we have verified) all the remaining intersection symbols, in particular the numbers of those points $(\alpha\beta)_1$ for which the significant ray is the same for each, but not the significant points thereon, as soon as the values of the remaining μ_a are known. They also give the intersections of the curve C^R, etc., and the curves (μ_a), and also the numbers of double points on each of the original curves, but for the sake of brevity we do not propose to record these calculations. The values of the μ_a are found as follows :

(i) Taking the figure of § 18, XI (i), and associating respectively the points (B'') and the planes $(CB''\,C'')$ in pairs the following relations are obtained :

$$2\{\mu_R+n(n-2)(n-3)\}(n-4)-2\{\mu+n(n-2)\}\binom{n-3}{2}=\mu_d+2\mu_a, \quad (47)$$

$$2\{\mu_{\bar{y}}+n(n-2)(n-3)\}(n-4)-2n(n-2)\binom{n-3}{2}-2\{\mu+n(n-2)\}\binom{n-3}{2}$$
$$=\mu_d+2\mu_e, \quad (48)$$

and, from the residual scroll for the curve (B''), (cf. § 18, II),

$$\{\mu_R+n(n-2)(n-3)\}(m+n)-\{\mu+n(n-2)\}(n-3)-\{\gamma+n(n-2)(n-3)\}$$
$$=\mu_w+\mu_{w'}+2\mu_a. \quad (49)$$

The consideration of the intersection of the focal curve of the scroll $(B''\,C'')$ and the curve (B'') gives

$$3\{\mu_R+n(n-2)(n-3)\}+\{\mu_\theta+3n(n-2)\}(n-3)-3\{\mu+n(n-2)\}(n-3)$$
$$=\mu_w+\mu_e. \quad (50)$$

Now let the rays $(B''\,C'')$ be met, in their turn, by rays $B'''\,C'''$ meeting ϖ, the points B''' being the points of intersection with the lines $B''\,C''$, and lying on a curve of order $n(n-2)\ (n-3)^2+(\gamma R)_1$, where $(\gamma R)_1$ is to

be interpreted as usual. Associating points B'', B''' on the same ray, we have

$$\{\mu_R+n(n-2)(n-3)\}(n-3)+\{n(n-2)(n-3)^2+(\gamma R)_1\}$$
$$-\{\gamma+n(n-2)(n-3)\}(n-3)=\mu_c+2\mu_a. \quad (51)$$

(ii) We now introduce the curve C^a of order $2z+m(n-2)(n-3)$, the locus of the pairs of points on lines of a pencil in which they are met by pairs of rays each met by the same third ray also meeting ϖ (say, in the point C). Let a pair of points of C^a on a line of the pencil be denoted by A_1 and A_2, and the points where the respective rays through them are met by the third ray by B_1 and B_2. By associating the pairs of points (C) in the figure of § 15 . 1, the locus of points (C) of the present figure is seen to be of order $u'=\mu(n-3)-\frac{1}{2}d$.

Aliter. Associating the points A_1, A_2, and B_1, B_2 respectively as the line varies give respectively

$$N_c=2u-2(nu)_1-(eR)^*=2\{2(nu)_1+\nu_R\}-2\{u'+(nu)_1\}-2N_a,$$

so that either gives the value of $(nu)_1$; or, preferably, we may consider the correspondence of pairs (X, Y) on a curve C^a such that the corresponding generators are met by the same third generator. The united points of the correspondence arise from the $nN+4n-6N$ generators tangent to the double curve, and hence

$$2(nu)_1=2(n-2)(n-3)n-nN-4n+6N.$$

(iii) From the argument of § 18, VI, applied to the curve C^a (using the pencil proper to the curve) we have

$$u(n-2)+2\{z(n-2)+m(nu)_1-u\}-2z(n-2)=N_{l',l''}+(uR)_1, \quad (i)$$

where $N_{l',l''}$ is the class of *the envelope formed by tangents to curves* C^n, *such that the generator at the point of contact and that at a point of simple intersection are met by the same third generator*, the loci of these points being respectively the curves $C^{l''}$, $C^{l'}$ of orders l'', l'. The value of (nl'), or (nl''), is obtained by associating the tangent to a curve C^a at a point X on it with the lines XY arising from X in connexion with the corresponding space, the notation being as in (ii). This gives

$$2(nu)_1+N(n-2)(n-3)-n(n-2)(n-3)=(nl')_1+(nN+4n-6N). \quad (ii)$$

Now associate the planes defined by a line of $N_{l',l''}$ with the respective significant rays at its (significant) points on $C^{l'}$ and $C^{l''}$. We obtain

$$N_{l,l''}+(nl')_1=D_1+2D_2. \quad (iii)$$

Finally, taking the figure of § 18, XI (v) for the curve C^R, and associating the points A, C'', we have

$$2R(n-3)(n-4)+(\gamma R)_1-(uR)_1=(cR)_1+2(eR)_1. \qquad \text{(iv)}$$

The equations (i) ... (iv) give the value of $(\gamma R)_1$.

The orders l', l'', not here required, are readily obtained by applying the argument of § 18, VIII, to the envelope of lines AC'' in the figure of (i) of the present section. Finally, taking the same figure, and considering the envelope of class $\mu_{\bar{y}}$ of lines CC'', and using the arguments of § 18, VIII, and IX, we obtain

$$\mu(n-3)(n-2)+\{\mu_{\bar{y}}n+m(n-2)(n-3)-\mu(n-3)-\gamma\}-\mu_{\bar{y}}(n-2)=\mu_{x''}+\mu_R(n-3), \qquad (52)$$

$$\gamma(n-2)+\{\mu_{\bar{y}}n+m(n-2)(n-3)-\mu(n-3)-\gamma\}-\mu_{\bar{\gamma}}(n-2)=\mu_{x'}+(\gamma R)_1, \qquad (53)$$

$$\{\mu_{\bar{y}}+n(n-2)(n-3)\}(n-2)=\mu_{x'}+\mu_{x''}+2\mu_e. \qquad (54)$$

The equations (47–54) determine successively the values of $\mu_u \ldots \mu_{x''}$.

21. The methods of §§ 12–20 are applicable, with certain exceptions, to any form of ∞^2 lines with a directrix plane, the characteristics being suitably interpreted (Cf. § 10). The exceptions referred to are all those dealing with focal properties (other than those of the form itself, § 6), and generally all methods which assume the existence of rays other than those of the form. A number of the symbols used accordingly cease to have any interpretation [w, w', all (aw), or (a, w'), etc.]. We have, however, verified that the formulae still applicable are sufficient to determine the values of all symbols with interpretations in the present case.

ON THE THEORY OF SCHLÖMILCH SERIES

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1. *Introduction.*

1.1. The theory of Schlömilch series is concerned with an examination of the conditions under which it is possible to expand an arbitrary function $f(x)$ in a series of the form

$$\sum_{m=1}^{\infty} A_m = \sum_{m=1}^{\infty} \frac{a_m J_\nu(mx) + b_m \mathbf{H}_\nu(mx)}{(\frac{1}{2}mx)^\nu}, \tag{1.11}$$

where $\mathbf{H}_\nu(z)$ is Struve's function, and the coefficients a_m and b_m are given suitably in terms of $f(x)$*; for brevity, we shall always employ the symbol A_m to denote

$$\frac{a_m J_\nu(mx) + b_m \mathbf{H}_\nu(mx)}{(\frac{1}{2}mx)^\nu}.$$

The theorem given by Watson† concerning such an expansion is as follows:

Let ν be a number such that $-\frac{1}{2} < \nu < \frac{1}{2}$, and let $f(x)$ be defined arbitrarily in the interval $(-\pi, \pi)$ subject to the following conditions:

(i) *The function $h(x)$, defined by the equation*

$$h(x) = 2\nu f(x) + x f'(x),$$

exists, and is continuous, in the closed interval $(-\pi, \pi)$.

(ii) *The function $h(x)$ is of bounded variation in $(-\pi, \pi)$.*

(iii) *If ν is negative, the integral*

$$\int_0^{\Delta} \frac{d}{dx}\left[|x|^{2\nu}\{f(x) - f(0)\}\right] dx$$

* The best account of what is at present known of these series is given by G. N. Watson, *Theory of Bessel functions* (1922), Chap. XIX.

† *Ibid.*, 629.

is absolutely convergent when Δ is a (small) number either positive or negative.

Then $f(x)$ admits of the expansion

$$f(x) = \frac{\frac{1}{2}a_0}{\Gamma(\nu+1)} + \sum_{m=1}^{\infty} A_m,$$

where

$$a_m = \int_{-\pi}^{\pi}\int_0^{\frac{1}{2}\pi} \frac{\sec^{2\nu+1}\phi}{\Gamma(\frac{1}{2}-\nu)\,\Gamma(\frac{1}{2})} \frac{d}{d\phi}\left[\sin^{2\nu}\phi\left\{f(u\sin\phi)-f(0)\right\}\right]\cos mu\,d\phi\,du,$$

$$b_m = \int_{-\pi}^{\pi}\int_0^{\frac{1}{2}\pi} \frac{\sec^{2\nu+1}\phi}{\Gamma(\frac{1}{2}-\nu)\,\Gamma(\frac{1}{2})} \frac{d}{d\phi}\left[\sin^{2\nu}\phi\left\{f(u\sin\phi)-f(0)\right\}\right]\sin mu\,d\phi\,du,$$

when $m > 0$; the value of a_0 is obtained by inserting an additional term $2\Gamma(\nu+1)f(0)$ on the right in the first equation of the above system.

In connexion with this theorem the following questions arise:

(i) Is there not some simpler and more natural method of obtaining the coefficients a_m, b_m than that given by Watson?

(ii) Can we apply Schlömilch expansions to a more general class of functions than those considered hitherto?

The first of these questions has been answered by Mr. E. C. Titchmarsh; the second he very kindly proposed that I should investigate.

The method employed by Titchmarsh to obtain the coefficients is substantially equivalent to that given by Nielsen*; but it is to be noted that Nielsen's results are incorrect.

Let us assume the formal expansion

$$f(x) = \sum_{m=1}^{\infty} A_m. \tag{1.12}$$

We transform the series into a Fourier series by using the formulae (obtained by term-by-term integration)

$$\int_0^{\frac{1}{2}\pi} J_\nu(mx\sin\theta)\sin^{\nu+1}\theta\cos^{-2\nu}\theta\,d\theta = \frac{\Gamma(\frac{1}{2}-\nu)}{2^\nu\Gamma(\frac{1}{2})}\frac{\sin mx}{(mx)^{1-\nu}},$$

$$\int_0^{\frac{1}{2}\pi} \mathbf{H}_\nu(mx\sin\theta)\sin^{\nu+1}\theta\cos^{-2\nu}\theta\,d\theta = \frac{\Gamma(\frac{1}{2}-\nu)}{2^\nu\Gamma(\frac{1}{2})}\frac{(1-\cos mx)}{(mx)^{1-\nu}},$$

both of which are certainly valid for $|\nu|<\frac{1}{2}$. Put $x\sin\theta$ for x in (1.12), multiply by $\sin^{2\nu+1}\theta\cos^{-2\nu}\theta$, and integrate term-by-term from

* N. Nielsen, *Handbuch der Theorie der Cylinderfunktionen* (1904), 348

0 to $\frac{1}{2}\pi$. We obtain

$$\int_0^{\frac{1}{2}\pi} f(x\sin\theta)\sin^{2\nu+1}\theta\cos^{-2\nu}\theta\,d\theta = \frac{\Gamma(\frac{1}{2}-\nu)}{2^\nu\Gamma(\frac{1}{2})}\sum_{m=1}^{\infty}\frac{a_m\sin mx+b_m(1-\cos mx)}{(\frac{1}{2}mx)^\nu(mx)^{1-\nu}},$$

or

$$\frac{x\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\nu)}\int_0^{\frac{1}{2}\pi} f(x\sin\theta)\sin^{2\nu+1}\theta\cos^{-2\nu}\theta\,d\theta = \sum_{m=1}^{\infty}\frac{a_m\sin mx+b_m(1-\cos mx)}{m}. \tag{1.13}$$

From (1.13) we can find the coefficients a_m/m, b_m/m by the Fourier method; we thus get

$$a_m = \frac{m}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)}\int_{-\pi}^{\pi} t\phi(t)\sin mt\,dt, \tag{1.14}$$

$$b_m = -\frac{m}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)}\int_{-\pi}^{\pi} t\phi(t)\cos mt\,dt, \tag{1.15}$$

where
$$\phi(t) \equiv \int_0^{\frac{1}{2}\pi} f(t\sin\theta)\sin^{2\nu+1}\theta\cos^{-2\nu}\theta\,d\theta. \tag{1.16}$$

Now these formulae for the coefficients do not involve the derivative of $f(x)$. Hence it appeared quite possible that a theory of the series might be developed which would apply to a very wide class of functions; and in Theorem III of the present paper it is actually proved that, in ranges excluding the origin and the end-points π and $-\pi$, this class of functions is the class to which the theory of Fourier series applies.

The proof of Theorem III is on the lines of two papers by Young *, involving a free use of restricted Fourier series.

In Theorems IV and V, analogues of the Parseval and the Riesz-Fischer theorems for Schlömilch series are proved.

Theorem VI deals with the behaviour of the series at the origin, and use is made of the theory of fractional integration; in Theorem VII the Gibbs's phenomenon at the origin is discussed. Theorem VIII covers the end-points π, $-\pi$.

In the last section we first illustrate the theory of Theorem III by means of an example without in any way employing that theory. Then,

* W. H. Young, "On the connexion between Legendre series and Fourier series", *Proc. London Math. Soc.* (2), 18 (1919), 141–162, and "On series of Bessel functions", *Ibid.* (2), 18 (1919), 163–200.

finally, some new series of Schlömilch type are summed, and these are found to bear strict analogy to a well known and fundamental integral involving Bessel functions.

I am greatly indebted to Mr. E. C. Titchmarsh for many kindly suggestions in connexion with the paper.

1.2. Let $f(x)$ be any function integrable (in the sense of Lebesgue) over the closed interval $(-\pi, \pi)$, and let

$$\phi(t) \equiv \int_0^{\frac{1}{2}\pi} f(t \sin\theta) \sin^{2\nu+1}\theta \cos^{-2\nu}\theta \, d\theta.$$

Then if $|\nu| < \frac{1}{2}$, and a_m, b_m are given by the formulae (1.14), (1.15) respectively, the Schlömilch series to be considered is

$$\Sigma A_m.$$

Where limits of summation are omitted, it is to be understood that they are $m = 1$ to $m = \infty$; when the limits are not 1 and ∞, they will be explicitly stated.

2. *Preliminary theorems.*

2.1. THEOREM 1. *If*

$$\frac{\pi t \phi(t)}{\Gamma(\frac{1}{2}-\nu)\,\Gamma(\frac{1}{2})} \sim \tfrac{1}{2}C_0 + \Sigma m^{-1}(a_m \sin mt - b_m \cos mt),$$

then

$$\Sigma \frac{b_m}{m} = \tfrac{1}{2}C_0 = \frac{1}{2\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)} \int_{-\pi}^{\pi} t\phi(t)\,dt.$$

This amounts to stating that the specified series converges to zero at $t = 0$.

This is the case, by Dini's classical test for convergence of Fourier series*, since $\phi(t) - \phi(-t)$ is integrable over $(-\pi, \pi)$.

2.2. THEOREM II. THE RIEMANN - LEBESGUE THEOREM FOR SCHLÖMILCH SERIES. *The terms of the Schlömilch series tend to zero.*

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* E. W. Hobson, *Theory of functions of a real variable*, 2nd ed., II, 525–6.

We have to prove that a_m and b_m are each $o(m^{\nu+\frac{1}{2}})$. From the definition of a_m,

$$a_m = \frac{m}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)}\left[\int_0^\pi u^{2\nu+1} f(u)\,du \int_u^\pi \frac{\sin mt}{(t^2-u^2)^{\nu+\frac{1}{2}}}\,dt + \int_{-\pi}^0 u^{2\nu+1} f(u)\,du \int_{-\pi}^u \frac{\sin mt}{(t^2-u^2)^{\nu+\frac{1}{2}}}\,dt\right].$$

It is sufficient to deal with the first repeated integral only. We have

$$\begin{aligned}\int_u^\pi \frac{\sin mt}{(t^2-u^2)^{\nu+\frac{1}{2}}}\,dt &= \frac{1}{u^{2\nu}}\int_0^{(\pi/u)-1} \frac{\sin\{mu(1+z)\}}{(z^2+2z)^{\nu+\frac{1}{2}}}\,dz \\ &= \frac{1}{u^{2\nu}}\int_0^{(\pi/u)-1} \left\{\frac{1}{(z^2+2z)^{\nu+\frac{1}{2}}} - \frac{1}{(2z)^{\nu+\frac{1}{2}}}\right\} \sin\{mu(1+z)\}\,dz \\ &\quad + \frac{\sin mu}{u^{2\nu}}\int_0^{(\pi/u)-1} \frac{\cos muz}{(2z)^{\nu+\frac{1}{2}}}\,dz + \frac{\cos mu}{u^{2\nu}}\int_0^{(\pi/u)-1} \frac{\sin muz}{(2z)^{\nu+\frac{1}{2}}}\,dz \\ &= I_1+I_2+I_3, \text{ say.}\end{aligned}$$

Now $I_1 = O(1/mu^{2\nu+1})$, since

$$\frac{1}{(z^2+2z)^{\nu+\frac{1}{2}}} - \frac{1}{(2z)^{\nu+\frac{1}{2}}}$$

is of bounded variation over $(0, \infty)$.

Also, for $\pi-u < 1/m$,

$$I_2 = \frac{\sin mu}{u^{2\nu}}\cdot\frac{1}{(mu)^{\frac{1}{2}-\nu}}\int_0^{m(\pi-u)} \frac{\cos\theta}{(2\theta)^{\nu+\frac{1}{2}}}\,d\theta = O\left(\frac{1}{m^{\frac{1}{2}-\nu}.u^{\frac{1}{2}+\nu}}\right);$$

while, for $\pi-u \geqslant 1/m$,

$$\begin{aligned}I_2 &= \frac{\sin mu}{m^{\frac{1}{2}-\nu}.u^{\frac{1}{2}+\nu}}\left[\int_0^\infty \frac{\cos\theta}{(2\theta)^{\nu+\frac{1}{2}}}\,d\theta - \int_{m(\pi-u)}^\infty \frac{\cos\theta}{(2\theta)^{\nu+\frac{1}{2}}}\,d\theta\right] \\ &= \frac{A\sin mu}{m^{\frac{1}{2}-\nu}.u^{\frac{1}{2}+\nu}} + O\left\{\frac{1}{m^{\nu+\frac{1}{2}}(\pi-u)^{\nu+\frac{1}{2}}}\right\},\end{aligned}$$

where A is an absolute constant.

Similarly for I_3 in both cases.

Hence the integral considered is equal to

$$\begin{aligned}&m\int_0^\pi u^{2\nu+1} f(u)\,O\left(\frac{1}{mu^{2\nu+1}}\right)du + m\int_{\pi-(1/m)}^\pi u^{2\nu+1} f(u)\,O\left(\frac{1}{m^{\frac{1}{2}-\nu}u^{\frac{1}{2}+\nu}}\right)du \\ &\quad + m\int_0^{\pi-(1/m)} u^{2\nu+1} f(u)\,\frac{A\sin mu + B\cos mu}{m^{\frac{1}{2}-\nu}.u^{\frac{1}{2}+\nu}}\,du \\ &\quad + m\int_0^{\pi-(1/m)} u^{2\nu+1} f(u)\,O\left\{\frac{1}{m^{\nu+\frac{1}{2}}(\pi-u)^{\nu+\frac{1}{2}}}\right\}du \\ &= J_1+J_2+J_3+J_4, \text{ say.}\end{aligned}$$

Then $J_1 = O(1)$,

$$J_2 = O\left(m^{\nu+\frac{1}{2}}\int_{\pi-(1/m)}^{\pi} |f(u)|\, u^{\nu+\frac{1}{2}}\, du\right) = o(m^{\nu+\frac{1}{2}}),$$

$J_3 = o(m^{\nu+\frac{1}{2}})$, by the Riemann-Lebesgue theorem, and

$$\begin{aligned} J_4 &= m\left[\int_0^{\pi-\delta} + \int_{\pi-\delta}^{\pi-(1/m)}\right] u^{2\nu+1} f(u)\, O\left\{\frac{1}{(m^{\nu+\frac{1}{2}}(\pi-u)^{\nu+\frac{1}{2}}}\right\} du \\ &= m\int_0^{\pi-\delta} u^{\nu+\frac{1}{2}} f(u)\, O\left(\frac{1}{m\delta^{\nu+\frac{1}{2}}}\right) du + m\int_{\pi-\delta}^{\pi-(1/m)} u^{\nu+\frac{1}{2}} f(u)\, O\left(\frac{1}{m^{\frac{1}{2}-\nu}}\right) du \\ &= O(\delta^{-\nu-\frac{1}{2}}) + O\left(m^{\nu+\frac{1}{2}}\int_{\pi-\delta}^{\pi} u^{\nu+\frac{1}{2}} |f(u)|\, du\right) \\ &= o(m^{\nu+\frac{1}{2}}), \end{aligned}$$

if, for example, $\delta = m^{-\frac{1}{2}}$. Similarly b_m is $o(m^{\nu+\frac{1}{2}})$.

This proves the theorem.

2.3. We can see by an example that the above is the *best possible* theorem of its kind.

For consider the series

$$\Sigma \frac{\{\psi(m)\}^{\nu+\frac{1}{2}}}{m^2} \cdot \frac{J_\nu\{\psi(m)\, x\}}{\{\frac{1}{2}\psi(m)\, x\}^\nu},$$

where $\psi(m)$ is an integer which tends rapidly to infinity with m.

The series is a Schlömilch series (see Theorem V of the present paper), and, employing the asymptotic formula*

$$J_\nu(z) = \sqrt{\left(\frac{2}{\pi z}\right)} \cdot \cos(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + O(z^{-\frac{3}{2}}), \qquad (2.31)$$

we see that the given series is equal to

$$\frac{2^{\nu+\frac{1}{2}}}{\sqrt{\pi}} \Sigma \frac{\cos\{\psi(m)\, x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\}}{m^2} + \Sigma O\left\{\frac{1}{m^2\psi(m)}\right\},$$

for a fixed value of x.

The first series is a trigonometrical series whose coefficients B_n are given by

$$B_n = 0 \text{ when } n \text{ is not a value of } \psi(m),$$

$$= \frac{1}{m^2} \text{ when } n \text{ is a value of } \psi(m).$$

* G. N. Watson, *loc. cit.*, 199, (1).

Hence we can choose $\psi(m)$ to tend to infinity so rapidly that the terms of this series (and consequently those of the given series) tend to zero as slowly as we please; *i.e.* given any sequence tending slowly to zero, we can make the terms of the given series tend to zero still more slowly.

3. *Fundamental convergence theorem.*

Let the half-open intervals $\epsilon \leqslant x < \pi$, $-\pi < x \leqslant -\epsilon$ $(\epsilon > 0)$ be denoted by e_1, e_2.

THEOREM III. *At any point of e_1 or e_2, the Schlömilch series behaves with respect to convergence, divergence, oscillation, uniform or otherwise, precisely like the Fourier series of $f(x)$.*

We have first to prove three lemmas.

3.1. LEMMA I. *The series*

$$\Sigma V_m \equiv \Sigma (\tfrac{1}{2}m)^{-\nu} \int_0^x z^{-2\nu-1}\, dz \int_0^z t^{\nu+1} \{a_m J_\nu(mt) + b_m \mathbf{H}_\nu(mt)\}\, dt$$

converges uniformly to $\displaystyle\int_0^x z^{-2\nu-1}\, dz \int_0^z t^{2\nu+1} f(t)\, dt$

in the intervals e_1, e_2.

We have*

$$\begin{aligned}\Sigma V_m &= \Sigma (\tfrac{1}{2}m)^{-\nu} \int_0^x z^{-2\nu-1} \cdot \frac{z^{\nu+1}}{m} \{a_m J_{\nu+1}(mz) + b_m \mathbf{H}_{\nu+1}(mz)\}\, dz \\ &= -\Sigma (\tfrac{1}{2}m)^{-\nu} \cdot \frac{1}{m^2} \left[a_m \left\{ x^{-\nu} J_\nu(mx) - \frac{m^\nu}{2^\nu \Gamma(\nu+1)} \right\} + b_m x^{-\nu} \mathbf{H}_\nu(mx) \right] \\ &\qquad + \frac{x}{\Gamma(\nu+\frac{3}{2})\,\Gamma(\frac{1}{2})} \Sigma \frac{b_m}{m} \dagger \\ &= \frac{1}{\Gamma(\nu+1)} \Sigma \frac{a_m}{m^2} + \frac{C_0 x}{2\Gamma(\nu+\frac{3}{2})\,\Gamma(\frac{1}{2})} - \Sigma \frac{A_m}{m^2}, \qquad (3.11)\end{aligned}$$

by Theorem I.

Now we have the asymptotic formulae‡

$$Y_\nu(z) = \sqrt{\left(\frac{2}{\pi z}\right)} \cdot \sin(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + O(z^{-\frac{3}{2}}), \qquad (3.12)$$

$$\mathbf{H}_\nu(z) = Y_\nu(z) + \frac{(\frac{1}{2}z)^{\nu-1}}{\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})} + O(z^{\nu-3}), \qquad (3.13)$$

and also the formula (2.31) for $J_\nu(z)$.

* G. N. Watson, *loc. cit.*, 132, (1), and 329.

† *Ibid.*, 132, (2), and 329.

‡ *Ibid.*, 199, (2), and 333, (1).

It is clear that these asymptotic forms will not apply in the neighbourhood of the origin, so that $x = 0$ is an exceptional point.

Now a_m and b_m are each $o(m^{\nu+\frac{1}{2}})$, by Theorem II; hence in e_1 or e_2 (where the asymptotic forms for J_ν and $\mathbf{H}_\nu$ may be employed) we have

$$\Sigma V_m = \Sigma o(m^{\nu-\frac{3}{2}}) + \Sigma o(m^{-2}) + \Sigma o(m^{\nu-\frac{5}{2}}),$$

and each of these series is uniformly convergent, so that ΣV_m converges uniformly in e_1 or e_2. Now

$$\frac{\pi t\phi(t)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)} \sim \tfrac{1}{2}C_0 + \Sigma m^{-1}(a_m \sin mt - b_m \cos mt).$$

Integrating term-by-term, we have

$$\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\nu)} \int_0^x t\phi(t)\,dt = \tfrac{1}{2}C_0 x - \Sigma m^{-2}(a_m \cos mx + b_m \sin mx) + \Sigma \frac{a_m}{m^2},$$

and this is uniformly convergent.

Change x into $x \sin\theta$, multiply by $\cos^{2\nu}\theta$, and integrate with regard to θ from 0 to $\frac{1}{2}\pi$. We get

$$\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\nu)} \int_0^{\frac{1}{2}\pi} \cos^{2\nu}\theta\,d\theta \int_0^{x\sin\theta} t\phi(t)\,dt = \frac{C_0 x}{4\nu+2} + \frac{\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}{2\Gamma(\nu+1)} \Sigma \frac{a_m}{m^2}$$
$$- \tfrac{1}{2}\Gamma(\tfrac{1}{2})\,\Gamma(\nu+\tfrac{1}{2})\,\Sigma \frac{A_m}{m^2},$$

or
$$\frac{2}{\Gamma(\frac{1}{2}-\nu)\,\Gamma(\frac{1}{2}+\nu)} \int_0^{\frac{1}{2}\pi} \cos^{2\nu}\theta\,d\theta \int_0^{x\sin\theta} t\phi(t)\,dt = \frac{1}{\Gamma(\nu+1)} \Sigma \frac{a_m}{m^2}$$
$$+ \frac{C_0 x}{2\Gamma(\nu+\frac{3}{2})\,\Gamma(\frac{1}{2})} - \Sigma \frac{A_m}{m^2}. \quad (3.14)$$

Comparing (3.11) with (3.14), we see that the lemma is reduced to proving that

$$\frac{2}{\Gamma(\frac{1}{2}-\nu)\,\Gamma(\frac{1}{2}+\nu)} \int_0^{\frac{1}{2}\pi} \cos^{2\nu}\theta\,d\theta \int_0^{x\sin\theta} t\phi(t)\,dt = \int_0^x z^{-2\nu-1}\,dz \int_0^z t^{2\nu+1} f(t)\,dt. \quad (3.15)$$

Since
$$\phi(t) = \int_0^{\frac{1}{2}\pi} f(t\sin\chi)\sin^{2\nu+1}\chi\cos^{-2\nu}\chi\,d\chi,$$

the left-hand side of (3.15) becomes, when we write $t = z\sin\theta$,

$$\frac{2}{\Gamma(\frac{1}{2}-\nu)\,\Gamma(\frac{1}{2}+\nu)} \int_0^{\frac{1}{2}\pi} \cos^{2\nu}\theta\,\sin^2\theta\,d\theta \int_0^x z\,dz \int_0^{\frac{1}{2}\pi} f(z\sin\theta\sin\chi)\sin^{2\nu+1}\chi\cos^{-2\nu}\chi\,d\chi$$
$$= \frac{2}{\Gamma(\frac{1}{2}-\nu)\,\Gamma(\frac{1}{2}+\nu)} \int_0^x z\,dz \int_0^{\frac{1}{2}\pi} f(z\sin\psi)\sin^{2\nu+1}\psi\cos\psi\,d\psi \int_0^\theta \frac{\cos^{2\nu}\theta\sin\theta\,d\theta}{(\sin^2\theta-\sin^2\psi)^{\nu+\frac{1}{2}}},$$

where $\sin\theta\sin\chi=\sin\psi$; and this is equal to

$$\frac{2}{\Gamma(\frac{1}{2}-\nu)\,\Gamma(\frac{1}{2}+\nu)}\int_0^x z\,dz\int_0^{\frac{1}{2}\pi} f(z\sin\psi)\sin^{2\nu+1}\psi\cos\psi\,d\psi\int_\psi^{\frac{1}{2}\pi}\frac{\cos^{2\nu}\theta\sin\theta\,d\theta}{(\cos^2\psi-\cos^2\theta)^{\nu+\frac{1}{2}}}$$

$$=\frac{2}{\Gamma(\frac{1}{2}-\nu)\,\Gamma(\frac{1}{2}+\nu)}\int_0^x z\,dz\int_0^{\frac{1}{2}\pi} f(z\sin\psi)\sin^{2\nu+1}\psi\cos\psi\,d\psi\int_0^{\frac{1}{2}\pi}\tan^{2\nu}\omega\,d\omega,$$

where $\cos\psi\sin\omega=\cos\theta$; and this is equal to

$$\int_0^x z\,dz\int_0^{\frac{1}{2}\pi} f(z\sin\psi)\sin^{2\nu+1}\psi\cos\psi\,d\psi=\int_0^x z^{-2\nu+1}\,dz\int_0^z t^{2\nu+1}f(t)\,dt,$$

where $t=z\sin\psi$.

This proves the lemma.

3.2. LEMMA II. *Provided that ϵ is a suitably chosen positive number, the series*

$$\Sigma V_{m,\epsilon}\equiv\Sigma(\tfrac{1}{2}m)^{-\nu}\int_\epsilon^x z^{-2\nu-1}\,dz\int_\epsilon^z t^{\nu+1}\{a_m J_\nu(mt)+b_m\mathbf{H}_\nu(mt)\}\,dt$$

converges uniformly to

$$\int_\epsilon^x z^{-2\nu-1}\,dz\int_0^z t^{2\nu+1}f(t)\,dt-K\int_\epsilon^x z^{-2\nu-1}\,dz,$$

where K is independent of x.

We have, by Lemma I,

$$\Sigma V_{m,\epsilon}=\int_0^x z^{-2\nu-1}\,dz\int_0^z t^{2\nu+1}f(t)\,dt-\int_0^\epsilon z^{-2\nu-1}\,dz\int_0^z t^{2\nu+1}f(t)\,dt$$

$$-\Sigma(\tfrac{1}{2}m)^{-\nu}\int_\epsilon^x z^{-2\nu-1}\,dz\int_0^\epsilon t^{\nu+1}\{a_m J_\nu(mt)+b_m\mathbf{H}_\nu(mt)\}\,dt;\quad(3.21)$$

and, calling the latter sum S, we have

$$S=\Sigma(\tfrac{1}{2}m)^{-\nu}\int_\epsilon^x z^{-2\nu-1}.\frac{\epsilon^{\nu+1}}{m}\{a_m J_{\nu+1}(m\epsilon)+b_m\mathbf{H}_{\nu+1}(m\epsilon)\}\,dz$$

$$=\sqrt{\left(\frac{2}{\pi}\right)}.\epsilon^{\nu+\frac{1}{2}}\int_\epsilon^x z^{-2\nu-1}\,dz.$$

$$\times\left\{\Sigma\left[\frac{-a_m\sin(m\epsilon-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)+b_m\cos(m\epsilon-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)}{m^{\nu+\frac{1}{2}}}+O(m^{-2})\right]\right.$$

$$\left.+\frac{C_0\epsilon^{\nu+\frac{1}{2}}}{2^\nu\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{3}{2})}\right\},$$

by (2.31), (3.12), (3.13), and Theorem I.

Now the series

$$\Sigma \frac{-a_m \sin(m\epsilon - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) + b_m \cos(m\epsilon - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}{m^{\nu+\frac{1}{2}}} \qquad (3.22)$$

is a Fourier series, by the Riesz-Fischer theorem*. Moreover, the coefficients $a_m/m^{\nu+\frac{1}{2}}$, $b_m/m^{\nu+\frac{1}{2}}$ are each $o(1/m)$. Hence the series converges for almost all values of ϵ†. If, then, we choose ϵ so that it is not one of the exceptional points of the set of measure zero, the series (3.22) is convergent, and, by (3.21), the lemma stated then follows.

We shall henceforth always suppose ϵ to be so chosen.

3.3. DEFINITION‡. *A trigonometrical series $\Sigma(a_m \cos mx + b_m \sin mx)$ is said to be an ordinary restricted Fourier series, or ORF series, if it satisfies the conditions*

(1) $a_m = o(1)$, $b_m = o(1)$; *and*

(2) *the integrated series $\Sigma m^{-1}(a_m \sin mx - b_m \cos mx)$ converges in the open interval (α, β) contained in $(-\pi, \pi)$, or in each of a set of such open intervals, to a function $F(x)$, which is a Lebesgue integral. The function $dF(x)/dx$, defined almost everywhere in (α, β), is then said to be the function associated with the ORF series in the open interval $\alpha < x < \beta$.*

LEMMA III. *Let $S_1 \equiv \Sigma(C_m \cos mx + D_m \sin mx)$ be a trigonometrical series whose coefficients C_m, D_m tend to zero as $m \to \infty$, and let*

$$\int_\epsilon^z dy \int_\epsilon^y S_1 dx + Az + B,$$

where A and B are constants, converge uniformly to the integral of an integral in e_1 or e_2; then S_1 is an ORF series in these intervals§.

We have $$F(z) \equiv \int_\epsilon^z dy \int_\epsilon^y S_1 dx + Az + B, \text{ in } e_1 \text{ or } e_2,$$

where $F(z)$ is the integral of an integral.

* E. W. Hobson, *loc. cit.*, II, 577.

† *Ibid.*, 563.

‡ *Ibid.*, 686; see also W. H. Young, "On restricted Fourier series and the convergence of power series", *Proc. London Math. Soc.* (2), 17 (1918), 353–366.

§ It is here that we see the exceptional character of the points $x = \pm\pi$.

Now the once-integrated series

$$\Sigma m^{-1}(C_m \sin mx - D_m \cos mx) \qquad (3.31)$$

is a Fourier series, by the Riesz-Fischer theorem, and so it converges uniformly in any interval in which the function associated with the series is continuous. But

$$F'(z) = \int_\epsilon^z S_1 dx + A$$

is an integral, and is therefore continuous in e_1 or e_2, so that

$$\int_\epsilon^z S_1 dx + A$$

converges uniformly to an integral.

Also, with our properly selected ϵ,

$$\Sigma m^{-1}(C_m \sin m\epsilon - D_m \cos m\epsilon)$$

is convergent, as in Lemma II; hence the series (3.31) converges uniformly to an integral.

By the definition of an ORF series, the lemma then fellows.

PROOF OF THE THEOREM.

3.4. By the asymptotic forms (2.31), (3.12), (3.13), we have, in e_1 or e_2, using Theorem I,

$$\Sigma A_m = x^{-\nu-\frac{1}{2}}(S_1 + x^{-1}S_2 + x^{-2}S_3) + \frac{C_0}{x\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} + \Sigma O\left(\frac{1}{m^3 x^{\nu+\frac{7}{2}}}\right) + \Sigma O\left(\frac{b_m}{m^3 x^3}\right), \qquad (3.41)$$

where
$$S_1 = 2^\nu \sqrt{\left(\frac{2}{\pi}\right)} \Sigma \left(\frac{a_m \cos\psi_m + b_m \sin\psi_m}{m^{\nu+\frac{1}{2}}}\right),$$

$$S_2 = 2^\nu \sqrt{\left(\frac{2}{\pi}\right)} \cdot \tfrac{1}{2}(\nu^2 - \tfrac{1}{4}) \Sigma \frac{1}{m} \left(\frac{-a_m \sin\psi_m + b_m \cos\psi_m}{m^{\nu+\frac{1}{2}}}\right),$$

$$S_3 = -2^\nu \sqrt{\left(\frac{2}{\pi}\right)} \cdot \tfrac{1}{8}(\nu^2 - \tfrac{1}{4})(\nu^2 - \tfrac{9}{4}) \Sigma \frac{1}{m^2} \left(\frac{a_m \cos\psi_m + b_m \sin\psi_m}{m^{\nu+\frac{1}{2}}}\right),$$

and $\psi_m = mx - \frac{1}{2}\nu\pi - \frac{1}{4}\pi$; in the error term $\Sigma O(1/m^3 x^{\nu+\frac{7}{2}})$, we use the fact that a_m and b_m are each $o(m^{\nu+\frac{1}{2}})$.

The above formulae for S_1, S_2, and S_3 are symbolic; no convergence of the series involved is implied, but merely identity, when we replace the infinite series by the corresponding n-th partial summations.

3.5. *We now prove that S_1 is an ORF series in e_1 or e_2.*

By a theorem due to Young*, it follows from Lemma II that

$$\Sigma(\tfrac{1}{2}m)^{-\nu}\int_\epsilon^x dz\int_0^z t^{\frac{1}{2}}\{a_m J_\nu(mt)+b_m \mathbf{H}_\nu(mt)\}\,dt=\int_\epsilon^x dz\int_0^z t^{\nu+\frac{1}{2}}f(t)\,dt. \quad (3.51)$$

Also
$$I_1\equiv\int_0^\epsilon t^{\frac{1}{2}}\{a_m J_\nu(mt)+b_m \mathbf{H}_\nu(mt)\}\,dt$$
$$=\int_0^\epsilon t^{-\nu-\frac{1}{2}}.\,t^{\nu+1}\{a_m J_\nu(mt)+b_m \mathbf{H}_\nu(mt)\}\,dt$$
$$=\frac{\epsilon^{\frac{1}{2}}}{m}\{a_m J_{\nu+1}(m\epsilon)+b_m \mathbf{H}_{\nu+1}(m\epsilon)\}$$
$$+\frac{(\nu+\frac{1}{2})}{m}\int_0^\epsilon t^{-\frac{1}{2}}\{a_m J_{\nu+1}(mt)+b_m \mathbf{H}_{\nu+1}(mt)\}\,dt.$$

Now† $I_2=\int_0^\epsilon t^{-\frac{1}{2}}J_{\nu+1}(mt)\,dt=m^{-\frac{1}{2}}\int_0^{m\epsilon}u^{-\frac{1}{2}}J_{\nu+1}(u)\,du$

$$=m^{-\frac{1}{2}}\left[(\nu-\tfrac{1}{2})\,m\epsilon J_{\nu+1}(m\epsilon)\,S_{-\frac{3}{2},\nu}(m\epsilon)-m\epsilon J_\nu(m\epsilon)\,S_{-\frac{1}{2},\nu+1}(m\epsilon)\right],$$

where $S_{\mu,\nu}(z)$ is Lommel's function of the second kind.

Inserting the asymptotic formula for this function‡, we have

$$I_2=m^{-\frac{1}{2}}\left[(\nu-\tfrac{1}{2})(m\epsilon)^{-\frac{3}{2}}J_{\nu+1}(m\epsilon)-(m\epsilon)^{-\frac{1}{2}}J_\nu(m\epsilon)+O(m^{-2})\right].$$

Again§, $\mathbf{H}_\nu(z)=Y_\nu(z)+\dfrac{(\frac{1}{2}z)^{\nu-1}}{\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})}\displaystyle\int_0^\infty e^{-s}\left(1+\frac{s^2}{z^2}\right)^{\nu-\frac{1}{2}}ds,$

so that $I_3\equiv\int_0^\epsilon t^{-\frac{1}{2}}\mathbf{H}_{\nu+1}(mt)\,dt=I_4+I_5,$

where
$$I_4=\int_0^\epsilon t^{-\frac{1}{2}}Y_{\nu+1}(mt)\,dt$$
$$=m^{-\frac{1}{2}}\left[(\nu-\tfrac{1}{2})(m\epsilon)^{-\frac{3}{2}}Y_{\nu+1}(m\epsilon)-(m\epsilon)^{-\frac{1}{2}}Y_\nu(m\epsilon)+O(m^{-2})\right],$$

as for I_2‖; and

$$I_5=\frac{1}{2^\nu\,\Gamma(\nu+\frac{3}{2})\,\Gamma(\frac{1}{2})}\int_0^\epsilon t^{-\frac{1}{2}}.(mt)^\nu\int_0^\infty e^{-s}\left(1+\frac{s^2}{m^2t^2}\right)^{\nu+\frac{1}{2}}ds$$
$$=\frac{m^\nu\,\epsilon^{\nu+\frac{1}{2}}}{2^\nu(\nu+\frac{1}{2})\,\Gamma(\nu+\frac{3}{2})\,\Gamma(\frac{1}{2})}+O(m^{\nu-2}).$$

* W. H. Young, "On series of Bessel functions", *Proc. London Math. Soc.* (2), 18 (1919), 163-200; Lemma of §7.

† G. N. Watson, *loc. cit.*, 350, (3).

‡ *Ibid.*, 352.

§ *Ibid.*, 331, (4), putting $\beta=0$.

‖ See *Ibid.*, 350, (4).

Combining the results for I_1, I_2, I_3, I_4, I_5, we have

$$I_1 = \frac{\epsilon^{\frac{1}{2}}}{m}\{a_m J_{\nu+1}(m\epsilon)+b_m \mathbf{H}_{\nu+1}(m\epsilon)\}+O(m^{\nu-3})+O(m^{\nu-2}) + \frac{b_m m^{\nu-1}}{2^\nu \Gamma(\nu+\frac{3}{2})\,\Gamma(\frac{1}{2})}+O(m^{2\nu-5}),$$

where, in the error terms, we have employed the fact that a_m and b_m are each $o(m^{\nu+\frac{1}{2}})$. Hence

$$\Sigma(\tfrac{1}{2}m)^{-\nu}\int_\epsilon^x dz\int_0^\epsilon t^{\frac{1}{2}}\{a_m J_\nu(mt)+b_m \mathbf{H}_\nu(mt)\}\,dt = (x-\epsilon)\Big[\Sigma\frac{\epsilon^{\frac{1}{2}}\{a_m J_{\nu+1}(m\epsilon)+b_m \mathbf{H}_{\nu+1}(m\epsilon)\}}{m^{\nu+1}} + \frac{1}{\Gamma(\nu+\frac{3}{2})\,\Gamma(\frac{1}{2})}\Sigma\frac{b_m}{m}+\Sigma O(m^{-2})\Big].$$

With our properly chosen ϵ, the first series is convergent; and $\Sigma b_m/m$ converges by Theorem I.

Therefore

$$\Sigma(\tfrac{1}{2}m)^{-\nu}\int_\epsilon^x dz\int_0^\epsilon t^{\frac{1}{2}}\{a_m J_\nu(mt)+b_m \mathbf{H}_\nu(mt)\}\,dt = K(x-\epsilon), \qquad (3.52)$$

where K is independent of x.

Consequently, by (3.51) and (3.52),

$$\Sigma(\tfrac{1}{2}m)^{-\nu}\int_\epsilon^x dz\int_\epsilon^z t^{\frac{1}{2}}\{a_m J_\nu(mt)+b_m \mathbf{H}_\nu(mt)\}\,dt = \int_\epsilon^x dz\int_0^z t^{\nu+\frac{1}{2}}f(t)\,dt - Ax - B, \qquad (3.53)$$

where A and B are constants.

Now multiply both sides of (3.41) by $x^{\nu+\frac{1}{2}}$, and integrate with respect to x from ϵ to y; and then integrate with respect to y from ϵ to z; then, since in e_1 or e_2, S_3, $\Sigma O(1/m^3 x^{\nu+\frac{7}{2}})$ and $\Sigma O(b_m/m^3 x^3)$ all converge uniformly, it follows that, in these intervals,

$$\Sigma A_m - \frac{C_0}{x\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})} - x^{-\nu-\frac{1}{2}}S_3 - \Sigma O\left(\frac{1}{m^3 x^{\nu+\frac{7}{2}}}\right) - \Sigma O\left(\frac{b_m}{m^3 x^3}\right) = x^{-\nu-\frac{1}{2}}(S_1 + x^{-1}S_2)$$

converges uniformly to the integral of an integral $-Az-B$, when twice integrated term-by-term in the manner described above.

Now S_2 is, by the Riesz-Fischer theorem, the Fourier series of a function of integrable square. Hence, if $h(x)$ is the function associated with S_2, S_2 converges in mean to $h(x)$, and therefore

$$\int_\epsilon^y x^{-1} S_2 dx = \int_\epsilon^y x^{-1} h(x)\, dx.$$

Thus $\int_\epsilon^y x^{-1} S_2\, dx$ converges uniformly to an integral.

We have now proved that each of the series

$$\int_\epsilon^z dy \int_\epsilon^y (S_1 + x^{-1} S_2)\, dx, \quad \int_\epsilon^z dy \int_\epsilon^y x^{-1} S_2 dx$$

converges uniformly, to the integral of an integral $-Az-B$ in the case of the first, and to the integral of an integral in the case of the second. Consequently

$$\int_\epsilon^z dy \int_\epsilon^y S_1 dx$$

converges uniformly to $F(z)-Az-B$, where $F(z)$ is the integral of an integral.

Also the coefficients $a_m/m^{\nu+\frac{1}{2}}$, $b_m/m^{\nu+\frac{1}{2}}$ in S_1 are each $o(1)$.

Hence, by Lemma III, S_1 is an ORF series in e_1 or e_2.

3.6. It follows, from the definition of an ORF series and from the fact that, apart from a constant, S_2 is the series obtained by integrating S_1 term-by-term, that S_2 converges uniformly to an integral in e_1 or e_2.

3.7. We shall now prove that the sum of three of the series contained in (3.41), when multiplied by $x^{\nu+\frac{1}{2}}$, namely

$$x^{-2} S_3 + \Sigma O\left(\frac{1}{m^3 x^3}\right) + \Sigma O\left(\frac{b_m}{m^3 x^{\frac{5}{2}-\nu}}\right),$$

converges uniformly to an integral in e_1 or e_2.

(i) The series obtained by differentiating $x^{-2}S_3$ term-by-term, *i.e.*

$$-2x^{-3} S_3 - \tfrac{1}{4}(\nu^2 - \tfrac{9}{4})\, x^{-2} S_2,$$

is uniformly convergent, and hence $x^{-2}S_3$ converges uniformly to an integral in e_1 or e_2.

(ii) The term $O(1/m^3 x^3)$ is the remainder after three terms in the asymptotic expansion of

$$x^{\nu+\frac{1}{2}} \cdot \frac{a_m J_\nu(mx) + b_m Y_\nu(mx)}{(\frac{1}{2}mx)^\nu},$$

by (3 . 13) and (3 . 41).

Now the remainder after three terms in the asymptotic expansion of

$$H_\nu^{(1)}(mx) \equiv J_\nu(mx) + iY_\nu(mx)$$

is given by*

$$R_3(x) = \frac{1}{(2imx)^3} \cdot \frac{(\frac{1}{2}-\nu)_3}{2!} \left(\frac{2}{\pi mx}\right)^{\frac{1}{2}} \frac{e^{i(mx-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)}}{\Gamma(\nu+\frac{1}{2})}$$

$$\times \int_0^\infty e^{-u} u^{\nu+\frac{5}{2}}\, du \int_0^1 (1-t)^2 \left(1-\frac{ut}{2imx}\right)^{\nu-\frac{7}{2}} dt.$$

Now† $$|I| \equiv \left| \int_0^\infty e^{-u} u^{\nu+\frac{5}{2}}\, du \int_0^1 (1-t)^2 \left(1-\frac{ut}{2imx}\right)^{\nu-\frac{7}{2}} dt \right|$$

$$\leqslant K \int_0^\infty e^{-u} u^{\nu+\frac{5}{2}}\, du \int_0^1 (1-t)^2\, dt \leqslant M,$$

where K and M are constants, and

$$\frac{dI}{dx} = \frac{(\nu-\frac{7}{2})}{2imx^2} \int_0^\infty e^{-u} u^{\nu+\frac{7}{2}}\, du \int_0^1 t(1-t)^2 \left(1-\frac{ut}{2imx}\right)^{\nu-\frac{9}{2}} dt,$$

so that $$\left|\frac{dI}{dx}\right| = O\left(\frac{1}{mx^2}\right).$$

Hence $$\frac{d}{dx} R_3(x) = O(m^{-\frac{1}{2}}) \text{ in } e_1 \text{ or } e_2.$$

Thus the differential coefficient with respect to x of the remainder after three terms in the asymptotic expansion of

$$x^{\nu+\frac{1}{2}} \cdot \Sigma \frac{a_m J_\nu(mx) + b_m Y_\nu(mx)}{(\frac{1}{2}mx)^\nu}$$

is equal to $\Sigma o(m^{-2})$ in e_1 or e_2, since a_m and b_m are each $o(m^{\nu+\frac{1}{2}})$.

This is uniformly convergent, and hence the series under consideration, $\Sigma O(1/m^3 x^3)$, converges uniformly to an integral.

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* G. N. Watson, *loc. cit.*, 197.

† *Ibid.*, 197.

(iii) The term $O(b_m/m^3x^{\frac{3}{2}-\nu})$ arose out of the error term $O(z^{\nu-3})$ in (3 . 13); and, denoting this by $Q_\nu(z)$, we have

$$Q_\nu(z) = \mathbf{H}_\nu(z) - Y_\nu(z) - \frac{(\frac{1}{2}z)^{\nu-1}}{\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})}.$$

Now*

$$x^{\nu+\frac{1}{2}}.\frac{b_m Q_\nu(mx)}{(\frac{1}{2}mx)^\nu} = \frac{2b_m}{mx^{\frac{1}{2}-\nu}\,\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})}\left[\int_0^\infty e^{-u}\left(1+\frac{u^2}{m^2x^2}\right)^{\nu-\frac{1}{2}}du - 1\right]. \quad (3\,.\,71)$$

Also†

$$\left(1+\frac{u^2}{m^2x^2}\right)^{\nu-\frac{1}{2}} = 1+\frac{(\nu-\frac{1}{2})\,u^2}{m^2x^2}+(\tfrac{1}{2}-\nu)(\tfrac{3}{2}-\nu)\frac{u^4}{m^4x^4}\int_0^1(1-t)\left(1+\frac{u^2t}{m^2x^2}\right)^{\nu-\frac{5}{2}}dt.$$

But, for values of t on the path of integration,

$$\left(1+\frac{u^2t}{m^2x^2}\right)^{\nu-\frac{1}{2}} < K. \quad (3\,.\,72)$$

Hence

$$\frac{d}{dx}\left[\frac{x^{\nu+\frac{1}{2}}b_mQ_\nu(mx)}{(\frac{1}{2}mx)^\nu}\right] = -\frac{2b_m(\frac{1}{2}-\nu)}{mx^{\frac{3}{2}-\nu}\,\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})}\left[\int_0^\infty e^{-u}\left(1+\frac{u^2}{m^2x^2}\right)^{\nu-\frac{1}{2}}du - 1\right]$$

$$-\frac{4b_m}{m^3x^{\frac{3}{2}-\nu}\,\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})}\int_0^\infty e^{-u}u^2\left(1+\frac{u^2}{m^2x^2}\right)^{\nu-\frac{3}{2}}du$$

$$= -\frac{2b_m(\frac{1}{2}-\nu)}{mx^{\frac{3}{2}-\nu}\,\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})}\left[1+\frac{2\nu-1}{m^2x^2}+O\left(\frac{1}{m^4x^4}\right)+O\left(\frac{1}{m^2x^2}\right)-1\right],$$

by (3 . 72);

$$= O\left(\frac{b_m}{m^3}\right), \text{ in } e_1 \text{ or } e_2.$$

Since $b_m = o(m^{\nu+\frac{1}{2}})$, it follows that

$$\frac{d}{dx}\left[x^{\nu+\frac{1}{2}}\Sigma\frac{b_mQ_\nu(mx)}{(\frac{1}{2}mx)^\nu}\right]$$

converges uniformly, and hence, by integrating term-by-term,

$$x^{\nu+\frac{1}{2}}\Sigma\frac{b_mQ_\nu(mx)}{(\frac{1}{2}mx)^\nu}$$

converges uniformly to an integral.

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* G. N. Watson, *loc. cit.*, 331, (4), putting $\beta = 0$.

† *Ibid.*, 197.

3.8. Since $C_0/\{x\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})\}$ is an integral in e_1 or e_2, we have now proved that

$$\Sigma A_m = x^{-\nu-\frac{1}{2}}\{S_1+T(x)\}, \tag{3.81}$$

where S_1 is an ORF series, and $T(x)$ is an integral in e_1 or e_2.

Now multiply (3.81) by $x^{\nu+\frac{1}{2}}$, and integrate with respect to x from ϵ to y; then integrate with respect to y from ϵ to z. This is permissible in consequence of the uniform convergence of the difference between the left-hand side of (3.81) and the ORF series $x^{-\nu-\frac{1}{2}}S_1$, in e_1 or e_2.

By (3.53) we then get

$$\int_\epsilon^z dy \int_0^y t^{\nu+\frac{1}{2}} f(t)\,dt - Az - B = \int_\epsilon^z dy \int_\epsilon^y G(t)\,dt + \int_\epsilon^z dy \int_\epsilon^y T(t)\,dt,$$

where $G(x)$ is the function associated with the ORF series S_1.

Differentiating twice with respect to z, we have

$$f(z) = z^{-\nu-\frac{1}{2}}\{G(z)+T(z)\}. \tag{3.82}$$

Then, since $T(z)$ is an integral in e_1 or e_2, the ORF series S_1 will converge to the sum $G(z)$ under precisely similar conditions to those required for the convergence of the Fourier series of $f(z)$ in e_1 or e_2.

A similar relationship between the two series exists with regard to uniform convergence, divergence, or oscillation; in other words, the ORF series behaves precisely like the Fourier series of $f(x)$ in e_1 or e_2*.

Hence in e_1 or e_2, by (3.81),

$$\begin{aligned}\Sigma A_m &= x^{-\nu-\frac{1}{2}}\{G(x)+T(x)\}\\ &= f(x), \text{ by (3.82).}\end{aligned}$$

This proves the theorem.

4. *Analogues of the Parseval and Riesz-Fischer theorems for Schlömilch series*

4.1. In the theory of Fourier series, considerable simplification is effected by taking the Parseval and Riesz-Fischer theorems together as one theorem, and using the theory of mean convergence to deduce the former theorem from the latter.

But this proof of the theorems in question hinges upon the orthogonal properties of the trigonometrical functions; and since such properties are

* W. H. Young, "On restricted Fourier series and the convergence of power series", *Proc. London Math. Soc.* (2), 17 (1918), 353–366 (362), Theorem 2.

non-existent for Schlömilch series, the above simplification cannot be made in the present proofs. The two theorems have therefore to be considered separately.

4.2. THEOREM IV. ANALOGUE OF THE PARSEVAL THEOREM. *If the square of $x^{\nu+\frac{1}{2}}f(x)$ is integrable L over $(0, \pi)$, then $\Sigma(a_m^2/m^{2\nu+1})$, where a_m is the Schlömilch constant of $f(x)$, is convergent.*

We first prove the following lemma, an analogue of a theorem due to Hardy*.

LEMMA. *If the square of $F(x)$ is integrable over $(0, \pi)$, then $\Sigma\{S_1(m)\}^2$, $\Sigma\{S_2(m)\}^2$ are convergent, where*

$$S_1(m) \equiv \int_0^{1/m} F(t)\,dt, \quad S_2(m) \equiv \frac{1}{m}\int_{1/m}^{\pi} \frac{F(t)}{t}\,dt.$$

(i) Let $$\psi(x) \equiv \int_0^{\pi/x} F(t)\,dt \quad (x \geqslant 1),$$

so that $\psi(m\pi) = S_1(m)$.

Then if $m\pi < x \leqslant (m+1)\pi$,

$$S_1(m) - \psi(x) = \int_{\pi/x}^{1/m} F(t)\,dt \leqslant \left(\frac{1}{m} - \frac{\pi}{x}\right)^{\frac{1}{2}}\left[\int_{\pi/x}^{1/m} \{F(t)\}^2\,dt\right]^{\frac{1}{2}} = O\left(\frac{1}{m}\right).$$

If l is the value of x which gives $\psi(x)$ its least value in the range $m\pi < x \leqslant (m+1)\pi$, we have

$$S_1(m) = O\left(\frac{1}{m}\right) + \psi(l),$$

so that
$$S_1^2(m) \leqslant O\left(\frac{1}{m^2}\right) + 2\{\psi(l)\}^2$$
$$\leqslant O\left(\frac{1}{m^2}\right) + \frac{2}{\pi}\int_{m\pi}^{(m+1)\pi} \{\psi(t)\}^2\,dt.$$

Hence
$$\Sigma\{S_1(m)\}^2 \leqslant \Sigma O\left(\frac{1}{m^2}\right) + \frac{2}{\pi}\int_{\pi}^{\infty} \{\psi(t)\}^2\,dt.$$

But
$$\{\psi(x)\}^2 \leqslant \int_0^{\pi/x} \{F(t)\}^2\,dt \int_0^{\pi/x} dt = o\left(\frac{1}{x}\right) \quad \text{as } x \to \infty.$$

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* G. H. Hardy, *Messenger of Math.*, 48 (1918), 107–112, and 54 (1925), 150–156; and *Math. Zeitschrift*, 6 (1920), 314–317.

Therefore

$$\Sigma\{S_1(m)\}^2 \leqslant O(1)+\frac{2}{\pi}\int_1^\infty \{\psi(t)\}^2\,dt$$

$$= O(1)+\frac{2}{\pi}\left[t\{\psi(t)\}^2\right]_1^\infty +4\int_1^\infty \psi(t)\,F\left(\frac{\pi}{t}\right)\frac{dt}{t}$$

$$\leqslant O(1)+o(1)+4\left[\int_1^\infty \{\psi(t)\}^2\,dt\right]^{\frac{1}{2}}\left[\int_1^\infty \left\{F\left(\frac{\pi}{t}\right)\right\}^2\frac{dt}{t^2}\right]^{\frac{1}{2}}$$

$$\leqslant O(1)+4\left[O(1)+\Sigma\{S_1(m)\}^2\right]^{\frac{1}{2}}\left[\frac{1}{\pi}\int_0^\pi \{F(y)\}^2\,dy\right]^{\frac{1}{2}}.$$

Hence, dividing through by $[\Sigma\{S_1(m)\}^2]^{\frac{1}{2}}$, we see that $\Sigma\{S_1(m)\}^2$ is convergent.

(ii) Let
$$\chi(x)\equiv\frac{\pi}{x}\int_{\pi/x}^\pi \frac{F(t)}{t}\,dt,$$

so that
$$\chi(m\pi)=S_2(m).$$

Then, if $(m-1)\pi<x\leqslant m\pi$, we have

$$mS_2(m)-\frac{x}{\pi}\chi(x)=\int_{1/m}^{\pi/x}\frac{F(t)}{t}\,dt\leqslant\left(m-\frac{x}{\pi}\right)^{\frac{1}{2}}\left[\int_{1/m}^{\pi/x}\{F(t)\}^2\,dt\right]^{\frac{1}{2}}$$

so that $S_2(m)\leqslant\frac{x}{m\pi}|\chi(x)|+\left(\frac{1}{m}-\frac{x}{m^2\pi}\right)^{\frac{1}{2}}O(1)\leqslant|\chi(x)|+O\left(\frac{1}{m}\right)$.

Hence, as in (i),

$$\sum_{m=2}^{\infty}\{S_2(m)\}^2\leqslant O(1)+\frac{2}{\pi}\int_1^\infty \{t\chi(t)\}^2\frac{dt}{t^2}$$

$$= O(1)-\frac{2}{\pi}\left[t\{\chi(t)\}^2\right]_1^\infty +4\int_1^\infty \chi(t)\,F\left(\frac{\pi}{t}\right)\frac{dt}{t}.$$

But
$$\{\chi(x)\}^2\leqslant\left(\frac{\pi}{x}\right)^2\int_{\pi/x}^\pi\{F(t)\}^2\,dt\int_{\pi/x}^\pi\frac{dt}{t^2}=O\left(\frac{1}{x}\right).$$

Hence

$$\sum_{m=2}^{\infty}\{S_2(m)\}^2\leqslant O(1)+4\left[\int_1^\infty\{\chi(t)\}^2\,dt\right]^{\frac{1}{2}}\left[\int_1^\infty\left\{F\left(\frac{\pi}{t}\right)\right\}^2\frac{dt}{t^2}\right]^{\frac{1}{2}}$$

$$\leqslant O(1)+4\left[O(1)+\sum_{m=2}^{\infty}\{S_2(m)\}^2\right]^{\frac{1}{2}}\left[\frac{1}{\pi}\int_0^\pi\{F(y)\}^2\,dy\right]^{\frac{1}{2}}.$$

Hence $\Sigma\{S_2(m)\}^2$ is convergent.

Proof of the theorem. We have

$$a_m = \frac{2m}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)} \int_0^\pi t\phi(t) \sin mt\, dt$$

$$= \frac{2m}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)} \int_0^\pi u^{2\nu+1} f(u)\, du \int_u^\pi \frac{\sin mt}{(t^2-u^2)^{\nu+\frac{1}{2}}}\, dt.$$

We consider the two cases

$$\text{(i)}\quad \pi - u \leqslant \frac{1}{m}, \qquad \text{(ii)}\quad \pi - u > \frac{1}{m}.$$

Case (i). We have

$$\int_u^\pi \frac{\sin mt}{(t^2-u^2)^{\nu+\frac{1}{2}}}\, dt = \frac{1}{u^{2\nu}} \int_0^{(\pi/u)-1} \frac{\sin\{mu(1+z)\}}{(z^2+2z)^{\nu+\frac{1}{2}}}\, dz$$

$$= \frac{\sin mu}{u^{2\nu}} \int_0^{(\pi/u)-1} \frac{\cos muz}{(2z)^{\nu+\frac{1}{2}}}\, dz + \frac{\cos mu}{u^{2\nu}} \int_0^{(\pi/u)-1} \frac{\sin muz}{(2z)^{\nu+\frac{1}{2}}}\, dz$$

$$+ \frac{1}{u^{2\nu}} \int_0^{(\pi/u)-1} \left\{\frac{1}{(z^2+2z)^{\nu+\frac{1}{2}}} - \frac{1}{(2z)^{\nu+\frac{1}{2}}}\right\} \sin\{mu(1+z)\}\, dz$$

$$= I_1 + I_2 + I_3, \text{ say.}$$

Now $$I_1 = \frac{\sin mu}{m^{\frac{1}{2}-\nu}(2u)^{\nu+\frac{1}{2}}} \int_0^{m(\pi-u)} \frac{\cos\theta}{\theta^{\nu+\frac{1}{2}}}\, d\theta = O(m^{\nu-\frac{1}{2}} u^{-\nu-\frac{1}{2}}),$$

and similarly for I_2.

Also $I_3 = O(1/mu^{2\nu+1})$, since

$$\frac{1}{(z^2+2z)^{\nu+\frac{1}{2}}} - \frac{1}{(2z)^{\nu+\frac{1}{2}}}$$

is of bounded variation over $(0, \infty)$.

Case (ii). We have

$$\int_u^\pi \frac{\sin mt}{(t^2-u^2)^{\nu+\frac{1}{2}}}\, dt = \int_u^\infty \frac{\sin mt}{(t^2-u^2)^{\nu+\frac{1}{2}}}\, dt - \int_\pi^\infty \frac{\sin mt}{(t^2-u^2)^{\nu+\frac{1}{2}}}\, dt.$$

Denoting the second integral on the right by I_4, we have

$$I_4 = O\left\{\frac{1}{m(\pi^2-u^2)^{\nu+\frac{1}{2}}}\right\}.$$

Hence, from the two cases,

$$a_m = W_m + X_m + Y_m + Z_m,$$

where $$W_m = O(m^{\nu+\frac{1}{2}}) \int_{\pi-(1/m)}^{\pi} u^{\nu+\frac{1}{2}} f(u)\, du,$$

$$X_m = O\left\{\int_{\pi-(1/m)}^{\pi} f(u)\, du\right\},$$

$$Y_m = \frac{2m}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)} \int_0^{\pi-(1/m)} u^{2\nu+1} f(u)\, du \int_u^{\infty} \frac{\sin mt}{(t^2-u^2)^{\nu+\frac{1}{2}}}\, dt,$$

$$Z_m = O\left\{\int_0^{\pi-(1/m)} \frac{u^{2\nu+1} f(u)}{(\pi^2-u^2)^{\nu+\frac{1}{2}}}\, du\right\}.$$

Let us examine these in turn. We have

$$\int_{\pi-(1/m)}^{\pi} u^{\nu+\frac{1}{2}} f(u)\, du = \int_0^{1/m} F(v)\, dv,$$

where $F(v) \equiv (\pi-v)^{\nu+\frac{1}{2}} f(\pi-v)$, so that $[F(v)]^2$ is integrable over $(0,\ \pi)$; hence by the lemma, $\Sigma(W_m^2/m^{2\nu+1})$ is convergent.

Similarly $\Sigma(X_m^2/m^{2\nu+1})$ converges.

Next, $$Y_m = \frac{2m}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)} \int_0^{\pi-(1/m)} uf(u)\, du \int_1^{\infty} (v^2-1)^{-\nu-\frac{1}{2}} \sin muv\, dv$$

$$= 2^{-\nu} m^{\nu+1} \int_0^{\pi-(1/m)} u^{\nu+1} f(u)\, J_\nu(mu)\, du^*.$$

Now $$z^{\frac{1}{2}} J_\nu(z) - \sqrt{\left(\frac{2}{\pi}\right)} \cdot \cos(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) = R(z),$$

where $$|R(z)| = O(1), \quad z < 1,$$

$$= O(1/z), \quad z \geqslant 1.$$

Hence

$$Y_m = \frac{2^{\frac{1}{2}-\nu} m^{\nu+\frac{1}{2}}}{\sqrt{\pi}} \int_0^{\pi-(1/m)} u^{\nu+\frac{1}{2}} f(u) \cos(mu - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)\, du$$

$$+ 2^{-\nu} m^{\nu+\frac{1}{2}} \int_0^{1/m} u^{\nu+\frac{1}{2}} f(u)\, R(mu)\, du$$

$$+ 2^{-\nu} m^{\nu+\frac{1}{2}} \int_{1/m}^{\pi-(1/m)} u^{\nu+\frac{1}{2}} f(u)\, R(mu)\, du$$

$$= E_m + F_m + G_m, \text{ say.}$$

* G. N. Watson, *loc. cit.*, 170 (3).

By Parseval's theorem, $\Sigma(E_m^2/m^{2\nu+1})$ is convergent.

Also $$|F_m| = O\left\{m^{\nu+\frac{1}{2}}\int_0^{1/m} u^{\nu+\frac{1}{2}} f(u)\,du\right\},$$

so that $\Sigma(F_m^2/m^{2\nu+1})$ converges, by the lemma.

Again, $$|G_m| = O\left\{m^{\nu-\frac{1}{2}}\int_{1/m}^{\pi} u^{\nu-\frac{1}{2}} f(u)\,du\right\},$$

so that $\Sigma(G_m^2/m^{2\nu+1})$ converges, also by the lemma.

It remains only to deal with Z_m.

Now an obvious extension of the second part of the lemma (all the steps holding exactly as before) is :

If the square of $F(x)$ is integrable over $(0, \pi)$, then $\Sigma\{S_3(m)\}^2$ is convergent, where

$$S_3(m) \equiv \frac{1}{m^{\nu+\frac{1}{2}}}\int_{1/m}^{\pi} \frac{F(t)}{t^{\nu+\frac{1}{2}}}\,dt.$$

We have $$|Z_m| = O\left\{\int_{1/m}^{\pi} G(v)\,v^{-\nu-\frac{1}{2}}\,dv\right\},$$

where $$G(v) \equiv \frac{(\pi-v)^{2\nu+1} f(\pi-v)}{(2\pi-v)^{\nu+\frac{1}{2}}},$$

the square of which is integrable over $(0, \pi)$.

Hence, by the (extended) lemma, $\Sigma(Z_m^2/m^{2\nu+1})$ is convergent.

This proves the theorem.

4.3. THEOREM V. ANALOGUE OF THE RIESZ-FISCHER THEOREM. *If $\Sigma(a_m^2/m^{2\nu+1})$ is convergent, then the numbers a_m are the Schlömilch constants of a function $f(x)$, such that the square of $x^{\nu+\frac{1}{2}}f(x)$ is integrable L over $(0, \pi)$.*

The theorem requires the employment of a lemma, which is an analogue of the same theorem of Hardy, mentioned at the beginning of Theorem IV.

LEMMA. *If $S \equiv \Sigma(a_m^2/m^{2\nu+1})$ is convergent, and*

$$\psi(x) \equiv \frac{1}{x}\sum_{m=1}^{[x]} \frac{a_m}{m^{\nu+\frac{1}{2}}}, \quad \chi(x) \equiv \sum_{m=[x]+1}^{\infty} \left|\frac{a_m}{m^{\nu+\frac{1}{2}}}\right|,$$

where $[x]$ *means the greatest integer contained in* x*; then*

$$\int_1^\infty \{\psi(x)\}^2\,dx < KS, \quad \int_1^\infty \{\chi(x)\}^2\,dx < KS,$$

where K *is a constant.*

The proof is almost exactly similar to the proof given by Hardy in the papers cited, and may be left to the reader.

Proof of the theorem. (i) The first step is to show that the sequence

$$x^{\nu+\frac{1}{2}} f_n(x) \equiv x^{\nu+\frac{1}{2}} \sum_{m=1}^{n} \frac{a_m J_\nu(mx)}{(\frac{1}{2}mx)^\nu}$$

converges in mean to a function $F(x)$ of integrable square over $(0, \pi)$.

This is effected on the lines of a recent note by Titchmarsh on Hankel transforms*. We have

$$z^{\frac{1}{2}} J_\nu(z) - \sqrt{\left(\frac{2}{\pi}\right)} \cdot \cos(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) = R(z),$$

where $\quad |R(z)| < K_1 \quad (z \leqslant \pi), \quad |R(z)| < K_2/z \quad (z > \pi).$

Hence

$$x^{\nu+\frac{1}{2}} f_n(x) = \frac{2^{\nu+\frac{1}{2}}}{\sqrt{\pi}} \sum_{m=1}^{n} \frac{a_m}{m^{\nu+\frac{1}{2}}} \cos(mx - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)$$

$$+ 2^\nu \sum_{m=1}^{[\pi/x]} \frac{a_m}{m^{\nu+\frac{1}{2}}} R(mx) + 2^\nu \sum_{m=[\pi/x]+1}^{n} \frac{a_m}{m^{\nu+\frac{1}{2}}} R(mx), \quad (4.31)$$

where $[\pi/x]$ means the greatest integer contained in π/x, which is greater than or equal to 1, since $x \leqslant \pi$.

The first sum is known to converge in mean to a function $F_1(x)$, of integrable square in $(0, \pi)$, such that

$$\int_0^\pi \{F_1(x)\}^2\,dx < KS,$$

where K is a constant, and $S \equiv \Sigma(a_m^2/m^{2\nu+1})$, as in the lemma.

Calling the other two terms of (4.31), $F_2(x)$, $F_3(x)$, and using the notation of the lemma, we have

$$\psi\left(\frac{\pi}{y}\right) = \frac{y}{\pi} \sum_{m=1}^{[\pi/y]} \left| \frac{a_m}{m^{\nu+\frac{1}{2}}} \right| > Ky F_2(y);$$

* E. C. Titchmarsh, "A note on Hankel transforms", *Journal London Math. Soc.*, 1 (1926), 195, 196.

therefore $\int_0^\pi \{F_2(x)\}^2\,dx < K\int_0^\pi \left\{\psi\left(\frac{\pi}{x}\right)\right\}^2 \frac{dx}{x^2} = \frac{K}{\pi}\int_1^\infty \{\psi(t)\}^2\,dt < KS,$

by the lemma.

Similarly $\chi\left(\frac{\pi}{y}\right) = \sum_{m=[\pi/y]+1}^{\infty} \left|\frac{a_m}{m^{\nu+\frac{3}{2}}}\right| > Ky\,F_3(y)$:

consequently $\int_0^\pi \{F_3(x)\}^2\,dx < KS.$

Then, since $\left[\int_0^\pi \{F(x)\}^2\,dx\right]^{\frac{1}{2}} \leqslant \sum_{n=1}^{3}\left[\int_0^\pi \{F_n(x)\}^2\,dx\right]^{\frac{1}{2}},$

it follows that $\int_0^\pi \{F(x)\}^2\,dx < KS.$

Hence, also $\int_0^\pi x^{2\nu+1}\{f_q(x)-f_p(x)\}^2\,dx < K\sum_{m=p}^{q}\frac{a_m^2}{m^{2\nu+1}},$

which proves the result.

(ii) Let $t\phi_n(t) \equiv t\int_0^{\frac{1}{2}\pi} f_n(t\sin\theta)\sin^{2\nu+1}\theta\cos^{-2\nu}\theta\,d\theta$

$$= t\sum_{m=1}^{n}\frac{a_m}{(\frac{1}{2}mt)^\nu}\int_0^{\frac{1}{2}\pi} J_\nu(mt\sin\theta)\sin^{\nu+1}\theta\cos^{-2\nu}\theta\,d\theta$$

$$= \frac{\Gamma(\frac{1}{2}-\nu)}{\Gamma(\frac{1}{2})}\sum_{m=1}^{n}\frac{a_m}{m}\sin mt.$$

Now, since $\Sigma(a_m^2/m^{2\nu+1})$ is convergent, so is $\Sigma(a_m^2/m^2)$ *a fortiori* since $|\nu|<\frac{1}{2}$; hence, by the Riesz-Fischer theorem, $t\phi_n(t)$ converges in mean to a function $t\phi(t)$, such that

$$\int_0^\pi t\phi(t)\sin mt\,dt = \lim_{n\to\infty}\int_0^\pi t\phi_n(t)\sin mt\,dt = \tfrac{1}{2}\Gamma(\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}-\nu)\,\frac{a_m}{m}.$$

Hence, by this, and the result of (i), $t\phi_n(t)$ has for its mean limit

$$t\int_0^{\frac{1}{2}\pi} f(t\sin\theta)\sin^{2\nu+1}\theta\cos^{-2\nu}\theta\,d\theta.$$

This proves the theorem.

5. *Special theorems about the end-points.*

5.1. THEOREM VI. *If $f(x)$ is continuous, except (possibly) at $x=0$, and $x^\nu\{f(\sqrt{x})-f(+0)\}$ is an integral of order $(\nu+\frac{1}{2})$ of a bounded function $\phi(x)$ in the range $(0,\pi)$, and if $(-x)^\nu\{f(\sqrt{-x})-f(-0)\}$ satisfy similar*

conditions in the range $(-\pi, 0)$; *then the Schlömilch series oscillates at* $x = 0$, *unless* $\phi(\pi)+\phi(-\pi) = 0$, *in which case it converges to the sum* $\frac{1}{2}\{f(+0)+f(-0)\}$.

We have, in accordance with the definitions* of Liouville and Riemann of an integral of order $(\nu+\frac{1}{2})$,

$$x^{\nu}\{f(\sqrt{x})-f(+0)\} = \frac{1}{\Gamma(\nu+\frac{1}{2})}\int_0^x g(u)(x-u)^{\nu-\frac{1}{2}}\,du.$$

Also
$$t\phi(t) = \int_0^t u^{2\nu+1} f(u)(t^2-u^2)^{-\nu-\frac{1}{2}}\,du.$$

Put $t^2 = s$, $u^2 = v$: then

$$\sqrt{s}.\phi(\sqrt{s}) = \tfrac{1}{2}\int_0^s v^{\nu} f(\sqrt{v})(s-v)^{-\nu-\frac{1}{2}}\,dv$$

$$= \frac{1}{2\Gamma(\nu+\frac{1}{2})}\int_0^s (s-v)^{-\nu-\frac{1}{2}}\,dv\int_0^v g(u)(v-u)^{\nu-\frac{1}{2}}\,du$$

$$+\tfrac{1}{2}f(+0)\int_0^s v^{\nu}(s-v)^{-\nu-\frac{1}{2}}\,dv$$

$$= \tfrac{1}{2}\Gamma(\tfrac{1}{2}-\nu)\int_0^s g(u)\,du + \frac{\Gamma(\nu+1)\,\Gamma(\frac{1}{2}-\nu)}{\Gamma(\frac{1}{2})}\,f(+0)\sqrt{s}.$$

Hence
$$\phi(t) = \frac{1}{2t}\,\Gamma(\tfrac{1}{2}-\nu)\int_0^{t^2} g(u)\,du + \frac{\Gamma(\nu+1)\,\Gamma(\frac{1}{2}-\nu)}{\Gamma(\frac{1}{2})}\,f(+0)$$

$$= \tfrac{1}{2}\Gamma(\tfrac{1}{2}-\nu)\psi_{\nu}(t)+\phi(+0), \qquad (5.11)$$

where
$$\psi_{\nu}(t) \equiv \frac{1}{t}\int_0^{t^2} g(u)\,du.$$

Now when $x = 0$,

$$\sum_{m=1}^{n} A_m = \frac{1}{\Gamma(\nu+1)}\sum_{m=1}^{n} a_m$$

$$= \frac{1}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)\,\Gamma(\nu+1)}\int_{\pi}^{n} t\phi(t)\,.\sum_{m=1}^{n} m\sin mt\,.\,dt.$$

But
$$\sum_{m=1}^{n} m\sin mt = \frac{\sin nt - 2n\sin\frac{1}{2}t\cos(n+\frac{1}{2})\,t}{4\sin^2\frac{1}{2}t};$$

* See G. H. Hardy, "Notes on some points in the integral calculus (xlviii)"; *Messenger of Math.*, 47 (1917–18), 145.

hence, dealing first with the range $(0, \pi)$ of t, and employing (5.11), we have

$$\lim_{n\to\infty} \int_0^\pi t\psi(t) \frac{\sin nt}{4\sin^2 \frac{1}{2}t}\, dt = \lim_{n\to\infty} \int_0^\pi \psi(t) \frac{\sin nt}{t}\, dt = (0)$$

by the theorem of Riemann-Lebesgue, since $\psi(t) = O(t)$; and

$$\lim_{n\to\infty} \phi(+0) \int_0^\pi \frac{\sin nt}{t} = \tfrac{1}{2}\pi\phi(+0).$$

Also

$$\lim_{n\to\infty} n \int_0^\pi t\phi(t) . \frac{\cos(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\, dt$$

$$= \lim_{n\to\infty} n\left[\frac{t\phi(t)}{2\sin\frac{1}{2}t} . \frac{\sin(n+\frac{1}{2})t}{(n+\frac{1}{2})}\right]_0^\pi - \lim_{n\to\infty} \frac{n}{n+\frac{1}{2}} \int_0^\pi \sin(n+\tfrac{1}{2})t\, \frac{d}{dt}\left\{\frac{t\phi(t)}{2\sin\frac{1}{2}t}\right\} dt,$$

and it is easily seen from (5.11) that $\frac{d}{dt}\left\{\frac{t\phi(t)}{2\sin\frac{1}{2}t}\right\}$ is bounded, so that the last integral tends to zero by the theorem of Riemann-Lebesgue.

Similarly for the range $(-\pi, 0)$. Hence

$$\frac{1}{\Gamma(\nu+1)} \Sigma a_m = \frac{1}{2\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)\,\Gamma(\nu+1)}$$

$$\times \left[\pi\{\phi(+0)+\phi(-0)\} + \lim_{n\to\infty} \frac{n}{n+\frac{1}{2}}\pi \sin(n+\tfrac{1}{2})\pi\{\phi(\pi)+\phi(-\pi)\}\right].$$

This shows that, to avoid oscillation at the origin, we must have

$$\phi(\pi)+\phi(-\pi) = 0.$$

And in that case

$$\frac{1}{\Gamma(\nu+1)} \Sigma a_m = \text{sum of the series when } x = 0$$

$$= \frac{\pi\{\phi(+0)+\phi(-0)\}}{2\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)\,\Gamma(\nu+1)} = \tfrac{1}{2}\{f(+0)+f(-0)\}.$$

This proves the theorem.

5.2. *Gibbs's phenomenon in Schlömilch series.* It is clear that the nature of the Gibbs's phenomenon at any point of discontinuity in e_1 or e_2 is settled by Theorem III.

Thus the *Gibbs Ratio** at such a point is equal to

$$\frac{2}{\pi} \int_0^\pi \frac{\sin v}{v}\, dv,$$

* For the definition of this phrase, see R. G. Cooke, "Gibbs's phenomenon in Fourier-Bessel series and integrals", *Proc. London Math. Soc.* (2), 27 (1928), 171–192 (172 and 185).

this being the Gibbs ratio for the Fourier series of $f(x)$ at a point of discontinuity in $f(x)$.

We will now discuss the phenomenon at the origin.

We consider the *modified Schlömilch series* $xS_n(x)$, where

$$S_n(x) \equiv \sum_{m=1}^{n} A_m.$$

Here
$$a_m = \frac{m}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)} \int_{-\pi}^{\pi} t\phi(t) \sin mt\, dt,$$

$$b_m = -\frac{m}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)} \int_{-\pi}^{\pi} t\phi(t) \cos mt\, dt,$$

where
$$\phi(t) \equiv \frac{1}{t} \int_0^{\frac{1}{2}\pi} f(t \sin\theta) \sin^{2\nu}\theta \cos^{-2\nu}\theta\, d\theta.$$

Take $f(t) = 1 \quad (0 < t \leqslant \pi), \quad f(t) = -1 \quad (-\pi \leqslant t < 0).$

Then
$$t\phi(t) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}+\nu)\,\Gamma(\tfrac{1}{2}-\nu) \quad (0 < t \leqslant \pi);$$
$$= -\tfrac{1}{2}\Gamma(\tfrac{1}{2}+\nu)\,\Gamma(\tfrac{1}{2}-\nu) \quad (-\pi \leqslant t < 0).$$

Hence $a_m = \dfrac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})}\, m \displaystyle\int_0^{\pi} \sin mt\, dt = \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})} \{1-(-1)^m\},$

and $b_m = 0.$

Consequently $xS_n(x) = \dfrac{x\Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})} \displaystyle\sum_{m=1}^{n} \{1-(-1)^m\} \frac{J_\nu(mx)}{(\frac{1}{2}mx)^\nu}.$

Now* $J_\nu(z) = \dfrac{z^\nu}{2^{\nu-1}\,\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})} \displaystyle\int_0^{\frac{1}{2}\pi} \cos(z\sin\theta) \cos^{2\nu}\theta\, d\theta \quad (\nu > -\tfrac{1}{2}).$

Hence

$$\sum_{m=1}^{n} (-1)^m \frac{J_\nu(mx)}{(\frac{1}{2}mx)^\nu} = -\frac{1}{2\Gamma(\nu+1)} + \frac{2}{\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})} \int_0^{\frac{1}{2}\pi} \left\{\tfrac{1}{2} + \sum_{m=1}^{n} (-1)^m \cos(mx\sin\theta)\right\} \cos^{2\nu}\theta\, d\theta$$

$$= -\frac{1}{2\Gamma(\nu+1)} + \frac{(-1)^n}{\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})\,x^{2\nu}} \int_0^{x} \frac{\cos(n+\frac{1}{2})u}{\cos\frac{1}{2}u} (x^2-u^2)^{\nu-\frac{1}{2}}\, du.$$

This oscillates as $x \to 0, \ n \to \infty$ in any manner. Hence the series

$$x \sum_{m=1}^{n} (-1)^m \frac{J_\nu(mx)}{(\frac{1}{2}mx)^\nu}$$

converges uniformly to zero, as $n \to \infty$, in the neighbourhood of $x = 0$.

It remains to consider $\dfrac{x\Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})} \displaystyle\sum_{m=1}^{n} \frac{J_\nu(mx)}{(\frac{1}{2}mx)^\nu}.$

* G. N. Watson, *loc. cit.*, 48, (5).

This is equal to

$$\frac{2x}{\pi}\int_0^{\frac{1}{2}\pi}\cos^{2\nu}\theta\sum_{m=1}^{n}\cos(mx\sin\theta)\,d\theta$$

$$=\frac{x}{\pi}\int_0^{\pi}\frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}(x^2-u^2)^{\nu-\frac{1}{2}}\,du-\frac{x}{2\Gamma(\nu+1)}$$

$$=\frac{x^{1-2\nu}}{\pi}\int_0^{(n+\frac{1}{2})x}\frac{\sin v}{\sin\left[v/\{2(n+\frac{1}{2})\}\right]}\left\{x^2-\left(\frac{v}{n+\frac{1}{2}}\right)^2\right\}^{\nu-\frac{1}{2}}dv-\frac{x}{2\Gamma(\nu+1)}.$$

Hence the Gibbs ratio at $x=0$ is given by

$$G=\lim_{n\to\infty}\operatorname*{Max}_x\frac{2}{\pi}\int_0^{(n+\frac{1}{2})x}\frac{\sin v}{v}\left\{1-\frac{v^2}{(n+\frac{1}{2})^2x^2}\right\}^{\nu-\frac{1}{2}}dv.$$

Taking $x=\lambda/(n+\frac{1}{2})$, we require the greatest maximum of the integral

$$F(\lambda)\equiv\int_0^{\lambda}\frac{\sin v}{v}\left(1-\frac{v^2}{\lambda^2}\right)^{\nu-\frac{1}{2}}dv.$$

Now
$$F'(\lambda)=\lim_{v\to\lambda}\left[\frac{\sin v}{v}\left(1-\frac{v^2}{\lambda^2}\right)^{\nu-\frac{1}{2}}+(\nu-\tfrac{1}{2})\int_0^{v}\frac{\sin v}{v}\cdot\frac{2v^2}{\lambda^3}\left(1-\frac{v^2}{\lambda^2}\right)^{\nu-\frac{3}{2}}dv\right]$$

$$=\frac{1}{\lambda}\int_0^{\lambda}\left(1-\frac{v^2}{\lambda^2}\right)^{\nu-\frac{1}{2}}\cos v\,dv\qquad\text{(integrating by parts)}$$

$$=\int_0^{\frac{1}{2}\pi}\cos(\lambda\sin\theta)\cos^{2\nu}\theta\,d\theta$$

$$=2^{\nu-1}\lambda^{-\nu}\Gamma(\tfrac{1}{2})\,\Gamma(\nu+\tfrac{1}{2})\,J_\nu(\lambda).$$

Hence
$$F(\lambda)=2^{\nu-1}\Gamma(\tfrac{1}{2})\,\Gamma(\nu+\tfrac{1}{2})\int_0^{\lambda}v^{-\nu}J_\nu(v)\,dv,$$

and this has its greatest maximum when $\lambda=j_1$, the smallest positive zero of $J_\nu(v)$*. Consequently

$$G=\frac{2^\nu\,\Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_0^{j_1}v^{-\nu}J_\nu(v)\,dv,\qquad(5.21)$$

which reduces, in the special (limiting) case $\nu=\frac{1}{2}$, to

$$\frac{2}{\pi}\int_0^{\pi}\frac{\sin v}{v}\,dv.$$

* R. G. Cooke, *loc. cit.* See also "A case of Gibbs's phenomenon", *Journal London Math. Soc.*, 3 (1928), 92-98 (97). The result (5.21) for G agrees with the case $\alpha=1-\nu$ in the expressions for G at the origin in the Hankel integral and the Fourier-Bessel series, as given in the former paper.

We therefore have

THEOREM VII *The modified Schlömilch series*

$$x\Sigma A_m$$

has a Gibbs ratio, at $x = 0$, *equal to*

$$\frac{2^\nu \Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})} \int_0^{j_1} v^{-\nu} J_\nu(v)\, dv,$$

where j_1 *is the smallest positive zero of* $J_\nu(v)$.

5.3. THEOREM VIII. *In general the Schlömilch series diverges at the points* $x = \pm\pi$; *but if* $\frac{d}{dt}\{t\phi(t)\}$ *is of bounded variation over* $(-\pi, \pi)$ *and* $\phi(\pi)+\phi(-\pi) = 0$, *then the series converges at* $x = \pi$ *to the sum* $f(\pi)$, *and at* $x = -\pi$ *to the sum* $f(-\pi)$.

We have

$$\sum_{m=1}^{n} A_m = \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-\nu)} \sum_{m=1}^{n} \left[\frac{mJ_\nu(mx)}{(\frac{1}{2}mx)^\nu} \int_{-\pi}^{\pi} t\phi(t) \sin mt\, dt \right.$$
$$\left. - \frac{m\mathbf{H}_\nu(mx)}{(\frac{1}{2}mx)^\nu} \int_{-\pi}^{\pi} t\phi(t) \cos mt\, dt \right].$$

Now $\sum_{m=1}^{n} \frac{mJ_\nu(mx)}{(\frac{1}{2}mx)^\nu} \int_{-\pi}^{\pi} t\phi(t) \sin mt\, dt$

$$= \int_{-\pi}^{\pi} t\phi(t) \left\{ \sum_{m=1}^{n} \frac{mJ_\nu(mx) \sin mt}{(\frac{1}{2}mx)^\nu} \right\} dt$$

$$= -\left[t\phi(t) \sum_{m=1}^{n} \frac{J_\nu(mx) \cos mt}{(\frac{1}{2}mx)^\nu} \right]_{-\pi}^{\pi}$$
$$+ \int_{-\pi}^{\pi} \frac{d}{dt}\{t\phi(t)\} \left\{ \sum_{m=1}^{n} \frac{J_\nu(mx) \cos mt}{(\frac{1}{2}mx)^\nu} \right\} dt.$$

The integrated part vanishes if $\phi(\pi)+\phi(-\pi) = 0$; otherwise, as x tends to π or $-\pi$, this would tend to infinity with n.

Since $\frac{d}{dt}\{t\phi(t)\}$ is of bounded variation over $(-\pi, \pi)$,

$$\int_{-\pi}^{\pi} \frac{d}{dt}\{t\phi(t)\} \left\{ \sum_{m=1}^{n} \frac{J_\nu(mx) \cos mt}{(\frac{1}{2}mx)^\nu} \right\} dt$$
$$= \sum_{m=1}^{n} \frac{J_\nu(mx)}{(\frac{1}{2}mx)^\nu} \int_{-\pi}^{\pi} \frac{d}{dt}\{t\phi(t)\} \cos mt\, dt$$
$$= \sum_{m=1}^{n} \frac{J_\nu(mx)}{(\frac{1}{2}mx)^\nu} O\left(\frac{1}{m}\right) = \sum_{m=1}^{n} O\left(\frac{1}{m^{\nu+\frac{1}{2}}}\right)$$

by the second mean-value theorem.

This converges, as $n \to \infty$, uniformly in the neighbourhood of $x = \pi$ and of $x = -\pi$.

We may deal similarly with

$$\sum_{m=1}^{n} \frac{m \mathbf{H}_\nu(m\pi)}{(\frac{1}{2}m\pi)^\nu} \int_{-\pi}^{\pi} t\phi(t) \cos mt \, dt.$$

The sum of the series ΣA_m is thus continuous in the neighbourhood of $x = \pi$, and therefore, by Theorem III, the sum at $x = \pi$ is $f(\pi)$. Similarly, the sum at $x = -\pi$ is $f(-\pi)$.

This proves the theorem.

6. *Examples.*

6.1. It is clearly of interest to illustrate the theory expounded in Theorem III by means of an example, independently of that theory.

The series selected as the example is

$$S \equiv \sum_{m=1}^{\infty} (-1)^m J_0\{(2m-1)\alpha\} \, J_1\{(2m-1)\beta\},$$

where $\alpha \geqslant 0$, $\beta \geqslant 0$, $\alpha+\beta \leqslant \pi$.

Here $\nu = 0$, and $$\Sigma \frac{a_m^2}{m^{2\nu+1}} = \Sigma O\left(\frac{1}{m^2}\right),$$

which is convergent.

Hence the series S is the Schlömilch series of its sum, by Theorem V.

Integrate S term-by-term with respect to β; then, C being an arbitrary constant,

$$S' \equiv \int S \, d\beta = C - \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m-1)} J_0\{(2m-1)\alpha\} \, J_0\{(2m-1)\beta\} \qquad (6.11)$$

$$= C - \frac{4}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m-1)} \int_0^{\frac{1}{2}\pi} \cos\{(2m-1)\alpha \sin\theta\} \, d\theta \int_0^{\frac{1}{2}\pi} \cos\{(2m-1)\beta \sin\phi\} \, d\phi$$

$$= C - \frac{2}{\pi^2} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m-1)} \Big[\cos\{(2m-1)(\alpha \sin\theta + \beta \sin\phi)\}$$

$$+ \cos\{(2m-1)(\beta \sin\phi - \alpha \sin\theta)\} \Big] d\theta \, d\phi.^* \qquad (6.12)$$

Now $$\sum_{m=1}^{\infty} \frac{(-1)^m \cos(2m-1)x}{2m-1}$$

$$= -\tfrac{1}{4}\pi \ (0 \leqslant x < \tfrac{1}{2}\pi), \quad 0 \ (x = \tfrac{1}{2}\pi), \quad \tfrac{1}{4}\pi \ (\tfrac{1}{2}\pi < x \leqslant \pi).$$

* The inversion of the order of summation and integration is legitimate; see Hobson, *loc. cit.*, II, 304, (2).

(i) Let $\beta > a$, and $\beta \leqslant \frac{1}{2}\pi$. Then, if

$$A = \arcsin\{(\pi-2\beta)/2a\}, \quad B = \arcsin\{(\pi-2a\sin\theta)/2\beta\},$$

$$D = \arcsin(a\sin\theta/\beta),$$

we have

$$S' = C + \frac{1}{2\pi}\left[\int_A^{\frac{1}{2}\pi}\int_0^{D} + \int_0^{A}\int_0^{\frac{1}{2}\pi} - \int_A^{\frac{1}{2}\pi}\int_B^{\frac{1}{2}\pi} + \int_0^{B}\int_D^{\frac{1}{2}\pi} - \int_0^{B}\int_0^{D}\right] d\theta\, d\phi$$

$$= C - \frac{1}{\pi}\int_0^{B} D\,d\theta + \frac{1}{\pi}\int_A^{\frac{1}{2}\pi} B\,d\theta + \tfrac{1}{2}A. \quad (6.13)$$

Hence, differentiating with respect to β, we have, when $a < \beta \leqslant \frac{1}{2}\pi$,

$$S = \frac{a}{\pi\beta^2}\int_0^{B}\frac{\sin\theta\,d\theta}{\sqrt{\left(1-\frac{a^2}{\beta^2}\sin^2\theta\right)}} - \frac{1}{\pi\beta^2}\int_A^{\frac{1}{2}\pi}\frac{(\frac{1}{2}\pi - a\sin\theta)\,d\theta}{\sqrt{\left\{1-\frac{(\pi-2a\sin\theta)^2}{4\beta^2}\right\}}}.$$

(ii) Let $a > \beta$, and $a \leqslant \frac{1}{2}\pi$.

Then S' is obtained by interchanging a and β in (6.13), since the expression (6 11) for S' is symmetrical in a and β. Thus

$$S' = C - \frac{1}{\pi}\int_0^{A} E\,d\theta + \frac{1}{\pi}\int_F^{\frac{1}{2}\pi} G\,d\theta + F,$$

where $E = \arcsin(\beta\sin\theta/a), \quad F = \arcsin\{(\pi-2a)/2\beta\},$

$$G = \arcsin\{(\pi-2\beta\sin\theta)/2a\}.$$

Hence, differentiating with respect to β, we have, when $\beta < a \leqslant \frac{1}{2}\pi$,

$$S = -\frac{1}{\pi a}\int_0^{A}\frac{\sin\theta\,d\theta}{\sqrt{\left(1-\frac{\beta^2}{a^2}\sin^2\theta\right)}} - \frac{1}{\pi a}\int_F^{\frac{1}{2}\pi}\frac{\sin\theta\,d\theta}{\sqrt{\left\{1-\frac{(\pi-2\beta\sin\theta)^2}{4a^2}\right\}}}.$$

If the sum of the series at any point $a = x$ $(x \leqslant \frac{1}{2}\pi)$ is $S(x)$, then

$$\tfrac{1}{2}\{S(\beta+0)+S(\beta-0)\} = -\frac{1}{4\beta^2}\int_{\arcsin\{(\pi-2\beta)/2\beta\}}^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{\left\{1-\frac{(\pi-2\beta\sin\theta)^2}{4\beta^2}\right\}}}.$$

By making the successive substitutions

$$t = \frac{\pi}{4\beta} - \sin\theta, \quad s = \frac{4\beta t}{\pi - 4\beta},$$

this is easily seen to reduce to

$$-\frac{2}{\beta(\pi+4\beta)}K\left(\frac{4\beta-\pi}{4\beta+\pi}\right) \quad (\tfrac{1}{4}\pi < \beta \leqslant \tfrac{1}{2}\pi),$$

with the notation of complete elliptic integrals, and this, by Landen's

transformation*, is equal to

$$-\frac{1}{4\beta^2}K\left\{\sqrt{\left(1-\frac{\pi^2}{16\beta^2}\right)}\right\}. \tag{6.14}$$

We will now verify by direct summation that, when $\alpha=\beta$, the series has the sum (6.14).

When $\alpha=\beta$,

$$\begin{aligned}S &= \sum_{m=1}^{\infty}(-1)^m J_0\{(2m-1)\beta\}\,J_1\{(2m-1)\beta\}\\ &= \frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^m\int_0^{\frac{1}{2}\pi}J_1\{(4m-2)\beta\cos\theta\}\cos\theta\,d\theta\dagger.\end{aligned}$$

Integrating with respect to β, we have

$$\begin{aligned}S' &= \int S\,d\beta = C-\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{(4m-2)}\int_0^{\frac{1}{2}\pi}J_0\{(4m-2)\beta\cos\theta\}\,d\theta\\ &= C-\frac{2}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m-1)}\cos\{(4m-2)\beta\sin\phi\cos\theta\}\,d\theta\,d\phi\ddagger\\ &= C-\frac{2}{\pi^2}\int_0^{M}\int_0^{N}(-\tfrac{1}{4}\pi)\,d\theta\,d\phi-\frac{2}{\pi^2}\int_M^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(-\tfrac{1}{4}\pi)\,d\theta\,d\phi\\ &\qquad -\frac{2}{\pi^2}\int_0^{M}\int_N^{\frac{1}{2}\pi}\tfrac{1}{4}\pi\,d\theta\,d\phi,\quad\text{if }\alpha>\tfrac{1}{4}\pi;\\ &= C+\frac{1}{\pi}\int_0^{M}N\,d\theta+\frac{\pi}{8}-\tfrac{1}{2}M,\end{aligned}$$

where $M=\arccos(\pi/4\beta)$, $N=\arcsin(\pi/4\beta\cos\theta)$.

Differentiate with respect to β, then

$$\begin{aligned}S &= \frac{1}{\pi}\int_0^{\arccos(\frac{1}{4}\pi/\beta)}\frac{(-\pi/4\beta^2\cos\theta)\,d\theta}{\sqrt{\{(1-\pi^2/(4\beta\cos\theta)^2\}}}\\ &= -\frac{1}{\pi\beta}\int_{\frac{1}{4}\pi/\beta}^{1}\frac{dt}{\sqrt{(1-t^2)}\sqrt{\left(\frac{16\beta^2t^2}{\pi^2}-1\right)}},\text{ where }t=\cos\theta;\\ &= -\frac{1}{4\beta^2}\int_1^{4\beta/\pi}\frac{ds}{\sqrt{(s^2-1)}\sqrt{\left(1-\frac{\pi^2s^2}{16\beta^2}\right)}},\text{ where }s=4\beta t/\pi;\\ &= -\frac{1}{4\beta^2}K\left\{\sqrt{\left(1-\frac{\pi^2}{16\beta^2}\right)}\right\};\end{aligned}$$

and this agrees with (6.14).

* Whittaker and Watson, *Modern analysis*, 3rd ed. (1920), 507.

† G. N. Watson, *loc. cit.*, 150.

‡ The inversion of the order of integration and summation is justifiable, as before.

When $\alpha < \frac{1}{4}\pi$, the sum of the series is zero. We have therefore verified that, at the point of discontinuity, $\alpha = \beta$, the sum of the series is

$$\tfrac{1}{2}\{S(\beta+0)+S(\beta-0)\}.$$

6.2. Finally, we will establish some results for the sum of series of Schlömilch type analogous to those obtained for the Weber-Schafheitlin integral*,

$$\int_0^\infty \frac{J_\mu(at)\,J_\nu(\beta t)}{t^{\mu+\nu}}\,dt.$$

Since

$$J_\nu(z) = \frac{z^\nu}{2^{\nu-1}\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{1}{2}\pi} \cos(z\sin\theta)\cos^{2\nu}\theta\,d\theta,$$

when $\nu > -\frac{1}{2}$, it follows that

$$\sum_{m=1}^{n} \frac{J_\nu(mx)\cos mt}{(\frac{1}{2}mx)^\nu} = \frac{2}{\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{1}{2}\pi} \cos^{2\nu}\theta \left\{\sum_{m=1}^{n} \cos mt\cos(mx\sin\theta)\right\} d\theta$$

$$= \frac{1}{\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{1}{2}\pi} \cos^{2\nu}\theta \sum_{m=1}^{n} \{\cos m(t+x\sin\theta)+\cos m(t-x\sin\theta)\}\,d\theta$$

$$= \frac{1}{2x^{2\nu}\,\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}\int_{t-\pi}^{t+\pi} \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\{x^2-(u-t)^2\}^{\nu-\frac{1}{2}}\,du - \frac{1}{2\Gamma(\nu+1)}.$$

Now let $n \to \infty$, and we obtain the formulae

$$\sum_{m=1}^{\infty} \frac{J_\nu(mx)\cos mt}{(\frac{1}{2}mx)^\nu} = -\frac{1}{2\Gamma(\nu+1)} \quad (0 < x < t),$$

$$= -\frac{1}{2\Gamma(\nu+1)} + \frac{\Gamma(\frac{1}{2})}{x\Gamma(\nu+\frac{1}{2})}\left(1-\frac{t^2}{x^2}\right)^{\nu-\frac{1}{2}} \quad (t < x < \pi).$$

Similarly, starting with the formula†, valid for $\nu > -\frac{1}{2}$,

$$\mathbf{H}_\nu(z) = \frac{z^\nu}{2^{\nu-1}\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{1}{2}\pi} \sin(z\sin\theta)\cos^{2\nu}\theta\,d\theta,$$

we derive the formulae

$$\sum_{m=1}^{\infty} \frac{\mathbf{H}_\nu(mx)\sin mt}{(\frac{1}{2}mx)^\nu} = 0 \quad (0 < x < t),$$

$$= \frac{\Gamma(\frac{1}{2})}{x\Gamma(\nu+\frac{1}{2})}\left(1-\frac{t^2}{x^2}\right)^{\nu-\frac{1}{2}} \quad (t < x < \pi).$$

Both series diverge at $x = t$.

* G. N. Watson, *loc. cit.*, 401 (1) and (3), 403 (2), and 396 (1).

† *Ibid.*, 328 (1).

The series involving $J_\nu(mx)$ is a generalisation of the well known null series, discovered by Nielsen*.

Since, then, when $\mu > -\frac{1}{2}$,

$$\sum_{m=1}^{\infty} \frac{J_\mu(ma)}{(\frac{1}{2}ma)^\mu} \cos m\beta = -\frac{1}{2\Gamma(\mu+1)} \quad (\pi > \beta > a > 0),$$

$$= -\frac{1}{2\Gamma(\mu+1)} + \frac{\Gamma(\frac{1}{2})}{a\Gamma(\mu+\frac{1}{2})}\left(1-\frac{\beta^2}{a^2}\right)^{\mu-\frac{1}{2}} \quad (\pi > a > \beta > 0),$$

we have, on multiplying by $\cos^{2\nu}\theta$, changing β into $\beta \sin\theta$, and integrating term-by-term from 0 to $\frac{1}{2}\pi$,

$$\int_0^{\frac{1}{2}\pi} \sum_{m=1}^{\infty} \frac{J_\mu(ma)}{(\frac{1}{2}ma)^\mu} \cos(m\beta \sin\theta) \cos^{2\nu}\theta \, d\theta$$

$$= -\frac{1}{2\Gamma(\mu+1)} \int_0^{\frac{1}{2}\pi} \cos^{2\nu}\theta \, d\theta + \frac{\Gamma(\frac{1}{2})}{a\Gamma(\mu+\frac{1}{2})} \int_0^{\arcsin(a/\beta)} \cos^{2\nu}\theta \left(1-\frac{\beta^2}{a^2}\sin^2\theta\right)^{\mu-\frac{1}{2}} d\theta.$$

Hence, if μ and ν are each greater than $-\frac{1}{2}$,

$$\tfrac{1}{2}\Gamma(\tfrac{1}{2})\,\Gamma(\nu+\tfrac{1}{2}) \sum_{m=1}^{\infty} \frac{J_\mu(ma)}{(\frac{1}{2}ma)^\mu} \frac{J_\nu(m\beta)}{(\frac{1}{2}m\beta)^\nu}$$

$$= -\frac{\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}{4\Gamma(\mu+1)\,\Gamma(\nu+1)} + \frac{\Gamma(\frac{1}{2})}{\beta\Gamma(\mu+\frac{1}{2})} \int_0^1 (1-t^2)^{\mu-\frac{1}{2}} \left(1-\frac{a^2}{\beta^2}t^2\right)^{\nu-\frac{1}{2}} dt,$$

if $\pi > \beta > a > 0$; but it is equal to

$$-\frac{\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}{4\Gamma(\mu+1)\,\Gamma(\nu+1)} + \frac{\Gamma(\frac{1}{2})}{a\Gamma(\mu+\frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \left(1-\frac{\beta^2}{a^2}t^2\right)^{\mu-\frac{1}{2}} dt,$$

if $\pi > a > \beta > 0$.

Hence

$$\sum_{m=1}^{\infty} \frac{J_\mu(ma)}{(\frac{1}{2}ma)^\mu} \frac{J_\nu(m\beta)}{(\frac{1}{2}m\beta)^\nu}$$

$$= -\frac{1}{2\Gamma(\mu+1)\,\Gamma(\nu+1)} + \frac{\Gamma(\frac{1}{2})}{\beta\Gamma(\mu+1)\,\Gamma(\nu+\frac{1}{2})} {}_2F_1\left(\tfrac{1}{2}-\nu, \tfrac{1}{2}; \mu+1; \frac{a^2}{\beta^2}\right)$$

when $\pi > \beta > a > 0$; but it is equal to

$$-\frac{1}{2\Gamma(\mu+1)\,\Gamma(\nu+1)} + \frac{\Gamma(\frac{1}{2})}{a\Gamma(\mu+\frac{1}{2})\,\Gamma(\nu+1)} {}_2F_1\left(\tfrac{1}{2}-\mu, \tfrac{1}{2}; \nu+1; \frac{\beta^2}{a^2}\right),$$

when $\pi > a > \beta > 0$.

The series converges uniformly when $a \neq \beta$, provided that $\mu > -\frac{1}{2}$ and $\nu > -\frac{1}{2}$.

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* G. N. Watson, *loc. cit.*, 635 (2).

It also converges uniformly in the neighbourhood of $\alpha = \beta$, provided that $\mu+\nu > 0$.

Both expressions for the sum of the series then reduce, by Gauss's formula*, to

$$-\frac{1}{2\Gamma(\mu+1)\,\Gamma(\nu+1)}+\frac{\Gamma(\frac{1}{2})\,\Gamma(\mu+\nu)}{\beta\Gamma(\mu+\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})\,\Gamma(\mu+\nu+\frac{1}{2})}.$$

These results agree with the corresponding formulae for the Weber-Schafheitlin integral already quoted.

Similarly from the formulae

$$\sum_{m=1}^{\infty}\frac{\mathbf{H}_\mu(m\alpha)}{(\frac{1}{2}m\alpha)^\mu}\sin m\beta = 0 \quad (\pi > \beta > \alpha > 0),$$

$$= \frac{\Gamma(\frac{1}{2})}{\alpha\Gamma(\mu+\frac{1}{2})}\left(1-\frac{\beta^2}{\alpha^2}\right)^{\mu-\frac{1}{2}} \quad (\pi > \alpha > \beta > 0),$$

we derive the results, valid for $\mu > -\frac{1}{2}$, $\nu > -\frac{1}{2}$,

$$\sum_{m=1}^{\infty}\frac{\mathbf{H}_\mu(m\alpha)}{(\frac{1}{2}m\alpha)^\mu}\cdot\frac{\mathbf{H}_\nu(m\beta)}{(\frac{1}{2}m\beta)^\nu} = \frac{\Gamma(\frac{1}{2})}{\beta\Gamma(\mu+1)\,\Gamma(\nu+\frac{1}{2})}\,{}_2F_1\left(\tfrac{1}{2}-\nu,\ \tfrac{1}{2};\ \mu+1;\ \frac{\alpha^2}{\beta^2}\right)$$

when $\pi > \beta > \alpha > 0$; but it is equal to

$$\frac{\Gamma(\frac{1}{2})}{\alpha\Gamma(\mu+\frac{1}{2})\,\Gamma(\nu+1)}\,{}_2F_1\left(\tfrac{1}{2}-\mu,\ \tfrac{1}{2};\ \nu+1;\ \frac{\beta^2}{\alpha^2}\right)$$

when $\pi > \alpha > \beta > 0$; and, when $\alpha = \beta$, it is equal to

$$\frac{\Gamma(\frac{1}{2})\,\Gamma(\mu+\nu)}{\beta\Gamma(\mu+\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})\,\Gamma(\mu+\nu+\frac{1}{2})},$$

the last result being valid only if $\mu+\nu > 0$.

* Whittaker and Watson, *loc. cit.*, 282.

PRODUCTS OF GENERALIZED HYPERGEOMETRIC SERIES

By W. N. Bailey.

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Several formulae were given by Gauss and Kummer* expressing the product of two hypergeometric series as a hypergeometric series, such as $e^{-x}\,{}_1F_1(x)$ as a series of the type ${}_1F_1(-x)$, and

$$(1+x)^{-p}\,{}_2F_1\left(\frac{4x}{(1+x)^2}\right)$$

as a series of the type ${}_2F_1(x)$. Another theorem of a similar type, which was given by Ramanujan*, expressed

$${}_1F_1(\alpha\,;\ \rho\,;\ x)\times{}_1F_1(\alpha\,;\ \rho\,;\ -x)$$

as a series of the type ${}_2F_3$. This result was proved by Preece*, who obtained other results of a similar kind from the consideration of differential equations. As recently as 1927 Whipple* obtained a formula expressing

$$(1-x)^{-p}\,{}_3F_2\left(\frac{-4x}{(1-x)^2}\right)$$

as a hypergeometric series of the type ${}_3F_2(x)$. In this paper a systematic search is made for such formulae. Evidently if the product of two hypergeometric series can be expressed as a hypergeometric series with argument x, the coefficient of x^n in the product must be expressible in terms of gamma functions. In all the cases considered the coefficient of x^n in the product can be found as a hypergeometric series, and everything depends on being able to sum this series. The following formulae are used :

Gauss's formula (1).

$${}_2F_1(a,\ b\,;\ c\,;\ 1)=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}.$$

* References are given in the course of the paper.

Gauss's formula (2).

$$ {}_2F_1\{a,\ b;\ \tfrac{1}{2}(a+b+1);\ \tfrac{1}{2}\} = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(\frac{1}{2}a+\frac{1}{2})\,\Gamma(\frac{1}{2}b+\frac{1}{2})}. $$

*Kummer's formula**.

$$ {}_2F_1(a,\ b;\ 1+a-b;\ -1) = \frac{\Gamma(1+a-b)\,\Gamma(\frac{1}{2})}{2^a\,\Gamma(\frac{1}{2}a+\frac{1}{2})\,\Gamma(\frac{1}{2}a-b+1)}. $$

Saalschütz's formula†.

$$ {}_3F_2(a,\ b,\ c;\ d,\ e) = \frac{\Gamma(d)\,\Gamma(1-e+a)\,\Gamma(1-e+b)\,\Gamma(1-e+c)}{\Gamma(1-e)\,\Gamma(d-a)\,\Gamma(d-b)\,\Gamma(d-c)}, $$

provided that a, b, or c is a negative integer and $d+e=a+b+c+1$.

Dixon's formula‡.

$$ {}_3F_2(a,\ a-d+1,\ a-e+1;\ d,\ e) = \frac{\Gamma(d)\,\Gamma(e)\,\Gamma(d+e-\frac{3}{2}a-1)\,\Gamma(\frac{1}{2})}{2^a\,\Gamma(\frac{1}{2}a+\frac{1}{2})\,\Gamma(d-\frac{1}{2}a)\,\Gamma(e-\frac{1}{2}a)\,\Gamma(d+e-a+1)}. $$

Watson's formula§.

$$ {}_3F_2\left[\begin{matrix}a,\ b,\ c:\\ \frac{1}{2}(a+b+1),\ 2c\end{matrix}\right] = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+c)\,\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)\,\Gamma(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\,\Gamma(\frac{1}{2}+\frac{1}{2}b)\,\Gamma(\frac{1}{2}-\frac{1}{2}a+c)\,\Gamma(\frac{1}{2}-\frac{1}{2}b+c)}, $$

Whipple's formula‖.

$$ {}_3F_2[a,\ b,\ c;\ d,\ e] = \frac{\pi\Gamma(d)\,\Gamma(e)}{2^{2c-1}\,\Gamma(\frac{1}{2}a+\frac{1}{2}d)\,\Gamma(\frac{1}{2}a+\frac{1}{2}e)\,\Gamma(\frac{1}{2}b+\frac{1}{2}d)\,\Gamma(\frac{1}{2}b+\frac{1}{2}e)} $$

provided that $a+b=1$ and $d+e=2c+1$.

When these formulae are used in the following, the letters G 1, G 2, K, S, D, Wa, Wh respectively will be placed after the formula obtained.

The formulae used all refer to the summation of series of the type ${}_2F_1$ and ${}_3F_2$. There are formulae for summing series of higher order up to ${}_7F_6$ based on Dougall's¶ theorem, and also formulae for summing ${}_4F_3$,

* Kummer, *Journal für Math.*, 15 (1836), 78, Formula 53.

† Saalschütz, "Eine Summationsformel", *Zeitschrift für Math. u. Phys.*, 35 (1890), 186-188.

‡ A. C. Dixon, "Summation of a certain series", *Proc. London Math. Soc.* (1), 35 (1902), 284-289.

§ Watson, *Proc. London Math. Soc.* (2), 23 (1924), xiii-xv. Generalized by Whipple, *Proc. London Math. Soc.* (2), 23 (1924), 113, Formula 31.

‖ Whipple, *loc. cit.* 114, Formula 34.

¶ Dougall, "On Vandermonde's theorem and some more general expansions", *Proc. Edinburgh Math. Soc.*, 25 (1907), 114-132.

which can be obtained by equating coefficients of x^n in the formulae

$$\{{}_2F_1(a,\ \beta;\ a+\beta+\tfrac{1}{2};\ x)\}^2 = {}_3F_2\{2a,\ 2\beta,\ a+\beta;\ 2a+2\beta,\ a+\beta+\tfrac{1}{2};\ x\},$$

$${}_2F_1(a,\ \beta;\ a+\beta-\tfrac{1}{2};\ x)\,{}_2F_1(a,\ \beta;\ a+\beta+\tfrac{1}{2};\ x)$$
$$= {}_3F_2\{2a,\ 2\beta,\ a+\beta;\ 2a+2\beta-1,\ a+\beta+\tfrac{1}{2};\ x\},$$

$${}_2F_1(a,\ \beta;\ a+\beta-\tfrac{1}{2};\ x)\,{}_2F_1(a,\ \beta-1;\ a+\beta-\tfrac{1}{2};\ x)$$
$$= {}_3F_2\{2a-1,\ 2\beta,\ a+\beta-1;\ 2a+2\beta-2,\ a+\beta-\tfrac{1}{2};\ x\},$$

the first of which was given by Clausen* and the others by Orr†.

It is found, however, that these methods of summation lead to no new formulae.

In § 2 the product ${}_pF_q(x)\times{}_rF_s(cx)$ is considered, and expressed when possible (by using one of the above formulae) as a hypergeometric series. In § 3 the product ${}_pF_q(x)\times{}_rF_s(cx^\lambda)$ is considered, and in § 4 the product

$$(1-x)^p_r F_s\left(\frac{kx^\lambda}{(1-x)^\mu}\right).$$

In § 5 a formula is obtained which is of a type somewhat similar to Cayley's theorem on hypergeometric functions, and in § 6 the sum of a series ${}_4F_3$ is obtained in finite terms.

2. In the product

$${}_pF_q(a_1,\ a_2,\ \ldots,\ a_p;\ \rho_1,\ \rho_2,\ \ldots,\ \rho_q;\ x)\times{}_rF_s(\beta_1,\ \beta_2,\ \ldots,\ \beta_r;\ \sigma_1,\ \ldots,\ \sigma_s;\ cx)$$

the coefficient of x^n is

$$\sum_{m=0}^{n}\frac{(a_1)_{n-m}\ldots(a_p)_{n-m}}{(n-m)!\ (\rho_1)_{n-m}\ldots(\rho_q)_{n-m}}\cdot\frac{(\beta_1)_m\ldots(\beta_r)_m}{m!\ (\sigma_1)_m\ldots(\sigma_s)_m}c^m$$

$$= \frac{(a_1)_n\ldots(a_p)_n}{n!\ (\rho_1)_n\ldots(\rho_q)_n}$$

$$\times\sum_{m=0}^{n}(-1)^{m(p+q+1)}\frac{(-n)_m(1-\rho_1-n)_m\ldots(1-\rho_q-n)_m(\beta_1)_m\ldots(\beta_r)_m}{m!\ (\sigma_1)_m\ldots(\sigma_s)_m(1-a_1-n)_m\ldots(1-a_p-n)_m}c^m$$

$$= \frac{(a_1)_n\ldots(a_p)_n}{n!\ (\rho_1)_n\ldots(\rho_q)_n}$$

$$\times{}_{q+r+1}F_{p+s}\left[\begin{matrix}-n,\ \beta_1,\ \ldots,\ \beta_r,\ 1-\rho_1-n,\ \ldots,\ 1-\rho_q-n;\\ \sigma_1,\ \ldots,\ \sigma_s,\ 1-a_1-n,\ \ldots,\ 1-a_p-n;\end{matrix}\ \pm c\right]$$

* Clausen, "Ueber die Fälle u.s.w.", *Journal für Math.*, 3 (1828), 89–95. This formula can be deduced from Dougall's theorem.

† Orr, "Theorems relating to the product of two hypergeometric series", *Trans. Camb. Phil. Soc.*, 17 (1899), 1–15.

the positive or negative sign being correct according as $p+q$ is odd or even.

The product ${}_pF_q \times {}_rF_s$ can be expressed as a hypergeometric series only if we can sum the series ${}_{q+r+1}F_{p+s}$. With the formulae we are using this can only be done if $p+s = q+r = 0,\ 1,\ 2$.

Omitting the trivial cases ${}_0F_0 \times {}_0F_0$, ${}_1F_0 \times {}_1F_0$ we are led to the following formulae :

(2.01)* $$e^{-x}\,{}_1F_1(a;\ \rho;\ x) = {}_1F_1(\rho-a;\ \rho;\ -x) \qquad \text{(G 1)}$$

(2.02)† $$e^{-\frac{1}{2}x}\,{}_1F_1(a;\ 2a;\ x) = {}_0F_1(a+\tfrac{1}{2};\ \tfrac{1}{16}x^2) \qquad \text{(G 2)}$$

(2.03)‡ $${}_0F_1(\rho;\ x) \times {}_0F_1(\sigma;\ x)$$

$$= {}_2F_3\left[\tfrac{1}{2}(\rho+\sigma),\ \tfrac{1}{2}(\rho+\sigma-1);\ \rho,\ \sigma,\ \rho+\sigma-1;\ 4x\right], \qquad \text{(G 1)}$$

(2.04)§ $${}_0F_1(\rho;\ x) \times {}_0F_1(\rho;\ -x) = {}_0F_3(\rho,\ \tfrac{1}{2}\rho,\ \tfrac{1}{2}\rho+\tfrac{1}{2};\ -\tfrac{1}{4}x^2), \qquad \text{(K)}$$

(2.05) $${}_0F_1(\rho;\ x) \times {}_0F_1(2-\rho;\ -x)$$

$$= {}_0F_3\left[\tfrac{1}{2},\ \tfrac{1}{2}(\rho+1),\ \tfrac{1}{2}(3-\rho);\ -\tfrac{1}{4}x^2\right]$$

$$+ \frac{2(1-\rho)}{\rho(2-\rho)}\,x\ {}_0F_3(\tfrac{3}{2},\ 1+\tfrac{1}{2}\rho,\ 2-\tfrac{1}{2}\rho;\ -\tfrac{1}{4}x^2), \qquad \text{(K)}$$

(2.06)‖ $$(1-x)^{\alpha+\beta-\gamma}\,{}_2F_1(a,\ \beta;\ \gamma;\ x) = {}_2F_1(\gamma-a,\ \gamma-\beta;\ \gamma;\ x), \qquad \text{(S)}$$

(2.07) $${}_2F_0(a,\ \beta;\ x) \times {}_2F_0(a,\ \beta;\ -x)$$

$$= {}_4F_1\left[a,\ \beta,\ \tfrac{1}{2}(a+\beta),\ \tfrac{1}{2}(a+\beta+1);\ a+\beta;\ 4x^2\right], \qquad \text{(D)}$$

(2.08) $${}_2F_0(a,\ 1-a;\ x) \times {}_2F_0(\beta,\ 1-\beta;\ -x)$$

$$= {}_4F_1\left[\tfrac{1}{2}(1+a-\beta),\ \tfrac{1}{2}(1-a+\beta),\ \tfrac{1}{2}(a+\beta),\ \tfrac{1}{2}(2-a-\beta);\ \tfrac{1}{2};\ 4x^2\right]$$

$$-(a-\beta)(a+\beta-1)\,x\ {}_4F_1\left[\tfrac{1}{2}(2+a-\beta),\ \tfrac{1}{2}(2-a+\beta),\ \tfrac{1}{2}(a+\beta+1),\right.$$

$$\left.\tfrac{1}{2}(3-a-\beta);\ \tfrac{3}{2};\ 4x^2\right], \qquad \text{(Wh)}$$

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* Kummer, *Journal für Math.*, 15 (1836), 139 (4).

† Kummer, *loc. cit.*, 140 (8).

‡ Equivalent to the formula for $J_m(x) \times J_n(x)$.

§ Giving $J_n(x) \times I_n(x)$.

‖ Conversely from (2.06) (due to Euler) we can prove Saalschütz's Theorem, as was pointed out by Watson.

$$(2.09)^* \quad {}_1F_1(a;\ \rho;\ x)\times {}_1F_1(a;\ \rho;\ -x) = {}_2F_3(a,\ \rho-a;\ \rho,\ \tfrac{1}{2}\rho,\ \tfrac{1}{2}\rho+\tfrac{1}{2};\ \tfrac{1}{4}x^2), \qquad (D)$$

$$(2.10)^\dagger \quad {}_1F_1(a;\ \rho;\ x)\times {}_1F_1(a-\rho+1;\ 2-\rho;\ -x) = {}_2F_3(a-\tfrac{1}{2}\rho+\tfrac{1}{2},\ \tfrac{1}{2}\rho-a+\tfrac{1}{2};\ \tfrac{1}{2}\rho+\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{3}{2}-\tfrac{1}{2}\rho;\ \tfrac{1}{4}x^2) + \frac{(2a-\rho)(1-\rho)}{\rho(2-\rho)}\ x\ {}_2F_3(a-\tfrac{1}{2}\rho+1,\ \tfrac{1}{2}\rho-a+1;\ \tfrac{1}{2}\rho+1,\ 2-\tfrac{1}{2}\rho,\ \tfrac{3}{2};\ \tfrac{1}{4}x^2), \qquad (D)$$

$$(2.11)^\dagger \quad {}_1F_1(a;\ 2a;\ x)\times {}_1F_1(\beta;\ 2\beta;\ -x) = {}_2F_3\left[\tfrac{1}{2}(a+\beta),\ \tfrac{1}{2}(a+\beta+1);\ a+\tfrac{1}{2},\ \beta+\tfrac{1}{2},\ a+\beta;\ \tfrac{1}{4}x^2\right], \qquad (Wa)$$

$$(2.12)^* \quad {}_0F_2(\rho_1,\ \rho_2;\ x)\times {}_0F_2(\rho_1,\ \rho_2;\ -x) = {}_3F_8\left[\tfrac{1}{3}(\rho_1+\rho_2-1),\ \tfrac{1}{3}(\rho_1+\rho_2),\ \tfrac{1}{3}(\rho_1+\rho_2+1);\ \rho_1,\ \rho_2,\ \tfrac{1}{2}\rho_1,\ \tfrac{1}{2}\rho_2,\ \tfrac{1}{2}(\rho_1+1),\ \tfrac{1}{2}(\rho_2+1),\ \tfrac{1}{2}(\rho_1+\rho_2-1),\ \tfrac{1}{2}(\rho_1+\rho_2);\ -\tfrac{27}{64}x^2\right]. \qquad (D)$$

The series ${}_2F_0$ and ${}_4F_1$ never converge, and so (2.07) and (2.08) are only significant when all the series terminate. In (2.07) either a or β must be a negative integer, and then the series can be reversed, and we are led to a particular case of (2.09). In (2.08) a and β must both be integers, positive or negative. If, for example, they are both negative, we are led, after reversing the series, to a formula involving

$${}_1F_1(a;\ 2a;\ x)\times {}_1F_1(\beta;\ 2\beta;\ -x)$$

when a and β are negative integers. This is not a particular case of (2.11), as that formula is true only if a and β are not negative integers.

3. The coefficient of x^n in the product

$${}_pF_q(a_1,\ \ldots,\ a_p;\ \rho_1,\ \ldots,\ \rho_q;\ x)\times {}_rF_s(\beta_1,\ \ldots,\ \beta_r;\ \sigma_1,\ \ldots,\ \sigma_s;\ cx^\lambda),$$

* (2.09) and (2.12) were given by Ramanujan; see Hardy, "A chapter from Ramanujan's note-book", *Proc. Camb. Phil. Soc.*, 21 (1923), 492–503.

† (2.10) was given by Preece, "The product of two generalized hypergeometric functions", *Proc. London Math. Soc.* (2), 22 (1924), 370–380, but (2.11) was given only in the particular cases $\beta = a$, $\beta = 1-a$.

where λ is a positive integer, can be put in the form

$$(3.1)\quad \frac{(a_1)_n \dots (a_p)_n}{n!\,(\rho_1)_n \dots (\rho_q)_n}$$

$$\times {}_PF_Q\left[-\frac{n}{\lambda}, \frac{1-n}{\lambda}, \dots, \frac{\lambda-1-n}{\lambda}, \beta_1, \dots, \beta_r, \cdot\right.$$

$$\frac{1-\rho_1-n}{\lambda}, \frac{2-\rho_1-n}{\lambda}, \dots, \frac{\lambda-\rho_1-n}{\lambda}, \dots;$$

$$\left.\sigma_1, \dots, \sigma_s, \frac{1-a_1-n}{\lambda}, \frac{2-a_1-n}{\lambda}, \dots \frac{\lambda-a_1-n}{\lambda}, \dots; \pm c\lambda^{\lambda(q-p+1)}\right],$$

where the positive or negative sign is taken according as $\lambda(p+q+1)$ is even or odd, and

$$P = \lambda(q+1)+r, \quad Q = \lambda p+s.$$

The only formulae obtained by summing ${}_PF_Q$ are

$$(3.2)\quad e^x\,{}_0F_1(\tfrac{1}{2};\ -\tfrac{1}{4}x^2) = \sum_{n=0}^{\infty} 2^{\frac{1}{2}n} \cos\frac{n\pi}{4} \cdot \frac{x^n}{n!}, \qquad \text{(K)}$$

$$(3.3)\quad e^x\,{}_0F_1(\tfrac{3}{2};\ -\tfrac{1}{4}x^2) = \sum_{n=1}^{\infty} 2^{\frac{1}{2}n} \sin\frac{n\pi}{4} \cdot \frac{x^{n-1}}{n!}, \qquad \text{(K)}$$

$$(3.4)\quad e^x\,{}_0F_2\left(\tfrac{1}{3}, \tfrac{2}{3};\ -\frac{x^3}{27}\right) = 1+2\sum_{n=1}^{\infty} 3^{\frac{1}{2}n-1} \cos\frac{n\pi}{6} \cdot \frac{x^n}{n!}, \qquad \text{(D)}$$

$$(3.5)\quad e^x\,{}_0F_2\left(\tfrac{2}{3}, \tfrac{4}{3};\ -\frac{x^3}{27}\right) = 2\sum_{n=0}^{\infty} 3^{\frac{1}{2}(n-1)} \sin\frac{(n+2)\pi}{6} \cdot \frac{x^n}{(n+1)!}, \qquad \text{(D)}$$

$$(3.6)\quad e^x\,{}_0F_2\left(\tfrac{4}{3}, \tfrac{5}{3};\ -\frac{x^3}{27}\right) = 4\sum_{n=0}^{\infty} 3^{\frac{1}{2}n} \sin\frac{(n+1)\pi}{6} \cdot \frac{x^n}{(n+2)!}. \qquad \text{(D)}$$

and a formula equivalent to (2.02).

The expressions on the left of (3.2)–(3.6) can all be put in terms of elementary functions. These relations can be generalized by using an elementary argument. For, if $\omega \equiv e^{\pi i/\lambda}$, where λ is a prime,

$$\sum_{r=1}^{\lambda} e^{x\omega^{2r-1}} = \lambda\left[1-\frac{x^\lambda}{\lambda!}+\frac{x^{2\lambda}}{(2\lambda)!}-\dots\right]$$

$$= \lambda\,{}_0F_{\lambda-1}\left\{\frac{1}{\lambda}, \frac{2}{\lambda}, \dots, \frac{\lambda-1}{\lambda};\ -\left(\frac{x}{\lambda}\right)^\lambda\right\},$$

and so the coefficient of x^n in

$$e^x\,{}_0F_{\lambda-1}\left\{\frac{1}{\lambda}, \frac{2}{\lambda}, \dots, \frac{\lambda-1}{\lambda};\ -\left(\frac{x}{\lambda}\right)^\lambda\right\}$$

is, if $n > 1$,

$$\frac{1}{\lambda}\sum_{r=1}^{\lambda}\frac{(1+\omega^{2r-1})^n}{n!} = \frac{2^n}{\lambda(n)!}\sum_{r=1}^{\lambda}\cos^n\frac{(2r-1)\pi}{2\lambda}\cos\frac{n(2r-1)\pi}{2\lambda}.$$

We therefore have the formula

$$(3.7)\quad e^x\,{}_0F_{\lambda-1}\left\{\frac{1}{\lambda}, \frac{2}{\lambda}, \ldots, \frac{\lambda-1}{\lambda};\ -\left(\frac{x}{\lambda}\right)^{\lambda}\right\} = 1+\frac{1}{\lambda}\sum_{n=1}^{\infty}\left\{\sum_{r=1}^{\lambda}\cos^n\frac{(2r-1)\pi}{2\lambda}\cos\frac{n(2r-1)\pi}{2\lambda}\right\}\frac{2^n x^n}{n!}.$$

This is a generalization of (3.2) and (3.4), and similar generalizations of the other formulae could be obtained.

From (3.1) and (3.7) we derive the formula

$$(3.8)\quad {}_{\lambda}F_{\lambda-1}\left\{\frac{-n}{\lambda}, \frac{1-n}{\lambda}, \ldots, \frac{\lambda-1-n}{\lambda};\ \frac{1}{\lambda}, \frac{2}{\lambda}, \ldots, \frac{\lambda-1}{\lambda}\right\} = \frac{2^n}{\lambda}\sum_{r=1}^{\lambda}\cos^n\frac{(2r-1)\pi}{2\lambda}\cos\frac{n(2r-1)\pi}{2\lambda},$$

where n is a positive integer greater than unity, and λ an odd prime. The series on the left of (3.8) is well poised.

4. Now consider the product

$$(1-x)^p\,{}_rF_s\left(\alpha_1, \ldots, \alpha_r;\ \rho_1, \ldots, \rho_s;\ \frac{kx^{\lambda}}{(1-x)^{\mu}}\right)$$

where λ is a positive integer, μ any integer, and $p \neq 0$.

The coefficient of x^n in the product is

$$\sum_{m=0}^{[n/\lambda]}\frac{(\alpha_1)_m \ldots (\alpha_r)_m}{m!\,(\rho_1)_m \ldots (\rho_s)_m}\cdot\frac{(m\mu-p)_{n-\lambda m}}{(n-\lambda m)!}\,k^m.$$

If $\mu > \lambda$, this can be written

$$(4.01)\quad \frac{(-p)_n}{n!}\cdot {}_{r+\mu}F_{s+\mu}\left[\alpha_1, \ldots, \alpha_r, \frac{-n}{\lambda}, \frac{1-n}{\lambda}, \ldots, \frac{\lambda-1-n}{\lambda}, \frac{n-p}{\mu-\lambda}, \frac{1+n-p}{\mu-\lambda}, \ldots, \frac{\mu-\lambda-1+n-p}{\mu-\lambda};\ \rho_1, \ldots, \rho_s, \frac{-p}{\mu}, \frac{1-p}{\mu}, \ldots, \frac{\mu-1-p}{\mu};\ \frac{(-\lambda)^{\lambda}(\mu-\lambda)^{\mu-\lambda}k}{\mu^{\mu}}\right].$$

If $\mu = \lambda\ (>0)$, the coefficient of x^n is

$$(4.02)\quad \frac{(-p)_n}{n!}\,{}_{r+\lambda}F_{s+\lambda}\left[a_1,\ \ldots,\ a_r,\ \frac{-n}{\lambda},\ \frac{1-n}{\lambda},\ \ldots,\ \frac{\lambda-1-n}{\lambda};\right.$$

$$\left.\rho_1,\ \ldots\ \rho_s,\ \frac{-p}{\lambda},\ \frac{1-p}{\lambda},\ \ldots,\ \frac{\lambda-1-p}{\lambda};\ (-1)^\lambda k\right].$$

If $0 < \mu < \lambda$, the coefficient is

$$(4.03)\quad \frac{(-p)_n}{n!}\,{}_{r+\lambda}F_{s+\lambda}\left[a_1,\ \ldots,\ a_r,\ \frac{-n}{\lambda},\ \frac{1-n}{\lambda},\ \ldots,\ \frac{\lambda-1-n}{\lambda};\right.$$

$$\rho_1,\ \ldots,\ \rho_s,\ \frac{-p}{\mu},\ \frac{1-p}{\mu},\ \ldots,\ \frac{\mu-1-p}{\mu},\ \frac{1+p-n}{\lambda-\mu},$$

$$\left.\frac{2+p-n}{\lambda-\mu},\ \ldots,\ \frac{\lambda-\mu+p-n}{\lambda-\mu};\ \frac{(-1)^\mu k\lambda^\lambda}{\mu^\mu(\lambda-\mu)^{\lambda-\mu}}\right]$$

If $\mu = -\mu' < 0$, the coefficient of x^n is

$$(4.04)\quad \frac{(-p)_n}{n!}\,{}_{r+\lambda+\mu'}F_{s+\lambda+\mu'}\left[a_1,\ \ldots,\ a_r,\ \frac{-n}{\lambda},\ \frac{1-n}{\lambda},\ \ldots,\ \frac{\lambda-1-n}{\lambda},\right.$$

$$\frac{1+p}{\mu'},\ \frac{2+p}{\mu'},\ \ldots,\ \frac{\mu'+p}{\mu'};$$

$$\rho_1,\ \ldots,\ \rho_s,\ \frac{1+p-n}{\lambda+\mu'},\ \ldots,\ \frac{\lambda+\mu'+p-n}{\lambda+\mu'};$$

$$\left.\frac{k\lambda^\lambda\,\mu'^{\mu'}}{(\lambda+\mu')^{\lambda+\mu'}}\right].$$

We are thus led to the formulae :

$$(4.05)^*\quad (1-x)^{-3a}\,{}_3F_2\left(a,\ a+\tfrac{1}{3},\ a+\tfrac{2}{3};\ \rho_1,\ \rho_2;\ \frac{-27x}{4(1-x)^3}\right)$$

$$= {}_3F_2(3a,\ \tfrac{1}{2}+\rho_1-\rho_2,\ \tfrac{1}{2}+\rho_2-\rho_1;\ \rho_1,\ \rho_2;\ \tfrac{1}{4}x),\quad (S)$$

provided that $\rho_1+\rho_2 = 3a+\frac{3}{2}$.

$$(4.06)\quad (1-x)^{-3a}\,{}_3F_2\left(a,\ a+\tfrac{1}{3},\ a+\tfrac{2}{3};\ \rho_1,\ \rho_2;\ \frac{27x^2}{4(1-x)^3}\right)$$

$$= {}_3F_2(3a,\ \rho_1-\tfrac{1}{2},\ \rho_2-\tfrac{1}{2};\ 2\rho_1-1,\ 2\rho_2-1;\ 4x),\quad (S)$$

* Formulae involving ordinary hypergeometric functions are obtained from (4.05) and (4.06) by taking $\rho_1 = \alpha$, $a+\frac{1}{3}$, $a+\frac{2}{3}$.

provided that $\rho_1+\rho_2 = 3a+\frac{3}{2}$.

$$(4.07)\quad (1-x)^{-2a}\,{}_2F_1\left(a,\ b;\ a+b+\tfrac{1}{2};\ \frac{-4x}{(1-x)^2}\right) = {}_2F_1(2a,\ a-b+\tfrac{1}{2};\ a+b+\tfrac{1}{2};\ x), \quad (S)$$

$$(4.08)\quad (1-x)^{1-2a}\,{}_2F_1\left(a,\ b;\ a+b+\tfrac{1}{2};\ \frac{-4x}{(1-x)^2}\right) = {}_3F_2(2a-1,\ a+\tfrac{1}{2},\ a-b-\tfrac{1}{2};\ a+b+\tfrac{1}{2},\ a-\tfrac{1}{2};\ x), \quad (S)$$

$$(4.09)^*\quad (1+x)^{-2a}\,{}_2F_1\left(a,\ a+\tfrac{1}{2};\ c;\ \frac{4x}{(1+x)^2}\right) = {}_2F_1(2a,\ 2a+1-c;\ c;\ x), \quad (G\,1)$$

$$(4.10)^*\quad (1+x)^{-2a}\,{}_2F_1\left(a,\ b;\ 2b;\ \frac{4x}{(1+x)^2}\right) = {}_2F_1(a,\ a-b+\tfrac{1}{2};\ b+\tfrac{1}{2};\ x^2), \quad (Wa)$$

$$(4.11)\quad (1-x)^{-1}\,{}_2F_1\left(1,\ b;\ 2b+\tfrac{1}{2};\ \frac{-4x}{(1-x)^2}\right) = {}_2F_1(1,\ \tfrac{3}{4}-b;\ \tfrac{3}{4}+b;\ x^2)+\frac{x(4b-1)}{4b+1}\,{}_2F_1(1,\ \tfrac{5}{4}-b;\ \tfrac{5}{4}+b;\ x^2), \quad (Wh)$$

$$(4.12)^\dagger\quad (1-x)^{1-2a}\,{}_3F_2\left(a,\ a-\tfrac{1}{2},\ \rho_1+\rho_2-2a;\ \rho_1,\ \rho_2;\ \frac{-4x}{(1-x)^2}\right) = {}_3F_2[2a-1,\ 2a-\rho_1,\ 2a-\rho_2;\ \rho_1,\ \rho_2;\ x), \quad (S)$$

$$(4.13)\quad (1-x)^{-1}\,{}_3F_2\left[1,\ \tfrac{1}{2},\ \tfrac{1}{2}(\rho_1+\rho_2-1);\ \rho_1,\ \rho_2;\ \frac{-4x}{(1-x)^2}\right]$$

$$= {}_3F_2[1,\ 1-\tfrac{1}{2}\rho_1,\ 1-\tfrac{1}{2}\rho_2;\ \tfrac{1}{2}(1+\rho_1),\ \tfrac{1}{2}(1+\rho_2);\ x^2] + \frac{(\rho_1-1)(\rho_2-1)x}{\rho_1\rho_2}\,{}_3F_2[1,\ \tfrac{1}{2}(3-\rho_1),\ \tfrac{1}{2}(3-\rho_2);\ 1+\tfrac{1}{2}\rho_1,\ 1+\tfrac{1}{2}\rho_2;\ x^2]. \quad (Wh)$$

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* Gauss, "Disquisitiones generales circa seriem infinitam, etc." (supplementary part), *Werke* 3, Formulae 100 and 101. In (4.09) and (4.10) the sign of x has been changed to agree with Gauss's notation.

† (4.12) is equivalent to Formulae (7.1) and (7.4) of Whipple, "Some transformations of generalized hypergeometric series", *Proc. London Math. Soc.* (2), 26 (1927), 257–272, but the above is a much simpler proof. The formula includes (4.07) and (4.08) as particular cases.

$$(4.14)\quad (1-x)^{-3a}\,{}_3F_2\left(a,\ a+\tfrac{1}{3},\ a+\tfrac{2}{3};\ \tfrac{1}{3},\ \tfrac{2}{3};\ \frac{-x^3}{(1-x)^3}\right) = 1+\tfrac{2}{3}\sum_{n=1}^{\infty}\frac{(3a)_n}{n!}\cos\frac{n\pi}{6}\,.\,3^{\frac{1}{2}n}\,x^n, \quad (D)$$

$$(4.15)\quad (1-x)^{-3a}\,{}_3F_2\left(a,\ a+\tfrac{1}{3},\ a+\tfrac{2}{3};\ \tfrac{2}{3},\ \tfrac{4}{3};\ \frac{-x^3}{(1-x)^3}\right) = 2\sum_{n=0}^{\infty}\frac{(3a)_n}{(n+1)!}\sin\frac{(n+2)\pi}{6}\,3^{\frac{1}{2}(n-1)}\,x^n, \quad (D)$$

$$(4.16)\quad (1-x)^{-3a}\,{}_3F_2\left(a,\ a+\tfrac{1}{3},\ a+\tfrac{2}{3};\ \tfrac{4}{3},\ \tfrac{5}{3};\ \frac{-x^3}{(1-x)^3}\right) = 4\sum_{n=0}^{\infty}\sin\frac{(n+1)\pi}{6}\,\frac{(3a)_n}{(n+2)!}\,3^{\frac{1}{2}n}\,x^n, \quad (D)$$

$$(4.17)^{*}\quad (1-x)^{-a_1}\,{}_2F_1\left(a_1,\ a_2;\ \rho;\ -\frac{x}{1-x}\right) = {}_2F_1(a_1,\ \rho-a_2;\ \rho;\ x), \quad (G\ 1)$$

$$(4.18)^{\dagger}\quad (1-x)^{-2a}\,{}_2F_1\left(a,\ a+\tfrac{1}{2};\ \rho+\tfrac{1}{2};\ \frac{x^2}{(1-x)^2}\right) = {}_2F_1(2a,\ \rho;\ 2\rho;\ 2x), \quad (G\ 1)$$

$$(4.19)\quad (1-x)^{-2a}\,{}_2F_1\left(a,\ a+\tfrac{1}{2};\ \tfrac{1}{2};\ -\frac{x^2}{(1-x)^2}\right) = \sum_{n=0}^{\infty}\cos\frac{n\pi}{4}\,\frac{(2a)_n\,2^{\frac{1}{2}n}}{n!}\,x^n, \quad (K)$$

$$(4.20)\quad (1-x)^{-2a}\,{}_2F_1\left(a,\ a+\tfrac{1}{2};\ \tfrac{3}{2};\ -\frac{x^2}{(1-x)^2}\right) = \sum_{n=0}^{\infty}\sin\frac{(n+1)\pi}{4}\,\frac{(2a)_n\,2^{\frac{1}{2}(n+1)}}{(n+1)!}\,x^n, \quad (K)$$

$$(4.21)\quad (1-x)^{-2a}\,{}_2F_1\left(a,\ a+\tfrac{1}{2};\ 2a+1;\ \frac{4x^3}{27(1-x)^2}\right) = {}_2F_1\left(3a,\ 3a+1;\ 6a+1;\ \frac{4x}{3}\right), \quad (S)$$

$$(4.22)\quad (1-x)^{-a}\,{}_2F_1\left(a,\ \beta;\ a+\beta+\tfrac{1}{2};\ -\frac{x^2}{4(1-x)}\right) = {}_2F_1(2a,\ a+\beta;\ 2a+2\beta;\ x), \quad (S)$$

$$(4.23)\quad (1-x)^{a+\beta-\frac{1}{2}}\,{}_2F_1\{a,\ \beta;\ a+\beta+\tfrac{1}{2};\ 4x(1-x)\} = {}_2F_1(a-\beta+\tfrac{1}{2},\ \beta-a+\tfrac{1}{2};\ a+\beta+\tfrac{1}{2};\ x). \quad (S)$$

By combining (2 . 06) and (4 . 23) we obtain the quadratic transformation

$$(4.24)\quad {}_2F_1\{a,\ \beta;\ a+\beta+\tfrac{1}{2};\ 4x(1-x)\} = {}_2F_1(2a,\ 2\beta;\ a+\beta+\tfrac{1}{2};\ x)$$

* Gauss, *loc. cit.*, Formula 92.

† Kummer, *Journal für Math.*, 15 (1836), 78 (52).

due to Gauss*. This appears to be the only formula of any elegance, of the type considered in this paragraph, in which there is no factor of the form $(1-x)^p$.

5. By comparison of coefficients it is easily seen that, if

$$(5.1)\quad (1-x)^{-p}\,{}_rF_s\left[a_1,\ a_2,\ \ldots,\ a_r;\ \rho_1,\ \rho_2,\ \ldots,\ \rho_s;\ c\left(\frac{\lambda x}{1-x}\right)^{\lambda}\right] = a_0+a_1x+a_2x^2+\ldots.$$

then

$$(5.2)\quad e^x\,{}_rF_{s+\lambda}\left(a_1,\ a_2,\ \ldots,\ a_r;\ \rho_1,\ \rho_2,\ \ldots,\ \rho_s,\ \frac{p}{\lambda},\ \frac{1+p}{\lambda},\ \ldots,\ \frac{\lambda-1+p}{\lambda};\ cx^{\lambda}\right) = a_0+\frac{a_1x}{(p)_1}+\frac{a_2x^2}{(p)_2}+\ldots.$$

The particular case of this when $\lambda=s=1$, $r=2$ becomes, after using (4.17),

$$(5.3)\quad e^x\,{}_2F_2(\alpha,\ \rho-\beta;\ \rho,\ \alpha-\gamma;\ -x) = a_0+\frac{a_1x}{(\alpha-\gamma)_1}+\frac{a_2x^2}{(\alpha-\gamma)_2}+\ldots,$$

where $$(1-x)^{\gamma}\,{}_2F_1(\alpha,\ \beta;\ \rho;\ x) = a_0+a_1x+a_2x^2+\ldots.$$

If $\gamma=0$ we obtain Kummer's formula (2.01), so that (5.3) may be regarded as a generalization of that formula.

Similarly, taking $\lambda=2$ and using (4.18) we find that

$$(5.4)\quad e^x\,{}_2F_3\left\{\alpha,\ \alpha+\tfrac{1}{2};\ \rho+\tfrac{1}{2},\ \tfrac{1}{2}p,\ \tfrac{1}{2}(1+p);\ \tfrac{1}{4}x^2\right) = b_0+\frac{b_1x}{(p)_1}+\frac{b_2x^2}{(p)_2}+\ldots,$$

where

$$(1-x)^{2\alpha-p}\,{}_2F_1(2\alpha,\ \rho;\ 2\rho;\ 2x) = b_0+b_1x+b_2x^2+\ldots.$$

This is a generalization of (2.02), since the latter formula is obtained by taking $p=2\alpha$.

From (5.1) and (5.2) we have

$$(5.5)\quad (1-x)^{-q}\,{}_{r+\lambda}F_{s+\lambda}\left[a_1,\ \ldots,\ a_r,\ \frac{q}{\lambda},\ \frac{1+q}{\lambda},\ \ldots,\ \frac{\lambda-1+q}{\lambda};\ \rho_1,\ \ldots,\ \rho_s,\ \frac{p}{\lambda},\ \frac{1+p}{\lambda},\ \ldots,\ \frac{\lambda-1+p}{\lambda};\ c\left(\frac{\lambda x}{1-x}\right)^{\lambda}\right] = \Sigma a_n\frac{(q)_n}{(p)_n}x^n,$$

* Gauss, *loc. cit.*, Formula 102. The formula can also be proved directly by the use of Saalschütz's formula.

where

$$e^{x}\,{}_rF_{s+\lambda}\left[\begin{matrix}\alpha_1, & \ldots, & \alpha_r; \\ \rho_1, & \ldots, & \rho_s, \quad \frac{p}{\lambda}, \quad \frac{1+p}{\lambda}, \quad \ldots, \quad \frac{\lambda-1+p}{\lambda};\end{matrix}\; cx^{\lambda}\right] = \Sigma\,\frac{a_n x^n}{(p)_n},$$

again using (5 . 1) and (5 . 2), and the a's are given by (5 . 1).

As particular cases of this result we get (4 . 17) for $\lambda = r = 1$, $s = 0$, $c = -1$, and (4 . 18) for $\lambda = r = 1$, $s = 0$, $c = -2$.

If we take $\lambda = 1$, $r = 2$, $s = 1$, $c = -1$, and use (4 . 17) we find that

$$(5\,.\,6)\qquad (1-x)^{-\alpha_3}\,{}_3F_2\left[\begin{matrix}\alpha_1, & \alpha_2, & \alpha_3; \\ \rho_1, & \rho_2; & \end{matrix}\; -\frac{x}{1-x}\right] = \Sigma a_n\,\frac{(\alpha_3)_n}{(\rho_2)_n}\,x^n,$$

where $$(1-x)^{\alpha_1-\rho_2}\,{}_2F_1(\alpha_1,\ \rho_1-\alpha_2;\ \rho_1;\ x) = \Sigma a_n x^n.$$

Using (4 . 12) with $x/(x-1)$ for x, we get, with a change of notation,

$$(5\,.\,7)\qquad {}_3F_2\left[\tfrac{1}{2}(a+c),\ \tfrac{1}{2}(a+c-1),\ b;\ c,\ a+b;\ 4x(1-x)\right] = \Sigma a_n\,\frac{(a+c-1)_n}{(a+b)_n}\,x^n,$$

where $$(1-x)^{-b}\,{}_2F_1(a,\ b;\ c;\ x) = \Sigma a_n x^n.$$

This may be regarded as a generalization of the quadratic transformation (4 . 24), as may be seen by putting $c = a$.

The results of this paragraph* can be regarded as relations between the coefficients in the expansions. Thus (5 . 3) implies that

$$(5\,.\,31)\qquad {}_3F_2\left[\begin{matrix}a, & \rho-\beta, & -n; \\ \rho, & a-\gamma & \end{matrix}\right] = (-1)^n\,\frac{(1+\gamma-n)_n}{(a-\gamma)_n}\,{}_3F_2\left[\begin{matrix}a, & \beta, & -n; \\ \rho, & 1+\gamma-n & \end{matrix}\right],$$

which is a known result†, and (5 . 6) depends on the same formula.

The basis of (5 . 7) is

$$(5\,.\,71)\qquad {}_4F_3\left[\begin{matrix}1-n-c, & 1-n-a-b, & -\frac{1}{2}n, & \frac{1}{2}(1-n); \\ 1-n-\frac{1}{2}(a+c), & \frac{3}{2}-n-\frac{1}{2}(a+c), & 1-n-b & \end{matrix}\right] = \frac{(c)_n}{(a+c+n-1)_n}\,{}_3F_2\left[\begin{matrix}a, & b, & -n; \\ c, & 1-n-b & \end{matrix}\right].$$

The series on the right is nearly-poised and is transformable‡ into a Saalschützian series, which reduces to the series on the left§.

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* The following remarks in this paragraph are due to the referee.

† Whipple, "A group of generalized hypergeometric series, etc.", *Proc. London Math. Soc.* (2), 23 (1924), 104–114 (23).

‡ Whipple, "Some transformations of generalized hypergeometric series", *Proc. London Math. Soc.* (2), 26 (1927), 257–272 (6 . 6).

§ Using the table on p. 534 of Whipple, "Well-poised series, etc.", *Proc. London Math. Soc.* (2), 25 (1926).

The formula (5 . 4) requires

$$(5.41)\quad {}_4F_3\left[\begin{matrix} a,\ a+\frac{1}{2},\ -\frac{1}{2}n,\ \frac{1}{2}(1-n) \\ \rho+\frac{1}{2},\ \frac{1}{2}p,\ \frac{1}{2}(p+1)\end{matrix}\right] = \frac{(p-2a)_n}{(p)_n}\ {}_3F_2\left[\begin{matrix} 2a,\ \rho,\ -n \\ 2\rho,\ 1+2a-n-p\end{matrix};\ 2\right].$$

This is apparently a new identity, and it may be established by Dougall's method*.

6. From (4 . 08) it is evident that

$$(6.1)\quad {}_3F_2(2a-1,\ a+\tfrac{1}{2},\ a-b-\tfrac{1}{2};\ a+b+\tfrac{1}{2},\ a-\tfrac{1}{2};\ x) = (1-x)^{2b-2a}\ {}_3F_2(2b-1,\ b+\tfrac{1}{2},\ b-a-\tfrac{1}{2};\ a+b+\tfrac{1}{2},\ b-\tfrac{1}{2};\ x).$$

Equating coefficients of x^n and changing a, b into $a+\frac{1}{2}$, $b+\frac{1}{2}$, we obtain

$$(6.2)\quad {}_4F_3\left[\begin{matrix} 2b,\ b+1,\ b-a-\frac{1}{2},\ -n \\ b,\ b+a+\frac{3}{2},\ 1-2a+2b-n\end{matrix}\right] = \frac{(2a)_n\,(a+1)_n\,(a-b-\frac{1}{2})_n}{(2a-2b)_n\,(a+b+\frac{3}{2})_n\,(a)_n},$$

the series on the left being nearly poised.

The formula (6 . 2) may be derived† from the formula‡

$$(6.3)\quad {}_4F_3\left[\begin{matrix} x,\ y,\ z,\ -n \\ 1+x-y,\ 1+x-z,\ w\end{matrix}\right] = \frac{\Gamma(w)\,\Gamma(w-x+n)}{\Gamma(w+n)\,\Gamma(w-x)} \times {}_5F_4\left[\begin{matrix} 1+x-w,\ \frac{1}{2}(1+x),\ \frac{1}{2}x,\ 1+x-y-z,\ -n \\ 1+x-y,\ 1+x-z,\ 1+\frac{1}{2}(x-w-n),\ \frac{1}{2}(1+x-w-n)\end{matrix}\right]$$

which gives

$$(6.4)\quad {}_4F_3\left[\begin{matrix} 2b,\ b+1,\ b-a-\frac{1}{2},\ -n \\ b,\ b+a+\frac{3}{2},\ 1-2a+2b-n\end{matrix}\right] = \frac{\Gamma(1-2a+2b-n)\,\Gamma(1-2a)}{\Gamma(1-2a+2b)\,\Gamma(1-2a-n)}\ {}_3F_2\left[\begin{matrix} 2a+n,\ b+\frac{1}{2},\ -n \\ b+a+\frac{3}{2},\ a\end{matrix};\ 1\right].$$

The series on the right can be summed by Saalschütz's theorem, and we are led to (6 . 2).

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* Comparing coefficients of a^n and using Gauss's second formula quoted in § 1.

† This alternative proof is due to the referee.

‡ Whipple, "Some transformations of generalized hypergeometric series", *Proc. London Math. Soc.* (2), 26 (1927), 257–272 (6.5).

ON QUANTITATIVE SUBSTITUTIONAL ANALYSIS

(Third Paper)

By ALFRED YOUNG.

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Two former papers have appeared on this subject in these *Proceedings*, the first in Ser. 1, 33 (1901), 97, and the second in Ser. 1, 34 (1902), 361. The object of these papers is the discussion and solution of substitutional equations of the form

$$\Sigma \lambda_r s_r X = R,$$

where s_r represents a permutation of the variables, X is an unknown function, and R a known function, λ_r being numerical. In the former papers expressions PN were introduced, which were derived from certain tableaux, here styled F, which each contained all the letters arranged in rows and columns, every row starting from the same left-hand column, and there being no more letters in any row than in a row above. For example, we may have F of the form

$$\begin{array}{l} a_{1,1}\, a_{1,2} \ldots \quad \ldots a_{1,\alpha_1} \\ a_{2,1}\, a_{2,2} \ldots a_{2,\alpha_2} \\ a_{3,1} \ldots \\ \ldots \qquad \ldots \\ a_{h,1} \ldots a_{h,\alpha_h}. \end{array}$$

In this case

$$P = \{a_{1,1}\, a_{1,2} \ldots a_{1,\alpha_1}\}\{a_{2,1} \ldots a_{2,\alpha_2}\} \ldots \{a_{h,1} \ldots a_{h,\alpha_h}\},$$

$$N = \{a_{1,1}\, a_{2,1} \ldots a_{h,1}\}'\{a_{1,2}\, a_{2,2} \ldots\}' \ldots,$$

where $\{abc \ldots\}$ stands for the sum of the substitutions of the symmetric group of $a, b, c, \ldots$, which I call the positive symmetric group of $a, b, c, \ldots$. And $\{abc\ldots\}'$ is the sum of the even substitutions of the

symmetric group of a, b, c, ..., minus the sum of the odd substitutions of this group; I call this the negative symmetric group. Then P is the product of the positive symmetric groups of the rows of F, and N is the product of the negative symmetric groups of the columns. Further, $T_{\alpha_1, \alpha_2, \ldots, \alpha_h}$ was defined as ΣPN, the sum extending to all expressions PN formed from F's which have α_1 letters in the first row, α_2 in the second row, and so on—where $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \ldots \geqslant \alpha_h$. It was shown that

$$1 = \Sigma A_{\alpha_1, \alpha_2, \ldots, \alpha_h} T_{\alpha_1, \alpha_2, \ldots, \alpha_h},$$

where $A_{\alpha_1, \alpha_2, \ldots, \alpha_h}$ is a certain number; and that

$$PN \,.\, P'N' = 0,$$

when PN and $P'N'$ belong to different T's.

Thus a substitutional equation is equivalent to a number of entirely separate and independent equations

$$T \,.\, \Sigma \lambda_r s_r X = T \,.\, R,$$

there being no connexion between two such with different T's.

Unfortunately, belonging to any T a large number of expressions SPN made their appearance, which were connected by an intricate system of equations. The primary object of the present paper is to establish the fact that from among the possible tableaux F belonging to a given T a certain definite set (easily defined) F_1, F_2, ..., F_f can be selected, from which we may derive forms $P_r \sigma_{r,s} N'_s$ $(r, s = 1, \ldots, f)$, where N'_s is obtained by a slight alteration from N_s, and $\sigma_{r,s}$ is that substitution which transforms the tableau F_s into F_r; these forms are all linearly independent; moreover,

$$P_r \sigma_{r,s} N'_s P_t \sigma_{t,u} N'_u = P_r \sigma_{r,u} N'_u \text{ or } 0,$$

according as t is or is not equal to s.

It follows at once that the solution of the original equation may be reduced to a certain system of matrix equations.

From this is deduced a fact as to the number of different values a given function may assume by interchange of variables, § 6.

It is shown, further, that the substitutional equation gives a set of f equations for a given T, each identical in form, but no two having a common member. Then in §§ 7, 11 it is shown how the equations may be expressed as equations between forms defined by the tableaux F.

Since, in practice, it is necessary to use forms defined by tableaux other than those of the standard set F_1, ..., F_f, convenient forms of the relations between these are worked out, §§ 8, 9, 10.

The results are illustrated, first by seeking to find what modifications may be introduced when it is known that the functions considered are unaltered by a cyclical interchange of all the variables. It is shown that a much smaller set of tableaux may be used, giving what I call reduced standard forms. And a theorem is proved establishing a relation between the number of reduced standard forms corresponding to any transitive group and the number of standard forms for one degree less. A further illustration is supplied by quartic invariant types, § 16. And, again, the theory is applied in some degree to a certain class of invariant types of the binary sextic, § 17.

When writing the two former papers I suffered from the disadvantage of being unacquainted with the closely related researches of the late Prof. Frobenius, published in the Berlin *Sitzungsberichte*, and beginning with "Über Gruppencharaktere", 1896; a lucid and more elementary exposition of the main features of Frobenius's theory of group characters was given by Schur*. But in a paper published in 1903 in the same place, "Über die charakteristischen Einheiten der symmetrischen Gruppe", Herr Frobenius referred to my two papers on substitutional analysis, and expounded very clearly the connexion between their contents and his own work. In view of his exposition there is no occasion for me to go over the same ground. But I feel that it is necessary to say a word in justification of my not adopting his notation throughout my work. The justification lies in my purpose of attempting to solve substitutional equations; and for this purpose it seems to me that it is necessary to use a notation which exhibits the relative positions of the letters. For the purpose of the general theory of groups it is different. Indeed, so far as that is concerned the general result of reduction of the group to matrices was accomplished by Frobenius in his researches, notably in his two papers "Über die Darstellung der endlichen Gruppen durch lineare Substitutionen", I (1897), II (1899), and all I can claim is to have expressed the matrices in a convenient form for calculation and solution of actual equations. Following his exposition in "Über die char. Einheiten", it will be seen that my expressions $P_r \sigma_{r,s} N'_s$ will appear as the elements of one of his irreducible group matrices. As I have occasion several times to refer to his papers I have done so in shortened form by the letters F.B.S., followed by the year of the Berlin *Sitzungsberichte* and the page referred to. Equations $\Sigma \lambda s X = R$ might be considered in the same way where s represents an operation of any

* "Neue Begründung der Theorie der Gruppen charaktere", *Berliner Sitzungsberichte* (1905), 406.

given group H, other than the symmetric group. The study of Frobenius's work shows how these can be solved, in much the same way as those here considered, with the help of group characters.

On the other hand, it will be seen how group characters would naturally turn up if we started with the object of solving such equations—instead of appearing, as they did, in the course of the solution of the problem to express any group as a linear substitution group, § 18.

My own two papers are quoted as Q.S.A. I and II.

It is perhaps as well to emphasize the fact that the theory is primarily meant to apply to the behaviour of functions when some of the variables are permuted among themselves. In the application, then, a function is supposed to stand on the right-hand side of the substitutional expression, and it is supposed to be operated on first by those operations which stand next it, and then the result is operated on by those further away. Thus

$$PNX = P(NX).$$

1. The form PN is derived from a tableau F*

$$\begin{array}{l} a_{1,1}\, a_{1,2} \ldots \quad \ldots \quad \ldots a_{1,\alpha_1} \\ a_{2,1}\, a_{2,2} \ldots \quad \ldots a_{2,\alpha_2} \\ \ldots \quad \ldots \quad \ldots \\ a_{h,1}\, a_{h,2} \ldots a_{h,\alpha_h} \end{array}$$

which contains α_1 letters in the first row, α_2 in the second, and so on; $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \ldots \geqslant \alpha_h$; every row commences from the same left-hand column, and the letters are the letters $a_1, a_2, \ldots, a_n$ in some arrangement. Then P is the product of the positive symmetric groups of the rows, and N is the product of the negative symmetric groups of the columns. Given the numbers $\alpha_1, \alpha_2, \ldots$ there are, of course $n!$ possible tableaux F. Amongst these we define certain particular tableaux as *standard*.

For this purpose we arrange the letters $a_1, a_2, \ldots, a_n$ in an agreed sequence; naturally the most convenient is the sequence of the suffixes. Then *a standard tableau is one in which in every row the letters appear in the order of the agreed sequence, and also in every column.* Thus when $n = 5$, $\alpha_1 = 3$, $\alpha_2 = 2$, there are 5 standard tableaux, viz. :

$$\begin{array}{lllll} a_1 a_2 a_3 & a_1 a_2 a_4 & a_1 a_2 a_5 & a_1 a_3 a_4 & a_1 a_3 a_5 \\ a_4 a_5 & a_3 a_5 & a_3 a_4 & a_2 a_5 & a_2 a_4 \end{array} .$$

Evidently in a standard tableau a_1 must appear in the top left-hand

* Q.S.A. II, 369.

corner, and a_2 in the second place in the top row or the first place in the second.

THEOREM I. *All forms PN belonging to a particular set $T_{a_1, a_2, \ldots, a_h}$ can be expressed as a sum*

$$\sum_{r=1}^{\lambda} A_r P_r \sigma_r N,$$

where P_r is derived from a standard tableau, σ_r is a substitution which transforms P into P_r, and A_r is numerical.

Let

$$P = \{b_{1,1}\, b_{1,2} \ldots b_{1,a_1}\}\{b_{2,1} \ldots b_{2,a_2}\} \ldots \{b_{h,1} \ldots b_{h\,a_h}\}.$$

Let the letters, as arranged in the agreed sequence, be $a_1, a_2, \ldots, a_n$. If a_1 is not in the first group of P, then

$$[1+\Sigma(a_1\, b_{1,r})]\, PN = \frac{1}{a_1!}\{a_1\, b_{1,1} b_{1,2} \ldots b_{1,a_1}\}\, PN = 0.*$$

Hence $$PN = -\Sigma P'(a_1\, b_{1,r})\, N,$$

where P' contains a_1 in its first group.

Now we may suppose that, in any group of P, the letters $b_{r,1}\, b_{r,2} \ldots b_{r,a_r}$ follow the fixed sequence. Let us suppose that $b_{2,s}$ precedes $b_{1,s}$; but that for values of $r < s$, $b_{1,r}$ precedes $b_{2,r}$. Then

$$\{b_{2,1}\, b_{2,2} \ldots b_{2,s}\, b_{1,s}\, b_{1,s+1} \ldots b_{1,a_1}\}\, PN = 0,$$

for the positive symmetric group on the left is of degree greater than the degree of any of the groups of P†.

Hence $$PN = -\Sigma A P' \sigma N,$$

where the first $s-1$ letters of the first group of P' are the same as in P and the s-th is one of $b_{2,1}\, b_{2,2} \ldots b_{2,s}$, all of which precede the other letters contained in the first two groups. The first letter in the first group of P' which can follow (in the fixed sequence) the corresponding letter of the second group is the $(s+1)$-th. Then if $b'_{2,s+1}$ precedes $b'_{1,s+1}$ we proceed again in the same way. Thus in a_2 steps at most we can express PN in the form $\Sigma A P' \sigma N$ where the letter $b'_{1,r}$ always precedes $b'_{2,r}$ in the tableau for P', according to the fixed sequence. We now apply the process to the other pairs of rows. Whenever a letter in a row precedes the corresponding letter in a higher row of the tableau, we can

* See Q.S.A. II, 370, 371.

† See Q.S.A. I, 134, 135.

perform this process of raising an earlier letter to take the place of a later one higher up. This is only possible a finite number of times, and hence ultimately we find

$$PN = \sum_{r=1}^{\lambda} A_r P_r \sigma_r N,$$

where P_r is derived from a standard tableau.

COROLLARY. The same argument is applicable to prove that

$$NP = \sum_{r=1}^{\lambda} A_r N \sigma_r P_r;$$

$$NP = \sum_{r=1}^{\lambda} A_r N_r \sigma_r P;$$

$$PN = \sum_{r=1}^{\lambda} A_r P \sigma_r N_r;$$

where in each case P_r or N_r is derived from a standard tableau, σ_r is some substitution, and A_r is numerical.

2. THEOREM II. *The number of standard tableaux belonging to* $T_{a_1, a_2, \ldots, a_h}$ *is*

$$n! \frac{\prod_{r,s}(a_r - a_s - r + s)}{\prod_{r}(a_r + h - r)!}.$$

We shall proceed by induction, and assume that this is true for $n-1$ letters. Consider first the case when no two a's are equal and $a_h > 1$. Let a_n be the last letter of the fixed sequence. Then it may occupy the end position in any row in a standard tableau. By removing a_n a standard tableau of the remaining $n-1$ letters is obtained. Thus, assuming the truth of the formula for $n-1$ letters, we see that the number of standard tableaux is

$$(n-1)! \frac{\prod_{r,s}(a_r - a_s - r + s)}{\prod_{r}(a_r + h - r)!} \left[\sum_{r=1}^{h} (a_r + h - r) \frac{\prod_{s \neq r}(a_r - 1 - a_s - r + s)}{\prod_{s \neq r}(a_r - a_s - r + s)} \right].$$

Let us call the sum in the brackets X, and let us write

$$\hat{\xi}_r = a_r + h - r.$$

Then

$$X = \Sigma \xi \frac{\Pi(\hat{\xi}_r - 1 - \hat{\xi}_s)}{\Pi(\xi_r - \xi_s)} = -\Sigma \xi_r \frac{f(\hat{\xi}_r - 1)}{f'(\hat{\xi}_r)},$$

where $$f(x) = (x-\xi_1)(x-\xi_2)\dots(x-\xi_h);$$

it is easy to see that no two ξ's can be equal for $\xi_1 > \xi_2 \dots > \xi_h$. Now it is well known that if the order of $\Phi(x)$ is less than $h-1$*,

$$\Sigma \frac{\Phi(\xi_r)}{f'(\xi_r)} = 0.$$

Hence
$$X = -\Sigma \xi_r \frac{f(\xi_r) - f'(\xi_r) + \frac{1}{2}f''(\xi_r)}{f'(\xi_r)}$$
$$= \Sigma \xi_r - \binom{h}{2} = \Sigma a_r = n.$$

Thus the induction is proved when no two of the a's are equal and $a_h > 1$. When $a_r = a_{r+1}$ the letter $a_{\cdot\cdot}$ cannot occur in the a_r-th row of a standard tableau at all. But in this case the corresponding term in the sum X contains a factor $a_r - 1 - a_{r+1} + 1$ which is zero; so that the proof is unaffected, and we may remove the restriction that two a's must not be equal. And when $a_h = 1$, the term in X for $r = h$ is exactly that given by putting $a_h = 1$. The formula is obviously true for $n = 1, 2, 3, 4$, as can easily be seen by writing down the standard tableaux in these cases. Hence it is always true.

We shall call this number $f_{a_1, a_2, \dots, a_h}$, and drop the suffixes, when no confusion can arise; it is the number which Frobenius calls f or χ_0.

Moreover, in the series

$$1 = \Sigma A_{a_1, a_2, \dots, a_h} T_{a_1, a_2, \dots, a_h},$$

for which the coefficients were obtained in my second paper,

$$A = \left(\frac{f}{n!}\right)^2.$$

3. The proof given for the above theorem establishes, in fact, the useful formula

$$f_{a_1-1, a_2, \dots, a_h} + f_{a_1, a_2-1, a_3, \dots, a_h} + \dots + f_{a_1, a_2, \dots, a_h-1} = f_{a_1, a_2, \dots, a_h}. \qquad (1)$$

It is interesting and useful to point out that $f_{a_1, a_2, \dots, a_h}$ is the coefficient of $x_1^{a_1+h-1} x_2^{a_2+h-2} \dots x_h^{a_h}$ in the expansion of

$$E = \frac{\Delta(x_1, x_2, \dots, x_h)}{1 - x_1 - x_2 \dots - x_h} = \Delta\Sigma(x_1 + x_2 + \dots + x_h)^n,$$

* *E.g.* see Burnside and Panton, *Theory of equations*, 1 (1899), 172.

where

$$\Delta(x_1, x_2, \ldots, x_h) = (x_1-x_2)(x_1-x_3)\ldots(x_1-x_h)(x_2-x_3)\ldots(x_{h-1}-x_h)$$
$$= \{x_1 x_2 \ldots x_h\}' x_1^{h-1} x_2^{h-2} \ldots x_{h-1}.$$

If the sum $a_1+a_2+\ldots+a_h = n$, we may express the same thing by saying that f is the coefficient of $x_1^{a_1+h-1}\ldots x_h^{a_h}$ in the expansion of

$$\Delta(x_1+x_2+\ldots+x_h)^n.$$

When $n = 1$ or 2 this is easily verified. For further values the formula proves the induction from one value of n to the next.

This is proved by Frobenius (F.B.S. 1900, 522).

The formula

$$f_{a_1+1, a_2, \ldots, a_h}+f_{a_1, a_2+1, \ldots, a_h}+\ldots+f_{a_1, a_2, \ldots, a_h, 1} = (n+1)f_{a_1, a_2, \ldots, a_h} \qquad (2)$$

may be proved in the same way, or else from the fact that in the expansion of

$$\Delta(x_1, x_2, \ldots, x_{h-1})(x_1+x_2+\ldots+x_{h-1})^{n+1}$$

it is the coefficient of

$$x_1^{a_1+h+1} x_2^{a_2+h-1}\ldots x_h^{a_h+1}+x_1^{a_1+h} x_2^{a_2+h} x_3^{a_3+h-2}\ldots x_h^{a_h+1}$$
$$+\ldots+x_1^{a_1+h} x_2^{a_2+h-1} x_h^{a_h+1} x_{h+1}$$
$$= x_1^{a_1+h} x_2^{a_2+h-1}\ldots x_h^{a_h+1}\{x_1+x_2+\ldots+x_{h+1}\},$$

i.e. $n+1$ times the coefficient of $x_1^{a_1+h} x_2^{a_2+h-1}\ldots x_h^{a_h-1}$ in the expansion of

$$\Delta(x_1+x_2+\ldots+x_{h+1})^n = (n+1)f_{a_1, a_2, \ldots, a_h}.$$

4. Consider the f standard tableaux $F_1, F_2, \ldots, F_f$ belonging to $T_{a_1, a_2, \ldots, a_h}$. We can arrange them in a definite sequence following the sequence of letters $a_1, a_2, \ldots, a_n$. Let the t-th letter of the s-th row of F_λ be $a_{st}^{(\lambda)}$. Then when the first $s-1$ rows of F_λ and F_μ are identical and also the first $t-1$ letters of the s-th row, but $a_{s,t}^{(\lambda)}$ precedes $a_{s,t}^{(\mu)}$ in the sequence of letters, we say that F_λ precedes F_μ in the sequence of tableaux.

We shall suppose that the suffixes of $F_1, F_2, \ldots, F_f$ are chosen so that the tableau F_r precedes F_s in this fixed sequence when $r < s$. We shall also use P_r, N_r for the products of positive or negative symmetric groups obtained from the tableau F_r.

It will be noticed that, by giving the columns of the tableaux the precedence, instead of the rows, we should obtain a different sequence; but one, indeed, equally suitable for our purpose.

THEOREM III. *If* $r < s$,

$$N_s P_r = 0.$$

Let us suppose that the first $t-1$ letters of the first row in F_r and F^s are the same, but that $a^{(r)}_{1,t}$ precedes $a^{(s)}_{1,t}$. Then, since F_s is a standard tableau, $a^{(s)}_{1,t}$ precedes $a^{(s)}_{1,v}$ unless $v < t$. Thus the letter $a^{(r)}_{1,t}$ must appear in one of the first $t-1$ columns of F_s; and N_s has a factor $\{a^{(r)}_{1,t}\, a^{(r)}_{1,v}\}'$ where $v < t$; and hence $N_s P_r = 0$.* The argument is the same when the first rows of F_r, F_s are identical and the distinction between them arises first in a later row.

When $r = s$, we know that $N_r P_r$ is not zero; and when $r > s$ this equation does not always hold good.

In all cases $N_f P_r = 0$, unless $r = f$, for otherwise $r < f$. Let us consider N_f, N_{f-1}, ... in the reverse order, and let N_s be the first N we come to for which there is an inequality

$$N_s P_r \neq 0,$$

when $r \neq s$. Then we know that $r > s$. Also we know that P_r contains a substitution τ_{sr} which transforms N_r into N_s†, so that

$$N_s \tau_{sr} = \tau_{sr} N_r.$$

It is possible that there is more than one value of r for which the above inequality is true. Let these be r_1, r_2, ... Let us write

$$\begin{aligned} N'_s &= N_s(1 - \tau_{sr_1} - \tau_{sr_2} - \ldots) \\ &= N_s - \tau_{sr_1} N_{r_1} - \tau_{sr_2} N_{r_2} - \ldots. \end{aligned}$$

Then

$$N'_s P_r = 0$$

for all values of r other than s. For, by hypothesis, $N_t P_r$ is zero when t is less than s and not equal to r, and hence

$$N'_s P_{r_1} = N_s P_{r_1} - \tau_{sr_1} N_{r_1} P_{r_1} = 0.$$

Further

$$N'_s P_s = N_s P_s.$$

We call N'_s a *prepared* form.

Proceeding, we take the next value of s for which there exists a different number r such that $N_s P_r$ is not zero. We then replace N_s by

$$N_s = N_s - \tau_{sr_1} N'_{r_1} - \tau_{sr_2} N'_{r_2} - \ldots$$

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* For P_r has a factor $\{a^{(r)}_{1,t}\, a^{(r)}_{1,v}\}$ and $\{a^{(r)}_{1,t}\, a^{(r)}_{1,v}\}'\{a^{(r)}_{1,t}\, a^{(r)}_{1,v}\} = 0$.

† Q.S.A. II, 363, § 3.

in the same way, only now we use the N's already prepared instead of the original ones. Then, as before,

$$N'_s P_r = 0 \quad (r \neq s)$$

and

$$N'_s P_s = N_s P_s.$$

Thus we can replace each of $N_1, N_2, \ldots, N_f$ by expressions $N'_1, N'_2, \ldots, N'_f$ which have the properties

$$N'_s P_r = 0 \quad (r \neq s),$$

$$N'_s P_s = N_s P_s.$$

Moreover $N'_s = N_s[1-\tau_1-\tau_2-\ldots]$, where $\tau_1, \tau_2, \ldots$ represent certain substitutions.

Let now $\sigma_{s,r}$ be that substitution which transforms the tableau F_r into F_s. It will transform $P_r N_r$ into $P_s N_s$, so that

$$\sigma_{s,r} P_r \sigma_{s,r}^{-1} = P_s,$$

$$\sigma_{s,r} N_r \sigma_{s,r}^{-1} = N_s.$$

Then consider the f^2 expressions

$$P_r \sigma_{rs} N'_s = \sigma_{r,s} P_s N'_s$$

The product of any two is either zero or else a multiple of one of themselves. For

$$P_r \sigma_{r,s} N'_s \,.\, P_u \sigma_{u,v} N'_v$$

is zero unless $u = s$, and

$$\begin{aligned} P_r \sigma_{rs} N'_s P_s \sigma_{s,v} N'_v &= \sigma_{r,s} P_s N_s P_s \sigma_{sv} N_v [1-\tau_1-\tau_2\ldots] \\ &= \sigma_{r,s}\sigma_{sv} P_v N_v P_v N_v [1-\tau_1-\tau_2\ldots] \\ &= \frac{n!}{f} \sigma_{r,v} P_v N_v [1-\tau_1-\tau_2-\ldots]^* \\ &= \frac{n!}{f} \sigma_{r,v} P_v N'_v = \frac{n!}{f} P_r \sigma_{r,v} N'_v. \end{aligned}$$

None of these f^2 expressions vanishes; for otherwise

$$P_r \sigma_{r,s} N'_s P_s N_s = \sigma_{r,s} P_s N_s P_s N_s = \frac{n!}{f} \sigma_{r,s} P_s N_s$$

would be zero, which we know is not the case.

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* Q.S.A. II, 366.

There can be no linear relation between them. For let us suppose that

$$\Sigma\lambda_{r,s} P_r \sigma_{r,s} N'_s = 0,$$

where $\lambda_{r,s}$ is numerical.

Multiply this equation on the right-hand side by $P_s N'_s$ and on the left-hand side by $P_r N'_r$; then every term will vanish except

$$\lambda_{r,s}\left(\frac{n!}{f}\right)^2 P_r \sigma_{r,s} N'_s,$$

and hence $\lambda_{r,s}$ vanishes for all values of r and s.

These expressions are then linearly independent. There are f^2 of them belonging to each T. The number $(f/n!)^2$ is, as we have said, the coefficient of $T_{\alpha_1, \alpha_2, \ldots, \alpha_h}$ in the series

$$1 = \Sigma A_{\alpha_1, \alpha_2, \ldots, \alpha_h} T_{\alpha_1, \alpha_2, \ldots, \alpha_h};$$

also

$$\frac{1}{n!} = \Sigma A_{\alpha_1, \alpha_2, \ldots, \alpha_h}{}^*.$$

Hence

$$\Sigma f^2 = n!$$

THEOREM IV. *Associated with every* $T_{\alpha_1, \alpha_2, \ldots, \alpha_h}$ *there are* $f^2_{\alpha_1, \alpha_2, \ldots, \alpha_h}$ *expressions* $P_r \sigma_{r,s} N'_s$ *and, taking all the possible values of* $\alpha_1, \alpha_2, \ldots, \alpha_h$, *there are exactly* $n!$ *of these expressions. They are all linearly independent, and in terms of them every substitution, and every substitutional expression, can be expressed with purely numerical coefficients.*

5. Let

$$T' = P_1 N'_1 + P_2 N'_2 + \ldots + P_f N'_f.$$

Then

$$T' P_r \sigma_{r,s} N'_s = P_r N'_r P_r \sigma_{r,s} N'_s = \left(\frac{n!}{f}\right) P_r \sigma_{r,s} N'_s.$$

Now

$$TP_r \sigma_{r,s} N'_s = TP_r N_r [1 - \tau_1 \ldots] \sigma_{r,s}$$

$$= \left(\frac{n!}{f}\right)^2 P_r \sigma_{r,s} N'_s \dagger.$$

Hence

$$\left[T - \left(\frac{n!}{f}\right) T'\right] S = 0,$$

where S is any substitutional expression whatever of the n letters. We may take $S = 1$; hence

$$T = \frac{n!}{f} T'.$$

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* Q.S.A. II, § 8, 368.

† Q.S.A. II, § 3.

THEOREM V. *The expression* $T_{a_1, a_2, \ldots, a_h}$ *is equal to*

$$\frac{n!}{f}[P_1 N_1' + P_2 N_2' + \ldots + P_f N_f'] = \frac{n!}{f} T',$$

that is a sum of terms, each one of which is obtained from a standard tableau. And the series may be written

$$n! = \Sigma f_{a_1, a_2, \ldots, a_h} T'_{a_1, a_2, \ldots, a_h}.$$

6. A substitutional equation may now be presented as a number of different equations each belonging to a different T, and each of the form

$$\Sigma \lambda_{r,s} P_r \sigma_{r,s} N_s' X = 0;$$

where X is the unknown function sought for. The solution desired is of the form

$$X = \Sigma \mu_{r,s} P_r \sigma_{r,s} N_s' U,$$

where U is an arbitrary function.

Substituting for X in the equation we obtain f^2 linear equations of the form

$$\sum_{s=1}^{f} \lambda_{r,s} \mu_{s,t} = 0$$

for determining the f^2 quantities $\mu_{r,s}$. Indeed, we may write substitutional expressions belonging to a particular T as matrices. They obey all the laws of matrices for addition, multiplication, &c., and, in particular, for solution of equations.

Thus we may write simply $[\lambda_{rs}]$ for $\Sigma \lambda_{r,s} P_r \sigma_{r,s} N_s'$, and consider the equation

$$[\lambda_{rs}][\mu_{rs}] = 0,$$

where λ_{rs} is known and μ_{rs} is to be found.

It is to be noticed that N_s' is never changed by any left-hand multiplication of $P_r \sigma_{r,s} N_s'$; so that no relation will connect terms $P_r \sigma_{r,s} N_s' U$ with terms having different values of s. Thus the equations we consider are really f sets of equations, there being f equations in each set, between f unknowns. The f sets of equations are identical, that is the coefficients in each set are the same; but the f unknowns in one set are distinct from those in any other set, except that they satisfy these equations. They may be called cogredient. The same remark is clear from the consideration of the matrix equations, when the unknown matrix is always on the right-hand side.

Let ρ be the rank of $[\lambda_{r,s}]$; then $(f-\rho)$ is the rank of $[\mu_{r,s}]$, and for a given value of s the equations for $\mu_{r,s}$ may be solved, the solution

having $f-\rho$ undetermined quantities. Thus the expression TX has $f(f-\rho)$ undetermined quantities, and TX is a function such that the functions obtained by interchange of letters from it may be expressed linearly in terms of $f(f-\rho)$ of them. If there is more than one equation we can, by introduction of arbitrary multipliers, reduce the equations to a single one.

THEOREM VI. *The number of linearly independent values obtained from a function TX by interchange of letters is ρf where $f \geqq \rho \geqq 0$.*

There are two cases in which f is unity, viz. f_n and $f_{1,1,\ldots,1}$. These correspond to the symmetric and alternating functions respectively. And, as is well known, the general two-valued function is of the form

$$AT_nU+BT_{1,1,\ldots,1}U.$$

In all other cases when $n>4$, the minimum value of f is $n-1$, which is the value of $f_{n-1,1}$ and of $f_{2,1,1,\ldots,1}$. This, of course, does not lead to an $(n-1)$-valued function in the sense required by the theory of equations; in that sense we obtain an n-valued function from a $T_{n-1,1}U = X$ which satisfies an equation $[\lambda_{r,s}]X = 0$ of rank $n-2$. But the sum of the n values is zero, and so the functions obtained can be expressed linearly in terms of $n-1$ of them. On the other hand, substitutional equations may arise when the unknown quantity appears either on the right-hand side or the left; such equations would appear if it were asked what substitutional expressions formed from the symmetric group form a group. In such cases the number of solutions TX may be any number from 0 to f^2.

What we have called T' is, when expressed as a matrix, the unit matrix; strictly speaking

$$T' = \frac{n!}{f} E_f.$$

7. In order to use the theory thus developed, it is necessary to express any substitution τ in terms of expressions $\lambda P_r \sigma_{rs} N'_s$. To do this we use the series

$$\tau = \Sigma \frac{f}{n!} \tau T',$$

and then evaluate

$$\tau T' = \sum_{r=1}^{f} \tau P_r N'_r.$$

When

$$\tau P_r \tau^{-1} = P_s,$$

where P_s is equal to the product of the positive symmetric groups formed

from the rows of a standard tableau F_s, then

$$\tau = \sigma\sigma_{sr},$$

where σ is a substitution which interchanges the letters only in the same row of F_s, and hence σ belongs to P_s. Thus

$$\tau P_r N'_r = P_s \sigma_{sr} N'_r ;$$

and hence, if τ changes P_r into P_s, which is of standard form, the nature of $\tau P_r N'_r$ is at once determined by operating on P_r with τ.

In the case of $T_{n-1,1}$, for example, where there is only one letter in the second row of the tableau and all the rest in the first, the tableau F_r is determined by (a_{n+1-r}), the letter in the second row. If τ does not contain a_1, $\tau P_r \tau^{-1} = P_s$, which is a standard P, and hence

$$\tau P_r N'_r = P_s \sigma_{sr} N'_r ;$$

where, indeed, $$\tau(a_{n+1-r})\,\tau^{-1} = (a_{n+1-s}).$$

Here $$\sigma_{sr} = (\mathrm{a_{n+1-s}\, a_{n-s}\, a_{n-1-s} \dots a_{n+1-r}}) \quad \text{for} \quad r > s,$$

and $$\sigma_{sr} = (\mathrm{a_{n+1-s}\, a_{n+2-s} \dots a_{n+1-r}}) \quad \text{for} \quad r < s.$$

Thus, when τ does not contain a_1, $\tau T'_{n-1,1}$ is represented by a matrix obtained from the unit matrix by an interchange of rows.

If τ changes the letter in the second row of F_r into a_1, we use the equation

$$\left[1 + \sum_{t=2}^{n} (\mathrm{a_1 a_t})\right] \{a_2 a_3 \dots a_n\}\, \tau N'_r = 0.$$

That is, in our case,

$$\begin{aligned} \tau P_r N'_r &= -\Sigma P_{n+1-t}(\mathrm{a_1 a_t})\, \tau N'_r \\ &= -\Sigma P_{n+1-t}\, \sigma_{n+1-t,\,r}\, N'_r \end{aligned}$$

for $(\mathrm{a_1 a_t})\tau$ transforms P_r into P_{n+1-t}.

In general, for any T we have

$$\tau P_r N'_r = Q \tau N'_r,$$

where Q is written to express any P belonging to T (not necessarily standard). Then, by § 1, we may express this as

$$Q\tau N'_r = \sum_{s=1}^{f} A_s P_s \varpi_s \tau N'_r,$$

where $\varpi_s \tau$ transforms P_r into P_s. Then, as we have seen, $\varpi_s \tau = \sigma \sigma_{s,r}$, where $\sigma_{s\ r}$ transforms F_r into F_s, and σ belongs to P_s, and hence

$$\tau P_r N'_r = \Sigma A_s P_s \sigma_{s,r} N'_r.$$

In practice it is important to notice when applying this that, if two rows of F contain the same number of letters, the forms Q obtained by interchanging the two rows are apparently identical, but they must be regarded as distinct (at least when the number of letters in either is odd). The simplest example is the case $n = 2$, $T_{1,1}$.

Here let $\tau = (\mathrm{a}_1 \mathrm{a}_2), \quad Q = \{a_2\}\{a_1\},$

$$[1 + (\mathrm{a}_1 \mathrm{a}_2)] Q \tau N' = 0, \quad i.e. \quad Q\tau N' = -\{a_1\}\{a_2\}(\mathrm{a}_1 \mathrm{a}_2)\tau N'$$

$$= -PN'.$$

The case appears trivial, but it is of real importance in dealing with equal rows of several letters (see § 9). With this proviso we may use the tableau F_r to define any entity $P_r \sigma_{r,s} N_s$; it stands equally for f different entities, which are indistinguishable for our purposes and are independent; then $\tau F_r = \tau P_r \sigma_{r,s} N$ is simply the tableau obtained by operating on F_r with τ.

8. It is of interest to inquire what form PN will take when another letter is added. That is to express $P_r \sigma_{r,s} N_s$ for a definite set of n letters in terms of the corresponding entities when an additional letter is present. We prove

THEOREM VII. *If* $T_{a_1, a_2, \ldots, a_h}$ *refer to n letters, and* $T'_{a'_1, a'_2, \ldots, a'_{h'}}$ *to these n letters and one more, then* $T_{a_1, \ldots, a_h} . T'_{a'_1, \ldots, a'_{h'}} = 0$, *unless the numbers* $a'_1, a'_2, \ldots, a'_h$ *are the same as the numbers* $a_1, a_2, \ldots, a_h$ *except that one of them is increased by unity.*

This statement is, of course, intended to include the case when $h' = h+1$, and the suffixes of T' are $a_1, a_2, \ldots, a_h, 1$. Calling the additional letter x, we may write T' as a sum of standard entities, the sequence of letters having x as the last.

Let $T = \Sigma PN$, PN being formed from a tableau F, and let $T' = \Sigma P'N'$, $P'N'$ being formed from a tableau F'. Then in F' the letter x appears at the end of a row and also at the end of a column, for F' is a standard tableau. If, then, x be suppressed in F' we have a tableau of the original n letters, from which we may form an entity $P''N''$ belonging

to $T_{\beta_1, \beta_2, \ldots, \beta_{h'}}$, where the suffixes are the suffixes $\alpha_1', \alpha_2', \ldots, \alpha_{h'}'$ except that one of them is reduced by unity. Also $P''P' = AP'$ and $P'N'' = BN'$, where A and B are numerical and not zero. Moreover, $TP'' = 0$ or $TN'' = 0$ unless $T_{\alpha_1} \ldots \equiv T_{\beta_1} \ldots$.

Hence, unless $\beta_1, \beta_2, \ldots$ are the same as $\alpha_1, \alpha_2, \ldots$, $TP'N' = 0$, and therefore $TT' = 0$.

This proves the theorem.

COROLLARY. *If T refers to n letters and T' to the same n letters and m more, then $TT' = 0$ unless T' is such that its tableaux can be formed from the tableaux of T by adding letters to the rows and columns. In the contrary case if PN be any entity of T, and $P'N'$ of T', then*

$$PN \,.\, P'N' = 0.$$

For example, $$\begin{Bmatrix} a_1 a_2 \\ a_3 a_4 \end{Bmatrix} . T_{n-r, 1, 1, 1, \ldots, 1} = 0.$$

We see the truth of this by building up the expressions one letter at a time.

9. As in the use of this calculus it is frequently necessary to use non-standard forms, it is important to be able to write down quickly the relations between them, and, in particular, to reduce them to standard forms. For this purpose we prove the following

THEOREM VIII. *A form*

$$\begin{aligned} PN &\equiv \{a_1 a_2 \ldots a_r\} \{b_1 b_2 \ldots b_s\} QN \\ &= (-)^t \Sigma \{b_1 b_2 \ldots b_t c_{t+1} c_{t+2} \ldots c_r\} \{c_1 c_2 \ldots c_t b_{t+1} b_{t+2} \ldots b_s\} Q\tau N, \end{aligned}$$

where $c_1, c_2, \ldots, c_r$ are the letters $a_1, a_2, \ldots, a_r$ in any order, the sum extending to all terms obtainable by selecting any t of the a's for $c_1, c_2, \ldots, c_t$; and τ is that substitution which changes P into that of the particular term, and $r \geqslant s$.

When $t = 1$ the theorem gives the ordinary identity

$$\left[1 + \sum_{\rho=1}^{r} (b_1 a_\rho)\right] PN = 0,$$

and is known already.

Assume the theorem to be true up to $t = t$; then

$$0 = \{a_1 a_2 \ldots a_r b_{t+1}\} . \{b_1 b_2 \ldots b_t a_{t+1} a_{t+2} \ldots a_r\} \{a_1 a_2 \ldots a_t b_{t+1} b_{t+2} \ldots b_s\} Q\tau N;$$

hence, expanding the first positive symmetric group, we have

$$0 = \Sigma \{b_1 b_2 \dots b_t c_{t+1} c_{t+2} \dots c_r\} \{c_1 c_2 \dots c_t b_{t+1} b_{t+2} \dots b_s\} Q\tau N)$$
$$+\Sigma \{b_1 b_2 \dots b_{t+1} c_{t+2} c_{t+3} \dots c_r\} \{c_1 c_2 \dots c_{t+1} b_{t+2} \dots b_s\} Q\tau N,$$

and the theorem is true for $t+1$ if it is true for t; it is therefore always true.

COROLLARY. *When* $r = s = t$,

$$\{a_1 a_2 \dots a_r\} \{b_1 b_2 \dots b_r\} QN = (-)^r \{b_1 b_2 \dots b_r\} \{a_1 a_2 \dots a_r\} Q\tau N,$$

where τ interchanges the a's and b's; an illustration of the remark made at the end of the last paragraph.

We may extend this result thus:

THEOREM IX.

$$PN \equiv \{a_1 a_2 \dots a_r\} \{b_1 b_2 \dots b_s\} QN$$
$$= (-)^{t+1} \Sigma \{a_1 b_1 \dots b_{u-1} b_{u+1} \dots b_t c_{t+1} \dots c_r\} \{b_u c_2 \dots c_t b_{t+1} \dots b_s\} Q\tau N$$
$$+(-)^{t+1}(t-1) \Sigma \{a_1 b_1 \dots b_t c_{t+1} \dots c_r\} \{c_2 \dots c_{t+1} b_{t+1} \dots b_s\} Q\tau N,$$

where $c_2 \dots c_r$ *are the letters* $a_2 \dots a_r$ *in any order, and* u *has the values* 1, 2, ..., t, *and* $r \geqslant s$.

For

$$PN = (-)^t \Sigma \{b_1 \dots b_t a_1 c_{t+2} \dots c_r\} \{c_2 \dots c_{t+1} b_{t+1} \dots b_s\} Q\tau N$$
$$+(-)^t \Sigma \{b_1 \dots b_t c_{t+1} \dots c_r\} \{a_1 c_2 \dots c_t b_{t+1} \dots b_s\} Q\tau N$$

by the last theorem.

To every term T of the second sum we apply the result

$$T = -\sum_{u=1}^{t} (a_1 b_u) T - \sum_{u=t+1}^{r} (a_1 c_u) T,$$

and the result follows.

In the same way we obtain

$$PN = (-)^{t+2} \left[\binom{t-1}{2} \Sigma \{a_1 a_2 b_1 \dots b_t a'_{t+3} \dots a'_r\} \{a'_3 \dots a'_{t+2} b_{t+1} \dots b_s\} Q\tau N \right.$$
$$+(t-2) \Sigma \{a_1 a_2 b'_2 \dots b'_t a'_{t+2} \dots a'_r\} \{a'_3 \dots a'_{t+1} b'_1 b_{t+1} \dots b_s\} Q\tau N$$
$$\left. +\Sigma \{a_1 a_2 b'_3 \dots b'_t a'_{t+1} \dots a'_r\} \{a'_3 \dots a'_t b'_1 b'_2 b_{t+1} \dots b_s\} Q\tau N \right],$$

where $a_3' \dots a_{t+2}'$ are selected from $a_3 a_4 \dots a_r$ in all possible ways, and b_1' or $b_1' b_2'$ are selected from $b_1 b_2 \dots b_t$ in all possible ways.

Further extensions are obvious.

10. The product of a positive symmetric group into a form PN was considered in Q.S.A. II, §§ 18, 19. It will be useful to detail the result here.

First consider

$$\{a_1 a_2 \dots a_\lambda\, b_1 b_2 \dots b_\mu\} \begin{pmatrix} a_1 a_2 \dots a_r \\ b_1 b_2 \dots b_s \end{pmatrix}$$

$$= [1+(b_\mu a_1)+(b_\mu a_2)+\dots+(b_\mu a_\lambda)+(b_\mu b_1)+\dots+(b_\mu b_{\mu-1})]$$
$$\times [1+(b_{\mu-1} a_1)+\dots+(b_{\mu-1} a_\lambda)+(b_{\mu-1} b_1)+\dots+(b_{\mu-1} b_{\mu-2}]$$
$$\dots [1+(b_1 a_1)+(b_1 a_2)+\dots+(b_1 a_\lambda)]\, \lambda ! \begin{pmatrix} a_1 a_2 \dots a_r \\ b_1 b_2 \dots b_s \end{pmatrix}$$

$$= -[1+(b_\mu a_1)+(b_\mu a_2)+\dots+(b_\mu a_\lambda)+(b_\mu b_1)+\dots+(b_\mu b_{\mu-1})]$$
$$\dots [1+(b_2 a_1)+\dots+(b_2 a_\lambda)+(b_2 b_1)]$$
$$\times [(b_1 a_{\lambda+1})+(b_1 a_{\lambda+2})+\dots+(b_1 a_r)]\, \lambda ! \begin{pmatrix} a_1 a_2 \dots a_r \\ b_1 b_2 \dots b_s \end{pmatrix}$$

$$= (-)^\mu \lambda !\, \mu !\, \Sigma \begin{pmatrix} a_1 a_2 \dots a_\lambda\, b_1 b_2 \dots b_\mu\, c_1 \dots c_{r-\lambda-\mu} \\ c_{r-\lambda-\mu+1} \dots c_{r-\lambda}\, b_{\mu+1} \dots b_s \end{pmatrix},$$

where $c_1 \dots c_{r-\lambda}$ are the letters $a_{\lambda+1} \dots a_r$ in some order, and the Σ extends to all possible ways of distributing them between the two rows. Of course, the result is zero when $\lambda+\mu > r$.

Now

$$\{a_1 a_2 \dots a_\lambda\, b_1 \dots b_\mu\} \begin{pmatrix} a_1 \dots a_r \\ b_1 \dots b_s \end{pmatrix} = \lambda !\, \mu !\, \Sigma \begin{pmatrix} d_1 \dots d_\lambda\, a_{\lambda+1} \dots a_r \\ d_{\lambda+1} \dots d_{\lambda+\mu}\, b_{\mu+1} \dots b_s \end{pmatrix},$$

where $d_1 \dots d_{\lambda+\mu}$ are the letters $a_1 a_2 \dots a_\lambda\, b_1 \dots b_\mu$ and the sign of summation extends to the different ways of distributing them between the two rows.

Thus the group $\{a_1 a_2 \dots a_\lambda\, b_1 \dots b_\mu\}$ may be said to give us the relation

$$\Sigma \begin{pmatrix} d_1 \dots d_\lambda\, a_{\lambda+1} \dots a_r \\ d_{\lambda+1} \dots d_{\lambda+\mu}\, b_{\mu+1} \dots b_s \end{pmatrix} = (-)^\mu \Sigma \begin{pmatrix} a_1 \dots a_\lambda\, b_1 \dots b_\mu\, c_1 \dots c_{r-\lambda-\mu} \\ c_{r-\lambda-\mu-1} \dots c_{r-\lambda}\, b_{\mu+1} \dots b_s \end{pmatrix}.$$

11. In applying this theory, the kind of equation considered is

$$\Sigma \lambda_r s_r X = R;$$

X being the function the properties of which it is desired to know. The way we proceed, as a rule, is to operate on this equation with a *standard* form $N'_r P_r$; where N'_r has been prepared in the manner of § 4. Thus we derive an equation of the form

$$\Sigma \mu_t N'_r \sigma_{r,t} P_t X = N'_r P_r R.$$

It will be observed that the substitutional part of every term in this equation begins with N'_r. Operation on the left-hand side with $N'_\rho \sigma_{\rho,r} P_r$ will replace N'_r by N'_ρ, leaving the rest of the equation unaltered. Thus the particular N' which appears in the equation is quite immaterial. In fact, each term is sufficiently defined by the tableau for the P_t it contains together with the numerical coefficient. Moreover, we may in general write the term $N'_r P_r s X$ of the equations down very simply by replacing $N'_r P_r s$ by $N'_r s^{-1} P_{r,1}$ (where $P_{r,1}$ is obtained from P_r by the operation s), and then the term $N'_r s^{-1} P_{r,1} X$ by the tableau for $P_{r,1}$ (which is not, in general, standard). (The sequence PN' has been here changed to $N'P$; it is, of course, just a matter of convenience which we take.)

As an example, consider the quartic invariant type*

$$(a_1 a_2)^2 (a_2 a_3)^2 (a_3 a_4)^2 (a_4 a_1)^2 = X.$$

It is subject to three equations

$$\text{(i)} \qquad [1 - (\mathrm{a}_1 \mathrm{a}_2 \mathrm{a}_3 \mathrm{a}_4)] X = 0,$$

$$\text{(ii)} \qquad [1 - (\mathrm{a}_2 \mathrm{a}_4)] X = 0,$$

$$\text{(iii)} \qquad \{\mathrm{a}_2 \mathrm{a}_3 \mathrm{a}_4\} X = \tfrac{1}{2} \{\mathrm{a}_2 \mathrm{a}_3 \mathrm{a}_4\} (a_1 a_2)^4 (a_3 a_4)^4.$$

There are five classes of PN or NP :

(1) T_4, we operate on each equation with $\{\mathrm{a}_1 \mathrm{a}_2 \mathrm{a}_3 \mathrm{a}_4\}$, and the only result is

$$\{\mathrm{a}_1 \mathrm{a}_2 \mathrm{a}_3 \mathrm{a}_4\} X = \tfrac{1}{2} \{\mathrm{a}_1 \mathrm{a}_2 \mathrm{a}_3 \mathrm{a}_4\} (a_1 a_2)^4 (a_3 a_4)^4.$$

(2) $T_{3,1}$, we operate in turn with $N_r P_r$ ($r = 1, 2, 3$), where

$$F_1 = \frac{a_1 a_2 a_3}{a_4}, \qquad F_2 = \frac{a_1 a_2 a_4}{a_3}, \qquad F_3 = \frac{a_1 a_3 a_4}{a_2}.$$

* *Proc. London Math. Soc.* (1), 30 (1898), 290.

Writing $N_1 P_1 X$ in the form $\begin{pmatrix} a_1 a_2 a_3 \\ a_4 \end{pmatrix}$, (i) gives

$$\begin{pmatrix} a_1 a_2 a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} a_2 a_3 a_4 \\ a_1 \end{pmatrix} = \begin{pmatrix} a_3 a_4 a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_4 a_1 a_2 \\ a_3 \end{pmatrix} = 0;$$

because $$\begin{pmatrix} a_1 a_2 a_3 \\ a_4 \end{pmatrix} [1 + (a_1 a_4) + (a_2 a_4) + (a_3 a_4)] = 0.$$

Thus $T_{3\,1} X = 0$, and we need go no further with this class.

(3) $T_{2,2}$, we operate with $N_1 P_1$, $N_2 P_2$, where

$$F_1 = \begin{matrix} a_1 a_2 \\ a_3 a_4 \end{matrix}, \qquad F_2 = \begin{matrix} a_1 a_3 \\ a_2 a_4 \end{matrix},$$

(i) $$\begin{pmatrix} a_1 a_2 \\ a_3 a_4 \end{pmatrix} = \begin{pmatrix} a_2 a_3 \\ a_4 a_1 \end{pmatrix} = -\begin{pmatrix} a_1 a_2 \\ a_3 a_4 \end{pmatrix} - \begin{pmatrix} a_1 a_3 \\ a_2 a_4 \end{pmatrix} \quad (\S\ 1)$$

or $$2\begin{pmatrix} a_1 a_2 \\ a_3 a_4 \end{pmatrix} + \begin{pmatrix} a_1 a_3 \\ a_2 a_4 \end{pmatrix} = 0;$$

(ii) gives the same equation again;

(iii) gives nothing, because

$$T_{2,2}\{a_2 a_3 a_4\} = 0.$$

Thus in this class we have one type of form, and, since $f = 2$, we have two independent forms of this type.

(4) $T_{2,1,1}$, there are three standard tableaux,

$$\begin{matrix} a_1 a_2 \\ a_3 \\ a_4 \end{matrix}, \quad \begin{matrix} a_1 a_3 \\ a_2 \\ a_4 \end{matrix}, \quad \begin{matrix} a_1 a_4 \\ a_2 \\ a_3 \end{matrix},$$

(i) $$\begin{pmatrix} a_1 a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} a_4 a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 a_3 \\ a_4 \\ a_1 \end{pmatrix} = -\begin{pmatrix} a_1 a_2 \\ a_4 \\ a_3 \end{pmatrix} - \begin{pmatrix} a_1 a_3 \\ a_4 \\ a_2 \end{pmatrix},$$

and since $$-\begin{pmatrix} a_1 a_2 \\ a_4 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 a_2 \\ a_3 \\ a_4 \end{pmatrix},$$

we have $$\begin{pmatrix} a_1 a_3 \\ a_2 \\ a_4 \end{pmatrix} = 0;$$

(ii) $$\begin{pmatrix} a_1 a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} a_1 a_4 \\ a_3 \\ a_2 \end{pmatrix} = -\begin{pmatrix} a_1 a_4 \\ a_2 \\ a_3 \end{pmatrix} = 0,$$

on comparing with the equation obtained from (i). Thus

$$T_{2,1,1}X = 0.$$

(5) Equation (i) gives at once

$$T_{1,1,1,1}X = 0.$$

Thus the quartic invariant types of degree 4 give two irreducible forms of a single type of class (3) (*i.e.* $T_{2,2}$); and a product form of class (1) (*i.e.* T_4). The products are substitutionally one form of class (1) and a pair of a single type of class (3). The fact that there are just three products shows at once that this is the only possible arrangement.

More generally the quartic invariant type $(a_1a_2)^2(a_2a_3)^2\dots(a_na_1)^2$ is unchanged by the substitution $(a_1 a_2 \dots a_n)$; and also if the sequence of the letters is reversed.

Consider the expression of the property that X is unchanged by $(a_1a_2\dots a_n)$. For $T_{n-1,1}$, we have

$$\binom{a_1a_2\dots a_{n-1}}{a_n} = \binom{a_1a_2\dots a_{n-2}a_n}{a_{n-1}} = \dots = \binom{a_2\dots a_n}{a_1} = 0,$$

on taking the sum and using the identity. Hence $T_{n-1,1}X = 0$.

In general, we find that we obtain a certain number of relations between the standard forms NP_rX, so that all can be expressed in terms of a smaller number. In practice we use the relations to express later standard forms in terms of earlier ones. Thus for $n = 2, 3, 4, 5, 6, 7$, when X is unchanged by $(a_1 a_2 \dots a_n)$, we can express everything in terms of the following forms (we need only write the suffixes of the letters):

n

2: $(1\,2)$ for $\binom{1}{2} = 0.$

3: $(1\,2\,3) : 0 : \begin{pmatrix}1\\2\\3\end{pmatrix}.$

4: $(1\,2\,3\,4) : 0 : \binom{1\,2}{3\,4} : \begin{pmatrix}1\,2\\3\\4\end{pmatrix} : 0.$

5: $(1\,2\,3\,4\,5) : 0 : \binom{1\,2\,3}{4\,5} : \begin{pmatrix}1\,2\,3\\4\\5\end{pmatrix}, \begin{pmatrix}1\,2\,4\\3\\5\end{pmatrix} : \binom{1\,2}{3\,4} : 0 : \begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}.$

$$6;\quad (1\,2\,3\,4\,5\,6):0:\begin{pmatrix}1&2&3&4\\5&6\end{pmatrix},\ \begin{pmatrix}1&2&3&5\\4&6\end{pmatrix}:\begin{pmatrix}1&2&3&4\\5\\6\end{pmatrix},\ \begin{pmatrix}1&2&3&5\\4\\6\end{pmatrix}:\begin{pmatrix}1&2&4\\3&5&6\end{pmatrix}:$$

$$\begin{pmatrix}1&2&3\\4&5\\6\end{pmatrix},\ \begin{pmatrix}1&2&3\\4&6\\5\end{pmatrix}:\begin{pmatrix}1&2&3\\4\\5\\6\end{pmatrix},\begin{pmatrix}1&2&4\\3\\5\\6\end{pmatrix}:$$

$$\begin{pmatrix}1&2\\3&4\\5&6\end{pmatrix},\ \begin{pmatrix}1&2\\3&5\\4&6\end{pmatrix}:\begin{pmatrix}1&2\\3&4\\5\\6\end{pmatrix}:\begin{pmatrix}1&2\\3\\4\\5\\6\end{pmatrix}:0.$$

$$7;\quad (1\,2\,3\,4\,5\,6\,7):0:\begin{pmatrix}1&2&3&4&5\\6&7\end{pmatrix},\ \begin{pmatrix}1&2&3&4&6\\5&7\end{pmatrix}:$$

$$\begin{pmatrix}1&2&3&4&5\\6\\7\end{pmatrix},\ \begin{pmatrix}1&2&3&4&6\\5\\7\end{pmatrix},\ \begin{pmatrix}1&2&3&5&6\\4\\7\end{pmatrix}:\begin{pmatrix}1&2&3&4\\5&6&7\end{pmatrix},\ \begin{pmatrix}1&2&3&5\\4&6&7\end{pmatrix}:$$

$$\begin{pmatrix}1&2&3&4\\5&6\\7\end{pmatrix},\ \begin{pmatrix}1&2&3&4\\5&7\\6\end{pmatrix},\ \begin{pmatrix}1&2&3&5\\4&6\\7\end{pmatrix},\ \begin{pmatrix}1&2&3&5\\4&7\\6\end{pmatrix},\ \begin{pmatrix}1&2&3&6\\4&5\\7\end{pmatrix}:$$

$$\begin{pmatrix}1&2&3&4\\5\\6\\7\end{pmatrix},\ \begin{pmatrix}1&2&3&5\\4\\6\\7\end{pmatrix}:\begin{pmatrix}1&2&3\\4&5&6\\7\end{pmatrix},\ \begin{pmatrix}1&2&3\\4&5&7\\6\end{pmatrix},\ \begin{pmatrix}1&2&4\\3&5&6\\7\end{pmatrix}:$$

$$\begin{pmatrix}1&2&3\\4&5\\6&7\end{pmatrix},\ \begin{pmatrix}1&2&3\\4&6\\5&7\end{pmatrix},\ \begin{pmatrix}1&2&4\\3&5\\6&7\end{pmatrix}:$$

$$\begin{pmatrix}1&2&3\\4&5\\6\\7\end{pmatrix},\ \begin{pmatrix}1&2&3\\4&6\\5\\7\end{pmatrix},\ \begin{pmatrix}1&2&3\\4&7\\5\\6\end{pmatrix},\ \begin{pmatrix}1&2&4\\3&5\\6\\7\end{pmatrix},\ \begin{pmatrix}1&2&4\\3&6\\5\\7\end{pmatrix}:$$

$$\begin{pmatrix}1&2&3\\4\\5\\6\\7\end{pmatrix},\ \begin{pmatrix}1&2&4\\3\\5\\6\\7\end{pmatrix},\ \begin{pmatrix}1&2&5\\3\\4\\6\\7\end{pmatrix}:$$

$$\begin{pmatrix}1&2\\3&4\\5&6\\7\end{pmatrix},\ \begin{pmatrix}1&2\\3&5\\4&6\\7\end{pmatrix}:\begin{pmatrix}1&2\\3&4\\5\\6\\7\end{pmatrix},\ \begin{pmatrix}1&2\\3&5\\4\\6\\7\end{pmatrix}:0:\left\langle\begin{matrix}1\\2\\3\\4\\5\\6\\7\end{matrix}\right\rangle.$$

We notice that when NP belongs to $T_{a_1, a_2, \ldots, a_h}$ it stands for $f_{a_1, a_2, \ldots, a_h}$ independent forms. Now the total number of independent forms obtained from X by interchanges of letters, when X is unaltered by $(a_1 a_2 \ldots a_n)$, is $(n-1)!$ It is easy to count up in each of the cases above that the forms given give the right number. Thus for $n = 5$ they represent $1+0+5+2 \cdot 6+5+0+1 = 4!$ independent forms.

Let us call these reduced standard forms.

It is easy to see that in every case, except that of $T_{1, 1, \ldots, 1}$, 2 must occupy the second place of the first row. Otherwise, let ab be the nearest pair of letters in the first row, then, if c is a letter between them, we may introduce c into the top row by means of the ordinary identity. Then the pair ac or the pair bc is now in the top row; *i.e.* a pair nearer together than ab; we can proceed thus until we have a consecutive pair in the top row; and then, by means of the permutation $(a_1 a_2 \ldots a_n)$, we may make the consecutive pair to be $a_1 a_2$. We remark that, in our agreed sequence of tableaux, a tableau with 12 in the first row precedes all those which do not contain these digits there.

The change in the sign of every transposition is equivalent to the interchange of N's and P's, and in the case of tableaux to the interchange of columns and rows. This change leaves the substitution $(a_1 a_2 \ldots a_n)$ unaltered when n is odd, but changes its sign when n is even. Hence, when n is odd, $T_{2, 1, \ldots, 1} = 0$.

When n is even the last T is obviously zero.

12. It will be useful to give the relations which express some of the ordinary standard forms for degree 7 in terms of the reduced forms in this case. These relations are sufficiently obvious for the first four T's.

By using the group $\{4\,5\,6\,7\}$ as in § 10, we find

$$\begin{pmatrix} 1\,2\,3\,6 \\ 4\,5\,7 \end{pmatrix} = -\begin{pmatrix} 1\,2\,3\,5 \\ 4\,6\,7 \end{pmatrix} - 3\begin{pmatrix} 1\,2\,3\,4 \\ 5\,6\,7 \end{pmatrix}.$$

In the same way from $\{4\,5\,6\}\begin{pmatrix} 1\,2\,3\,6 \\ 4\,5 \\ 7 \end{pmatrix}$ we obtain

$$\begin{pmatrix} 1\,2\,3\,6 \\ 4\,7 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 1\,2\,3\,6 \\ 4\,5 \\ 7 \end{pmatrix} + \begin{pmatrix} 1\,2\,3\,5 \\ 4\,7 \\ 6 \end{pmatrix} + 2\begin{pmatrix} 1\,2\,3\,5 \\ 4\,6 \\ 7 \end{pmatrix} + \begin{pmatrix} 1\,2\,3\,4 \\ 5\,7 \\ 6 \end{pmatrix} + 2\begin{pmatrix} 1\,2\,3\,4 \\ 5\,6 \\ 7 \end{pmatrix} = 0.$$

Using the ordinary identity two or three times we find

$$\begin{pmatrix}1\,2\,3\,6\\4\\5\\7\end{pmatrix}+\begin{pmatrix}1\,2\,3\,5\\4\\6\\7\end{pmatrix}+\begin{pmatrix}1\,2\,3\,4\\5\\6\\7\end{pmatrix}=0.$$

For $T_{3,3,1}$, we have

$$\begin{pmatrix}1\,2\,3\\5\,6\,7\\4\end{pmatrix}=\begin{pmatrix}4\,5\,6\\1\,2\,3\\7\end{pmatrix}=-\begin{pmatrix}1\,2\,3\\4\,5\,6\\7\end{pmatrix},$$

and hence

$$\begin{pmatrix}1\,2\,3\\4\,5\,7\\6\end{pmatrix}+\begin{pmatrix}1\,2\,3\\4\,6\,7\\5\end{pmatrix}=0.$$

Also we find

$$\begin{pmatrix}1\,2\,4\\3\,6\,7\\5\end{pmatrix}=\begin{pmatrix}1\,2\,4\\3\,5\,6\\7\end{pmatrix}+\begin{pmatrix}1\,2\,3\\4\,5\,6\\7\end{pmatrix}+\begin{pmatrix}123\\457\\6\end{pmatrix},$$

$$2\begin{pmatrix}1\,2\,4\\3\,5\,6\\7\end{pmatrix}+\begin{pmatrix}1\,2\,4\\3\,5\,7\\6\end{pmatrix}+\begin{pmatrix}1\,2\,3\\4\,5\,6\\7\end{pmatrix}=0,$$

$$\begin{pmatrix}1\,2\,5\\3\,4\,6\\7\end{pmatrix}=-\begin{pmatrix}1\,2\,4\\3\,6\,7\\5\end{pmatrix},\ \begin{pmatrix}1\,2\,5\\3\,4\,7\\6\end{pmatrix}=-\begin{pmatrix}1\,2\,5\\3\,6\,7\\4\end{pmatrix}=2\begin{pmatrix}1\,2\,4\\3\,5\,6\\7\end{pmatrix}-\begin{pmatrix}1\,2\,3\\4\,5\,7\\6\end{pmatrix},$$

$$\begin{pmatrix}1\,2\,6\\3\,4\,5\\7\end{pmatrix}=\begin{pmatrix}1\,2\,3\\4\,5\,7\\6\end{pmatrix},\ \begin{pmatrix}1\,2\,6\\3\,4\,7\\5\end{pmatrix}=-\begin{pmatrix}1\,2\,4\\3\,6\,7\\5\end{pmatrix},\ \begin{pmatrix}1\,2\,6\\3\,5\,7\\4\end{pmatrix}=-\begin{pmatrix}1\,2\,4\\3\,5\,7\\6\end{pmatrix}.$$

For $T_{3,2,2}$:

$$\begin{pmatrix}1\,2\,4\\3\,6\\5\,7\end{pmatrix}=-\begin{pmatrix}1\,2\,3\\4\,6\\5\,7\end{pmatrix},\ \begin{pmatrix}1\,2\,5\\3\,4\\6\,7\end{pmatrix}=-\begin{pmatrix}1\,2\,4\\3\,5\\6\,7\end{pmatrix},$$

$$\begin{pmatrix}1\,2\,5\\3\,6\\4\,7\end{pmatrix}=3\begin{pmatrix}1\,2\,3\\4\,6\\5\,7\end{pmatrix}+\begin{pmatrix}1\,2\,3\\4\,5\\6\,7\end{pmatrix}-\begin{pmatrix}1\,2\,4\\3\,5\\6\,7\end{pmatrix},$$

$$\begin{pmatrix}1\,2\,6\\3\,4\\5\,7\end{pmatrix}=\begin{pmatrix}1\,2\,4\\3\,5\\6\,7\end{pmatrix},\ \begin{pmatrix}1\,2\,6\\3\,5\\4\,7\end{pmatrix}=-\begin{pmatrix}1\,2\,3\\4\,6\\5\,7\end{pmatrix}.$$

For $T_{3,2,1,1}$:

$$\begin{pmatrix}1\,2\,4\\3\,7\\5\\6\end{pmatrix}+\begin{pmatrix}1\,2\,4\\3\,6\\5\\7\end{pmatrix}+\begin{pmatrix}1\,2\,4\\3\,5\\6\\7\end{pmatrix}=-2\begin{pmatrix}1\,2\,3\\4\,5\\6\\7\end{pmatrix},$$

$$\begin{pmatrix}1\,2\,5\\3\,4\\6\\7\end{pmatrix}=-\begin{pmatrix}1\,2\,4\\3\,5\\6\\7\end{pmatrix}-\begin{pmatrix}1\,2\,3\\4\,5\\6\\7\end{pmatrix}+\begin{pmatrix}1\,2\,3\\4\,6\\5\\7\end{pmatrix}+\begin{pmatrix}1\,2\,3\\4\,7\\5\\6\end{pmatrix},$$

$$\begin{pmatrix}1\,2\,5\\3\,6\\4\\7\end{pmatrix}=\begin{pmatrix}1\,2\,4\\3\,7\\5\\6\end{pmatrix}+\begin{pmatrix}1\,2\,3\\4\,5\\6\\7\end{pmatrix},$$

$$\begin{pmatrix}1\,2\,5\\3\,7\\4\\6\end{pmatrix}=\begin{pmatrix}1\,2\,3\\4\,7\\5\\6\end{pmatrix},\quad\begin{pmatrix}1\,2\,6\\3\,4\\5\\7\end{pmatrix}=\begin{pmatrix}1\,2\,4\\3\,7\\5\\6\end{pmatrix}-\begin{pmatrix}1\,2\,3\\4\,6\\5\\7\end{pmatrix},$$

$$\begin{pmatrix}1\,2\,6\\3\,5\\4\\7\end{pmatrix}=\begin{pmatrix}1\,2\,4\\3\,5\\6\\7\end{pmatrix}-\begin{pmatrix}1\,2\,4\\3\,7\\5\\6\end{pmatrix}+\begin{pmatrix}1\,2\,3\\1\,5\\6\\7\end{pmatrix}-\begin{pmatrix}1\,2\,3\\4\,6\\5\\7\end{pmatrix}-\begin{pmatrix}1\,2\,3\\4\,7\\5\\6\end{pmatrix},$$

$$\begin{pmatrix}1\,2\,6\\3\,7\\4\\5\end{pmatrix}=-\begin{pmatrix}1\,2\,4\\3\,5\\6\\7\end{pmatrix}-\begin{pmatrix}1\,2\,3\\4\,5\\6\\7\end{pmatrix}-\begin{pmatrix}1\,2\,3\\4\,7\\5\\6\end{pmatrix}.$$

The remaining forms do not call for special remark.

13. Let us write $\bar{f}_{a_1, a_2, \ldots, a_h}$ or just $\bar{f}$ as the number of reduced standard forms belonging to $T_{a_1, a_2, \ldots, a_h}$. Then $f \,.\, \bar{f}$ is the total number of linearly independent forms obtained by interchanging the letters in TX.

If we ignore the letter a_n—or rather agree that it shall always occupy the same position in X (*e.g.* the last), which we can do—then we may regard X as a function of the first $n-1$ letters only, and express it and the functions derived by interchange of letters from it in terms of the corresponding forms $N'P'_rX$. These are expressible in terms of the forms NPX involving all the n letters, and hence in terms of the reduced standard forms. Let us use f' to denote the number of standard tableaux F' corresponding to T' which involves the first $n-1$ letters only. Then the number $\Sigma f'$ is considerably greater than the number of reduced

standard tableaux $\Sigma \bar{f}$, as we see at a glance of the particulars above; it is evident from the fact that the total number of linearly independent forms X is

$$\Sigma f\bar{f} = (n-1)! = \Sigma f'^2,$$

remembering that f is, in general, considerably greater than f'.

Now
$$N'_s P'_s X = \Sigma \frac{ff'}{n!(n-1)!} N'_s P'_s T N'_s P'_s X$$
$$= \Sigma \frac{ff'}{n!(n-1)!} N'_s P'_s N_s P_s N'_s P'_s X,$$

where the standard tableau F_s is obtained from the standard tableau F'_s by adding a_n, so that it occupies a position which is the end of a row and the bottom of a column.

In order to see the truth of this we observe that a_n must occupy such a position in any standard tableau F_r, and its suppression produces a standard tableau F'_r. Hence $N_r P_r = \lambda N'_r N_r P_r P'_r$, and, unless $r = s$, either

$$P'_s N'_r = 0 \quad \text{or} \quad P'_r N'_s = 0.$$

Now
$$N'_s P'_s N_s P_s N'_s P'_s X = \Sigma \lambda_{r,s} N'_s P'_s N_s \sigma_{sr} \bar{P}_r X,$$

where $\bar{P}_r$ is obtained from a reduced standard tableau. Thus

$$N'_s P'_s X = \Sigma' \Sigma \mu_{r,s} N'_s P'_s N_s \sigma_{sr} \bar{P}_r X,$$

where the first Σ' expresses the sum of the sets of terms according to the possible T's involved, viz. those which are obtained by the increase of one of the suffixes by unity of the T' concerned on the other side. In just the same way we obtain

$$N'_s \sigma'_{st} P'_t X = \Sigma' \Sigma \mu^{(t)}_{r,s} N'_s P'_s N_s \sigma_{sr} \bar{P}_r X.$$

There are thus $f'_{a_1, a_2, \ldots, a_h}$ possible linearly independent forms with the same N'_s on the left-hand side expressed in terms of

$$\bar{f}_{a_1+1, a_2, \ldots, a_h} + \bar{f}_{a_1, a_2+1, \ldots a_h} + \ldots + \bar{f}_{a_1, a_2, \ldots, a_h, 1} = \bar{S}_f \quad \text{(say)}$$

forms on the right-hand side. Hence

$$\bar{S}_f \geqslant f'_{a_1, a_2, \ldots, a_h}.$$

Now, by § 3 (1),

$$\Sigma \bar{f}_{\beta_1, \beta_2, \ldots, \beta_h} f_{\beta_1, \beta_2, \ldots, \beta_h}$$
$$= \Sigma \bar{f}_{\beta_1, \beta_2, \ldots, \beta_h} \{ f'_{\beta_1 - 1, \beta_2, \ldots, \beta_h} + f'_{\beta_1, \beta_2 - 1, \ldots, \beta_h} + \ldots + f'_{\beta_1, \beta_2, \ldots, \beta_h - 1} \}$$
$$= \Sigma f'_{\alpha_1, \alpha_2, \ldots, \alpha_h} \{ \bar{f}_{\alpha_1 + 1, \alpha_2, \ldots, \alpha_h} + \bar{f}_{\alpha_1, \alpha_2 + 1, \ldots, \alpha_h} + \ldots + \bar{f}_{\alpha_1, \alpha_2, \ldots, \alpha_h, 1} \}$$
$$= \Sigma f'^2_{\alpha_1, \alpha_2, \ldots, \alpha_h}.$$

Therefore $$\Sigma f'_{\alpha_1, \alpha_2, \ldots, \alpha_h} \{\bar{S}_f - f'_{\alpha_1, \alpha_2 \ldots, \alpha_h}\} = 0.$$

None of the terms on the right can be negative, as we have just seen, hence each is zero. Therefore

THEOREM X. *If* $\bar{f}_{\beta_1, \beta_2, \ldots, \beta_h}$ *be the number of reduced standard tableaux corresponding to* $T_{\beta_1, \beta_2, \ldots, \beta_h}$, *when the operand is unaltered by the substitution* $(a_1 a_2 \ldots a_n)$, *then*

$$\bar{f}_{\alpha_1+1, \alpha_2, \ldots, \alpha_h} + \bar{f}_{\alpha_1, \alpha_2+1, \ldots, \alpha_h} + \ldots + f_{\alpha_1, \alpha_2, \ldots, \alpha_h, 1} = f_{\alpha_1, \alpha_2, \ldots, \alpha_h}.$$

It should be noticed that the proof of this theorem applies equally well to the case when X is unaltered by the substitutions of any transitive group of degree n. In fact, it applies to the following:

THEOREM XI. *If* G *be a transitive group involving* n *letters* $a_1, a_2, \ldots, a_n$; *and if* Γ *be that subgroup of* G *which leaves* a_n *unchanged, and if* $\bar{f}_{\beta_1, \beta_2, \ldots, \beta_h}$ *be the number of reduced standard tableaux corresponding to* $T_{\beta_1, \beta_2, \ldots, \beta_h}$ *when the operand is unaltered by any substitution of* G; *and* $\bar{f}'_{\alpha_1, \alpha_2, \ldots, \alpha_h}$ *the number of reduced standard tableaux corresponding to* $T'_{\alpha_1, \alpha_2, \ldots, \alpha_h}$ *(involving the first* $n-1$ *letters only) when the operand is unaltered by any substitution of* Γ, *then*

$$\bar{f}_{\alpha_1+1, \alpha_2, \ldots, \alpha_h} + \bar{f}_{\alpha_1, \alpha_2+1, \ldots, \alpha_h} + \ldots + \bar{f}_{\alpha_1, \alpha_2, \ldots, \alpha_h, 1} = \bar{f}'_{\alpha_1, \alpha_2, \ldots, \alpha_h}.$$

13. When the object of discussion is not a function of the letters, but just a substitutional expression, in general the equations will involve both left- and right-hand multiplication. Thus we might seek to find those substitutional expressions which are unaltered by multiplication by $(a_1 a_2 \ldots a_n)$ on either side. We have called the number of independent forms NP belonging to T which are unchanged by multiplication on the right hand by $(a_1 a_2 \ldots a_n)$, N being supposed fixed, f; and thus, taking the f different forms N, we had $f\bar{f}$ independent forms. We might equally well have used the forms PN instead of NP, and we are bound to find the same number $\bar{f}$ in this case. Now, if we are dealing with expressions also unchanged by left-hand multiplication, there are only $\bar{f}$ reduced standard forms N in NP as well as that number for P. Thus the total number of linearly independent forms of this character belonging to T is $\bar{f}^2$. And the total number altogether of such forms is $\Sigma \bar{f}^2$. These, of course, include the sum of the substitutions of any group which contains $(a_1 a_2 \ldots a_n)$.

14. The quartic invariant types are unchanged by a reversal of the sequence of letters. We then seek for standard forms further reduced, which are appropriate to the discussion of functions unchanged by the operations of the group :

$$\{(1\,2\,3\,\ldots\,n),\ (1n)(2\,\overline{n-1}),\ \ldots\},$$

These are :

n

3 ; $(1\,2\,3) : 0 : 0,$

4 ; $(1\,2\,3\,4) : 0 : \begin{pmatrix} 1\,2 \\ 3\,4 \end{pmatrix} : 0 : 0,$

5 ; $(1\,2\,3\,4\,5) : 0 : \begin{pmatrix} 1\,2\,3 \\ 4\,5 \end{pmatrix} : 0 : \begin{pmatrix} 1\,2 \\ 3\,4 \\ 5 \end{pmatrix} : 0 : \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix},$

6 ; $(1\,2\,3\,4\,5\,6) : 0 : \begin{pmatrix} 1\,2\,3\,4 \\ 5\,6 \end{pmatrix}, \begin{pmatrix} 1\,2\,3\,5 \\ 4\,6 \end{pmatrix} : 0 : 0 : \begin{pmatrix} 1\,2\,3 \\ 4\,5 \\ 6 \end{pmatrix} : \begin{pmatrix} 1\,2\,4 \\ 3 \\ 5 \\ 6 \end{pmatrix} :$

$$\begin{pmatrix} 1\,2 \\ 3\,4 \\ 5\,6 \end{pmatrix}, \begin{pmatrix} 1\,2 \\ 3\,5 \\ 4\,6 \end{pmatrix} : 0 : \begin{pmatrix} 1\,2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} : 0.$$

7 ; $(1\,2\,3\,4\,5\,6\,7) : 0 : \begin{pmatrix} 1\,2\,3\,4\,5 \\ 6\,7 \end{pmatrix}, \begin{pmatrix} 1\,2\,3\,4\,6 \\ 5\,7 \end{pmatrix} : 0 : \begin{pmatrix} 1\,2\,3\,4 \\ 5\,6\,7 \end{pmatrix} :$

$$\begin{pmatrix} 1\,2\,3\,4 \\ 5\,6 \\ 7 \end{pmatrix}, \begin{pmatrix} 1\,2\,3\,5 \\ 4\,6 \\ 7 \end{pmatrix}, \begin{pmatrix} 1\,2\,3\,5 \\ 4\,7 \\ 6 \end{pmatrix} : \begin{pmatrix} 1\,2\,3\,5 \\ 4 \\ 6 \\ 7 \end{pmatrix} : 0 :$$

$$\begin{pmatrix} 1\,2\,3 \\ 4\,5 \\ 6\,7 \end{pmatrix}, \begin{pmatrix} 1\,2\,3 \\ 4\,6 \\ 5\,7 \end{pmatrix}, \begin{pmatrix} 1\,2\,4 \\ 3\,5 \\ 6\,7 \end{pmatrix} : \begin{pmatrix} 1\,2\,3 \\ 4\,5 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 1\,2\,4 \\ 3\,5 \\ 6 \\ 7 \end{pmatrix} :$$

$$\begin{pmatrix} 1\,2\,3 \\ 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 1\,2\,4 \\ 3 \\ 5 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 1\,2\,5 \\ 3 \\ 4 \\ 6 \\ 7 \end{pmatrix} : \begin{pmatrix} 1\,2 \\ 3\,4 \\ 5\,6 \\ 7 \end{pmatrix} : 0 : 0 : 0,$$

$$8;\ (1\,2\,3\,4\,5\,6\,7\,8):0:\begin{pmatrix}1\,2\,3\,4\,5\,6\\7\,8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\,4\,5\,7\\6\,8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\,4\,6\,7\\5\,8\end{pmatrix}:0:$$

$$\begin{pmatrix}1\,2\,3\,4\,5\\6\,7\,8\end{pmatrix}:\begin{pmatrix}1\,2\,3\,4\,5\\6\,7\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\,4\,6\\5\,7\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\,4\,6\\5\,8\\7\end{pmatrix},\ \begin{pmatrix}1\,2\,3\,5\,6\\4\,7\\8\end{pmatrix}:$$

$$\begin{pmatrix}1\,2\,3\,4\,6\\5\\7\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\,5\,6\\4\\7\\8\end{pmatrix}:\begin{pmatrix}1\,2\,3\,4\\5\,6\,7\,8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\,5\\4\,6\,7\,8\end{pmatrix},\ \begin{pmatrix}1\,2\,4\,6\\3\,5\,7\,8\end{pmatrix}:$$

$$\begin{pmatrix}1\,2\,3\,4\\5\,6\,7\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\,5\\4\,6\,7\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\,5\\4\,6\,8\\7\end{pmatrix}:\begin{pmatrix}1\,2\,3\,4\\5\,6\\7\,8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\,4\\5\,7\\6\,8\end{pmatrix},$$

$$\begin{pmatrix}1\,2\,3\,5\\4\,6\\7\,8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\,5\\4\,7\\6\,8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\,6\\4\,5\\7\,8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\,6\\4\,7\\5\,8\end{pmatrix},\ \begin{pmatrix}1\,2\,4\,6\\3\,5\\7\,8\end{pmatrix}:$$

$$\begin{pmatrix}1\,2\,3\,4\\5\,6\\7\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\,5\\4\,6\\7\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\,5\\4\,7\\6\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,4\,6\\3\,7\\5\\8\end{pmatrix}:$$

$$\begin{pmatrix}1\,2\,3\,4\\5\\6\\7\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\,5\\4\\6\\7\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\,6\\4\\5\\7\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,4\,6\\3\\5\\7\\8\end{pmatrix}:\begin{pmatrix}1\,2\,3\\4\,5\,7\\6\,8\end{pmatrix}:$$

$$\begin{pmatrix}1\,2\,3\\4\,5\,6\\7\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\\4\,5\,7\\6\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,4\\3\,5\,6\\7\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,4\\3\,5\,7\\6\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,4\\3\,5\,8\\6\\7\end{pmatrix}:$$

$$\begin{pmatrix}1\,2\,3\\4\,5\\6\,7\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\\4\,5\\6\,8\\7\end{pmatrix},\ \begin{pmatrix}1\,2\,3\\4\,6\\5\,7\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,4\\3\,5\\6\,7\\8\end{pmatrix}:$$

$$\begin{pmatrix}1\,2\,3\\4\,5\\6\\7\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,3\\4\,6\\5\\7\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,4\\3\,5\\6\\7\\8\end{pmatrix},\ \begin{pmatrix}1\,2\,4\\3\,6\\5\\7\\8\end{pmatrix}:\begin{pmatrix}1\,2\,3\\4\\5\\6\\7\\8\end{pmatrix}:$$

$$\begin{pmatrix}1\,2\\3\,4\\5\,6\\7\,8\end{pmatrix},\ \begin{pmatrix}1\,2\\3\,4\\5\,7\\6\,8\end{pmatrix},\ \begin{pmatrix}1\,2\\3\,6\\4\,7\\5\,8\end{pmatrix}:0:\begin{pmatrix}1\,2\\3\,4\\5\\6\\7\\8\end{pmatrix},\ \begin{pmatrix}1\,2\\3\,6\\4\\5\\7\\8\end{pmatrix}:0:0.$$

15. If we were to consider the corresponding relations for sextic invariant types of the form

$$(a_1 a_2)^3 (a_2 a_3)^3 \dots (a_n a_1)^3 ;$$

we should find that, when n is odd, a reversal of the sequence of letters changes the sign. The appropriate reduced standard forms then are :

n

$$3 ; \quad 0 : 0 : \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

$$5 ; \quad 0 : 0 : \begin{pmatrix} 1\,2\,3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 1\,2\,4 \\ 3 \\ 5 \end{pmatrix} : 0 : 0.$$

$$7 ; \quad 0 : 0 : 0 : \begin{pmatrix} 1\,2\,3\,4\,5 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 1\,2\,3\,4\,6 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} 1\,2\,3\,5\,6 \\ 4 \\ 7 \end{pmatrix} : \begin{pmatrix} 1\,2\,3\,5 \\ 4\,6\,7 \end{pmatrix} :$$

$$\begin{pmatrix} 1\,2\,3\,4 \\ 5\,6 \\ 7 \end{pmatrix}, \begin{pmatrix} 1\,2\,3\,5 \\ 4\,6 \\ 7 \end{pmatrix} : \begin{pmatrix} 1\,2\,3\,4 \\ 5 \\ 6 \\ 7 \end{pmatrix} : \begin{pmatrix} 1\,2\,3 \\ 4\,5\,6 \\ 7 \end{pmatrix}, \begin{pmatrix} 1\,2\,3 \\ 4\,5\,7 \\ 6 \end{pmatrix} : \begin{pmatrix} 1\,2\,4 \\ 3\,5\,6 \\ 7 \end{pmatrix} :$$

$$0 : \begin{pmatrix} 1\,2\,3 \\ 4\,5 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 1\,2\,3 \\ 4\,6 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} 1\,2\,4 \\ 3\,5 \\ 6 \\ 7 \end{pmatrix} :$$

$$0 : \begin{pmatrix} 1\,2 \\ 3\,5 \\ 4\,6 \\ 7 \end{pmatrix} : \begin{pmatrix} 1\,2 \\ 3\,4 \\ 5 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 1\,2 \\ 3\,5 \\ 4 \\ 6 \\ 7 \end{pmatrix} : 0 : \left\langle \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} \right\rangle.$$

16. The other equations which the quartic types satisfy are of the form $\{\mathrm{abc}\}\, I = R$, where R stands for a sum of products of invariants, and the symbol $\{\mathrm{abc}\}$ means, in the first place, the symmetric group of any three consecutive letters; and, in the second place, a, b, c may each stand for a set of several letters, and then the symbol $\{\mathrm{abc}\}$ is supposed to interchange the sets *en bloc* in I after the manner of the symmetric group of degree 3—(the sequence of the letters in each set being undisturbed)—it being understood that the three sets abc, as they stand,

form a consecutive set of letters in I^*. A further proviso is necessary, namely that there is at least one letter in I not contained in abc. Then we see at once that, for $n = 4$,

$$I_4 = \lambda \begin{pmatrix} 1\,2 \\ 3\,4 \end{pmatrix} I_4 + R,$$

and there are two independent irreducible forms. For $n = 5$

$$I_5 = \lambda \begin{pmatrix} 1\,2 \\ 3\,4 \\ 5 \end{pmatrix} I_5 + \mu \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} I_5 + R$$

giving 6 forms.

For $n = 6$, we have to introduce a second equation, which we may write $\{1, 2, 3\,4\}\, I = R$. If $I = [1\,2\,3\,4\,5\,6]$, this gives

$$\{1\,2\}\,\{[1\,2\,3\,4\,5\,6] + [1\,3\,4\,2\,5\,6] + [3\,4\,1\,2\,5\,6]\} = R.$$

Now when we operate with NP on $\Sigma \lambda s \,.\, I$, the result is represented by operating on the tableau for P with s^{-1}.

Thus $\begin{pmatrix} 1\,2 \\ 3\,4 \\ 5\,6 \end{pmatrix} \{1, 2, 3\,4\}\, I = R$, gives

$$\left[\begin{pmatrix} 1\,2 \\ 3\,4 \\ 5\,6 \end{pmatrix} + \begin{pmatrix} 1\,4 \\ 2\,3 \\ 5\,6 \end{pmatrix} + \begin{pmatrix} 3\,4 \\ 1\,2 \\ 5\,6 \end{pmatrix} \right] I = R$$

or

$$\left[\begin{pmatrix} 1\,2 \\ 3\,4 \\ 5\,6 \end{pmatrix} - \begin{pmatrix} 1\,2 \\ 3\,5 \\ 4\,6 \end{pmatrix} \right] I = R.$$

Similarly operating with $\begin{pmatrix} 1\,2 \\ 3\,5 \\ 4\,6 \end{pmatrix}$ we have

$$\left[\begin{pmatrix} 1\,2 \\ 3\,5 \\ 4\,6 \end{pmatrix} + \begin{pmatrix} 1\,4 \\ 2\,5 \\ 3\,6 \end{pmatrix} + \begin{pmatrix} 3\,4 \\ 1\,5 \\ 2\,6 \end{pmatrix} \right] I = R$$

or

$$\left[5 \begin{pmatrix} 1\,2 \\ 3\,5 \\ 4\,6 \end{pmatrix} + \begin{pmatrix} 1\,2 \\ 3\,5 \\ 4\,6 \end{pmatrix} \right] I = R.$$

Whence

$$T_{2,2,2} I = R.$$

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* *Proc. London Math. Soc.* (1), 30 (1898), 295.

Similarly we may prove that $T_{2,1,1,1,1}I = R$, though this is obvious for other reasons.

Hence $$I_6 = \lambda \begin{pmatrix} 1\,2\,4 \\ 3 \\ 5 \\ 6 \end{pmatrix} I_6 + R,$$

and there are 10 independent forms.

For $n = 7$, we may use $\{2\,3,\ 4\,5,\ 6\,7\}\,I = R$, *i.e.*

$$\{(2\,4)(3\,5),\ (4\,6)(5\,7)\}\,I = R\,;$$

operation with $\begin{pmatrix} 1\,2\,4 \\ 3\,5 \\ 6\,7 \end{pmatrix}$ reduces this.

The reduction of $\begin{pmatrix} 1\,2\,4 \\ 3\,5 \\ 6 \\ 7 \end{pmatrix}$ does not come with this operation. We may, however, obtain it by operating with $\begin{Bmatrix} 1\,2\,5 \\ 3\,4 \\ 6 \\ 7 \end{Bmatrix}$ on

$$\{1,\ 2,\ 3\,4\}\,I = R$$

and using the equations of § 9.

The reduction of $T_{3,1,1,1,1}$ is obvious, but it may be obtained, of course, from these equations. Thus $I_7 = R$.

Similarly for I_8.

17. For the sextic invariant types mentioned above, § 15, we have equations of exactly the same form, viz. :

$$\{\mathrm{a\,b\,c\,d}\}\,I = R,$$

where a, b, c, d each stands for a set of consecutive letters, all four sets being consecutive in I. There are no equations before degree 6. Thus

$$I_5 = \lambda \left[\begin{pmatrix} 1\,2\,3 \\ 4 \\ 5 \end{pmatrix} + \begin{pmatrix} 1\,2\,4 \\ 3 \\ 5 \end{pmatrix} \right] I_5$$

and there are 12 independent forms;

$$I_6 = \lambda \begin{pmatrix} 1\,2\,3\,5 \\ 4\,6 \end{pmatrix} I_6 + \ldots$$

and there are 50 forms.

Consider these sextic invariant types in general. If, in any row of P, there are 4 consecutive digits, then $NPI = R$. Further, the same is true when there are two triads of consecutive digits in any row. Let these be 1, 2, 3, and r, $r+1$, $r+2$, the result is obtained by operating with NP on the equation

$$\{1, 2, 3, 4\,5 \ldots (r-1)\} = R.$$

When there are three separated digits in a row of P, the form may be expressed in terms of such as have the *first* pair separated only by a single digit, and of forms which have two of the digits replaced by a consecutive pair. Let P have 1, r, s in a row, but none of the intervening letters in that row, then the result is given by

$$NP\{1,\ 2\,3 \ldots (r-2),\ (r-1)\,r\,(r+1) \ldots (s-1),\ s\} = R;$$

this equation introduces also terms in which the number of digits between the first and last of the three considered is diminished, we regard this as a reduction, and use the same process to the terms so reduced, thus obtaining the result. We may regard the digits 1, r, s as systems of consecutive digits for which the first and last of any system is in the row considered, and the argument holds good. Thus we obtain the result that, if a row contains four separated digits, the form may be expressed in terms of forms in which there are four separated digits, the first two pairs being separated by a single digit, and of forms in which two of the separated digits are replaced by a consecutive pair. We treat the last three digits first, and, by what we have just proved, we may suppose that the first pair of these are separated by a single digit. Then we treat this pair and the digit between as a single digit, and the result follows.

We are now in a position to prove that a form with a row which has four separated digits can be expressed in terms of forms in which two of the digits are replaced by a consecutive pair. We have shown that we may suppose that each of the first two pairs is separated by a single digit, thus we may suppose that the digits 1, 3, 5, r lie in a row of P. Then we may include in the generic symbol R all forms in which two of these digits are replaced by a consecutive pair. Then

$$NP\{1, 2, 3, 4\}\, I = R$$

gives

$$NP\{2, 4\}\, I = R.$$

Let us write

$$I = \begin{pmatrix} 1 & 3 & 5 & r \\ 2, & 4, & 6, & 7, \ \ldots, \ (r-1) \end{pmatrix},$$

the commas in the lower row indicating that any permutation of the

digits in the row must be supposed to change the function. The equation

$$\{4,\ 5,\ 6\ 7\ \ldots (r-1),\ r\}\, I = R$$

gives

$$\begin{pmatrix} 1 & 3 & 5 & r \\ 2, & 4, & 6, & 7, \ \ldots, \ (r-1) \end{pmatrix} + \begin{pmatrix} 1 & 3 & (r-2) & r \\ 2, & (r-1), & 4, & 5, \ \ldots, \ (r-3) \end{pmatrix} = R.$$

Also by

$$\{1,\ 2,\ 3,\ 4\ 5\ \ldots (r-3)\} \begin{pmatrix} 1 & 3 & (r-2) & r \\ 2, & (r-1), & 4, & 5, \ (r-3) \end{pmatrix} = R,$$

$$\begin{pmatrix} 1 & 3 & (r-2) & r \\ 2, & (r-1), & 4, & 5, \ \ldots, \ (r-3) \end{pmatrix} + \begin{pmatrix} 1 & (r-4) & (r-2) & r \\ (r-3), & (r-1), & 2, & 3, \ \ldots, \ (r-5) \end{pmatrix} = R;$$

and by

$$\{2\,3\,4\ \ldots (r-5),\ (r-4)\,(r-3)\,(r-2),\ (r-1),\ r\}$$

$$\times \begin{pmatrix} 1 & (r-4) & (r-2) & r \\ (r-3), & (r-1), & 2, & 3, \ \ldots, \ (r-5) \end{pmatrix} = R,$$

we have

$$\begin{pmatrix} 1 & (r-4) & (r-2) & r \\ (r-3), & (r-1), & 2, & 3, \ \ldots, \ (r-5) \end{pmatrix} + \begin{pmatrix} 1 & 3 & 5 & r \\ 2, & 4, & 6, & 7, \ \ldots, \ (r-1) \end{pmatrix} = R.$$

Hence $$\begin{pmatrix} 1 & 3 & 5 & r \\ 2, & 4, & 6, & 7, \ \ldots, \ (r-1) \end{pmatrix} = R.$$

The argument holds good when one or more of the four separated digits is replaced by a consecutive pair or a consecutive triad. We may, in fact, state the result thus: in any particular row of P we may suppose that the digits are separated by not more than two gaps. Naturally we shall take the particular row to be the top one. Let us suppose that this row contains four letters. In this case we may suppose that there is not more than one gap. For, including forms of the desired nature in the symbol R, $\{1, 2, 3, 4\}$ gives

$$\begin{pmatrix} 1 & 2 & 4 & r \\ 3, & 5, & 6, \ \ldots, & (r-1) \end{pmatrix} + \begin{pmatrix} 1 & 3 & 4 & r \\ 2, & 5, & 6, \ \ldots, & (r-1) \end{pmatrix} = R,$$

$\{3,\ 4,\ 5\,6\ \ldots (r-1),\ r\}$ gives

$$\begin{pmatrix} 1 & 3 & 4 & r \\ 2, & 5, & 6, \ \ldots, & (r-1) \end{pmatrix} + \begin{pmatrix} 1 & 3 & (r-1) & r \\ 2, & 4, & 5, \ \ldots, & (r-2) \end{pmatrix} = R$$

and $\{1,\ 2\,3\ \ldots (r-2),\ (r-1),\ r\}$ gives

$$\begin{pmatrix} 1 & 3 & (r-1) & r \\ 2, & 4, & 5, \ \ldots, & (r-2) \end{pmatrix} + \begin{pmatrix} 1 & 2 & 4 & r \\ 3, & 5, & 6, \ \ldots, & (r-1) \end{pmatrix} = R,$$

whence the result as stated.

The same thing is true when there are five letters. The process of proof is the same, and need not be detailed; the theorem is proved first for the case when there are two pairs of consecutive digits and a single one, and then using this result for the case of a triad and two single digits.

A form with two consecutive pairs in a row can be expressed in terms of forms with a triad and a single digit, for example, if P contains 1, 2, and r, $r+1$ in a row, we obtain the result from $\{1, 2, 3\,4 \ldots (r-1), r\}$.

Hence, when there are 6 digits in the top row of P we may suppose them to be a triad, a pair, and a single one; and the same process as used in the former cases may be now employed to show that we can express this in terms of forms in which there is only one gap in the top row. Now this means either two consecutive triads or a set of 4 consecutive letters, and we have already seen that either of these cases is reducible. Hence

$$T_{a_1, a_2 \ldots} I = R,$$

when $a_1 > 5$.

Moreover, when $a_1 = 4$ or 5, we need consider only forms with one gap in the top row, and, in particular, forms with a triad. For $a_1 = 3$, we have already shown that, owing to the fact that I is unchanged, or changed only in sign, by the permutation $(1\,2 \ldots n)$, we may suppose that the top row begins with the digits 1 2, and hence here, too, there is only one gap. Thus in all cases we need consider only forms with one gap in the top row.

Further, we may suppose that the single gap contains not more than three digits. For, writing a, b as the two consecutive sets in the top row and supposing that $a+1$, $a+2$, ..., $b-1$ are the single digits between them, we see that the equation

$$\{a\,(a+1),\ (a+2),\ (a+3)\,(a+4),\ (a+5) \ldots (b-1)\,b\}\, I = R$$

gives

$$\{1+(\overline{a+2}\ \overline{a+3}\ \overline{a+4})\}\, I = R,$$

where R includes forms with a smaller gap than in I. This equation, of course, reduces I, and the reduction holds good so long as there are at least 4 digits in I.

The same process yields the result, for $b = a+4$,

$$I = -(\overline{a+1}\ \overline{a+2})\, I + R = -(\overline{a+2}\ \overline{a+3})\, I + R$$
$$= -(\overline{a+1}\ \overline{a+3})\, I + R,$$

which means that we have a reduction for a gap of three digits unless each of those digits is in a different row of P.

Let Γ_r be a negative symmetric group of r digits. Then, by Peano's theorem*, $\Gamma_r I = 0$ when $r > 7$, and $\Gamma_7 I$ has a factor Δ the determinant of the seven quantics to which the letters of Γ_7 refer. In the case of the sextic Δ is reducible. For

$$\{\text{abcdefg}\}'(ab)^3(bc)^3(ca)^3.(de)^5(ef)(fg)^5(gd)$$

is not zero, and is therefore the invariant in question. Now it was shown in the proof of Peano's theorem, just quoted, that when this invariant is reducible for the binary n-ic, then also $\Gamma_n I$ is reducible. Hence, when there are more than 5 rows in P the form is certainly reducible. It will be noticed that this last reduction is not contained in our formal equations, for none of them reduce the invariant in question of degree 7. They would give the general result of a reduction for $\Gamma_7 I$ when the degree of I is greater than 7. Thus we can say from these equations alone, and so far as we have gone, that I is certainly reducible when the degree is greater than 30, for P cannot have more than 6 rows and more than 5 digits in a row, and with the added knowledge of the reducibility of Δ we can say that P is certainly reducible when the degree is greater than 25. Of course this system does not contain all the sextic invariant types; it does not, for instance, include the invariant of degree 15 for a single sextic. We have considered it rather as an example of substitutional equations.

18. We proceed to show briefly how group characters naturally arise in the investigation of equations of the form $\Sigma \lambda_s X = R$, where s is an operation of a group H. For this purpose we consider the operations of the group as capable of quantitative expressions, as heretofore. That is, we submit them to the ordinary laws of algebra, the commutative law alone excepted. Equations composed of sums of terms involving such operations may be interpreted in two ways. In the first place a function to be operated on—an operand—may always be implicitly understood to be present, so that the sum of the operations really represents a sum of functions. Or, in the second place, we may regard the operations as capable of independent existence, as, for instance, hypercomplex units, or else as matrices, thinking of the results as referring to matrix groups.

Frobenius first considers the k different classes into which the elements of a group H may be divided, all the elements conjugate to a given

* Q.S.A., I, 141–143.

element forming one class. The number of elements belonging to a class is called h_α, and the number of elements in whole group is h, so that

$$\Sigma h_\alpha = h.$$

If now we write S_α as the sum of the elements of the α-th class, the product $S_\alpha S_\beta$, when expanded as a sum of elements, contains all the elements of any given class in exactly the same way, so that

$$S_\alpha S_\beta = \sum_\gamma \frac{h_{\alpha\beta\gamma'}}{h_\gamma} S_\gamma,$$

where γ' is the class inverse to γ, that is, if s is an operation of γ, then γ' contains that operation s^{-1} which satisfies the equation $s \,.\, s^{-1} = 1$, and $h_{\alpha,\beta,\gamma'}$ is numerical—in fact, is defined by this equation [F.B.C. (1896), 987]. The expressions S_α obey all the laws of algebra, including the commutative law. And

$$\Sigma\lambda_\alpha S_\alpha \,.\, \Sigma\mu_\alpha S_\alpha = \Sigma\nu_\alpha S_\alpha,$$

whatever numerical values λ_α and μ_α may have. Moreover,

$$\Sigma\lambda_\alpha S_\alpha \,.\, \Sigma\mu_\alpha S_\alpha = \sum_\gamma \left[\sum_{\alpha,\beta} \frac{h_{\alpha\beta\gamma'}}{h_\gamma} \lambda_\alpha \mu_\beta \right] S_\gamma.$$

Let us solve the equation

$$[\Sigma\lambda_\alpha S_\alpha]^2 = \Sigma\lambda_\alpha S_\alpha.$$

This is done by the aid of the numbers $\chi_\alpha^{(\kappa)}$ called group characters. For consider $\Sigma\chi_{\alpha'} S_\alpha$ with the aid of the equations given by Frobenius*, quoted below :

$$[\Sigma\chi_{\alpha'} S_\alpha]^2 = \sum_\gamma \left[\sum_{\alpha,\beta} \chi_{\alpha'}\chi_{\beta'} \frac{h_{\alpha\beta\gamma'}}{h_\gamma} \right] S_\gamma$$

$$= \sum_\gamma \left[\sum_\beta \frac{h_\beta \chi_\beta \chi_{\gamma'}}{f} \chi_{\beta'} \right] S_\gamma \quad [\S\,1\,(5) \text{ and } \S\,2\,(6)]$$

$$= \sum_\gamma \frac{h\chi_{\gamma'}}{e} S_\gamma \quad [\S\,3\,(3)].$$

Let us write $$T_\kappa = \frac{e^{(\kappa)}}{h} \Sigma\chi_\alpha^{(\kappa)} S_\alpha,$$

then $$T_\kappa^2 = T_\kappa;$$

* F.B.C. (1896), 985 *et seq.*

and we obtain k possible solutions of the equation. Also

$$T_\kappa T_\lambda = \sum_\gamma \left[\sum_{\alpha,\beta} \frac{e^{(\kappa)} e^{(\lambda)}}{h^2} \chi_\alpha^{(\kappa)} \chi_{\beta'}^{(\lambda)} \frac{h_{\alpha\beta\gamma'}}{h_\gamma} \right] S_\gamma$$

$$= \sum_\gamma \left[\sum_\beta \frac{e^{(\kappa)} e^{(\lambda)}}{h^2 f^{(\kappa)}} \chi_{\beta'}^{(\lambda)} h_\beta \chi_\beta^{(\kappa)} \chi_{\gamma'}^{(\kappa)} \right] S_\gamma$$

$$= 0 \quad [\S\ 3\ (2)].$$

When H is the symmetric group these expressions T_κ are the same as those which, in my former papers, were written $A_{a_1, a_2, \ldots, a_h} T_{a_1, a_2, \ldots, a_h}$.

The expressions S_α can be written in terms of the T_κ's; thus using [§ 3, (4)],

$$S_\alpha = \sum_\kappa \frac{h_\alpha \chi_\alpha^{(\kappa)}}{f^{(\kappa)}} T_\kappa.$$

Hence

$$\Sigma x_\alpha S_\alpha = \sum_{\alpha,\kappa} \frac{h_\alpha \chi_\alpha^{(\kappa)} x_\alpha}{f^{(\kappa)}} T_\kappa$$

$$= \xi^{(\nu)} T_\kappa,$$

where $\xi^{(\kappa)}$ is the same as in [§ 2, 11].

Consider the specially simple form of equation

$$\Sigma x_\alpha S_\alpha X = 0,$$

which may be written

$$\Sigma \xi^{(\kappa)} T_\kappa X = 0.$$

A solution is sought of the form

$$\Sigma y_\alpha S_\alpha F = \Sigma \eta^{(\kappa)} T_\kappa F.$$

Then

$$(\Sigma \xi^{(\kappa)} T_\kappa)(\Sigma \eta^{(\kappa)} T_\kappa) F = \Sigma \xi^{(\kappa)} \eta^{(\kappa)} T_\kappa F,$$

and there is a solution other than zero when

$$\prod_\kappa \xi^{(\kappa)} = 0,$$

and only then.

In his paper *Über die Darstellung der endlichen Gruppen durch lineare Substitutionen* [B.S. (1897)], Frobenius treats functions of this kind of hypercomplex units in this way in § 6, and he further shows (p. 1010) that in general the multiplication of two such functions may be reduced to a matter of composition of matrices.

ON THE INTEGRAL $\int_a^b \frac{dF(x)}{x-t}$

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1. LEMMA. *If*

$$(a_1, b_1),\ (a_2, b_2),\ \ldots,\ (a_n, b_n),\ \ldots$$

is any set of intervals, not necessarily non-overlapping, with a finite total length, and if

$$\omega_1+\omega_2+\ldots+\omega_n+\ldots$$

is any convergent series of positive terms, then

$$\sum_{n=1}^{\infty} \frac{(b_n-a_n)\omega_n}{(t-a_n)(t-b_n)} \tag{1.1}$$

converges for almost all values of t.

Let q be an arbitrary positive number and write

$$\alpha_n = a_n-q(b_n-a_n), \quad \beta_n = b_n+q(b_n-a_n),$$

so that

$$\frac{a_n-b_n}{a_n-\alpha_n} = \frac{\beta_n-a_n}{\beta_n-b_n} = \frac{q+1}{q}. \tag{1.2}$$

Let S be the set of points covered by the intervals (a_n, b_n), and let S_q be the set covered by the intervals (α_n, β_n). Then

$$m(S_q-S) \leqslant 2q \sum_{n=1}^{\infty} (b_n-a_n) \to 0 \tag{1.3}$$

as $q \to 0$.

Write

$$\phi_n(t) = \frac{(b_n-a_n)\omega_n}{(t-a_n)(t-b_n)} \quad (t \leqslant a_n,\ t \geqslant \beta_n),$$

$$= 0 \qquad\qquad (a_n < t < \beta_n).$$

Then a simple calculation gives, in view of (1.2),

$$\int_{-\infty}^{\infty} \phi_n(t)\,dt = 2 \log\left(\frac{q+1}{q}\right)\omega_n,$$

so that, since series of positive functions are integrable term by term,

$$\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \phi_n(t)\,dt = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \phi_n(t)\,dt = 2\log\left(\frac{q+1}{q}\right) \sum_{n=1}^{\infty} \omega_n < \infty;$$

and this is impossible unless

$$\sum_{n-1}^{\infty} \phi_n(t) \tag{1.4}$$

converges almost everywhere.

Now, outside S_q, (1.4) and (1.1) are the same series, so that (1.1) converges almost everywhere outside S_q and therefore, in virtue of (1.3), almost everywhere outside S.

Since the removal of a finite number of the intervals (a_n, b_n), by means of which mS can be made arbitrarily small, does not affect the convergence of the series (1.1), it follows that (1.1) converges almost everywhere.

2. THEOREM I. *If $F(x)$ is a merely discontinuous increasing function, then*

$$\int_a^b \frac{dF(x)}{x-t} \tag{2.1}$$

exists as a principal value for almost all values of t.

It is evidently sufficient to consider only the values of t in $a < t < b$. Moreover, there is no loss of generality in assuming that $F(x)$ is continuous at a and b and constant in $x \leqslant a$ and also in $x \geqslant b$.

Take an arbitrary positive number η and enclose the discontinuities of $F(x)$, which are countable and so form a set of measure zero, in an open subset of (a, b) whose constituent intervals

$$(a_1, b_1),\ (a_2, b_2),\ \ldots,\ (a_n, b_n),\ \ldots$$

have a total length less than η.

Write $$\omega_n = F(b_n) - F(a_n).$$

Then $$\omega_1+\omega_2+\ldots+\omega_n+\ldots = F(b)-F(a) < \infty.$$

By a known theorem constants λ_n can be determined such that

$$\frac{\omega_n}{\lambda_n} \to 0, \quad \lambda = \lambda_1+\lambda_2+\ldots+\lambda_n+\ldots < \infty, \tag{2.2}$$

and there is no difficulty in determining them in such a way that λ is any arbitrarily small positive number given in advance.

Write $$\alpha_n = a_n - \lambda_n, \quad \beta_n = b_n + \lambda_n,$$

and denote by S_λ the set covered by the intervals (α_n, β_n). Evidently

$$mS_\lambda < \eta + 2\lambda. \tag{2.3}$$

Define now

$$g_n(x) = \frac{\omega_n}{b_n - a_n} \qquad (a_n \leqslant x \leqslant b_n),$$

$$= 0 \qquad (x < a_n,\ x > b_n)$$

$$G_n(x) = \int_{-\infty}^{x} g_n(t)\,dt,$$

$$G(x) = \sum_{n=1}^{\infty} G_n(x).$$

The series for $G(x)$ plainly converges with $\Sigma\,\omega_n$.

Let $F_n(x)$ be the merely discontinuous function which vanishes at a_n, which has the same discontinuities and oscillations in (a_n, b_n) as $F(x)$, and which is constant for $x \geqslant b_n$. Then

$$F(x) = F(a) + \sum_{n=1}^{\infty} F_n(x).$$

Take any point t of $a < t < b$ outside S_λ and let ϵ be any sufficiently small positive number. Then

$$\int_a^{t-\epsilon} \frac{d\{F(x) - G(x)\}}{x-t} = \sum_{n=1}^{\infty} \int_a^{t-\epsilon} \frac{d\{F_n(x) - G_n(x)\}}{x-t},$$

because $x - t \geqslant \epsilon$ in the range.

The series on the right is made up of contributions from whole intervals (a_n, b_n) in the range and, possibly, a contribution from part of the interval (a_n, b_n), if such exists, which encloses $t-\epsilon$.

Suppose first that $t-\epsilon \to t$ avoiding the intervals (a_n, b_n). Then there is no contribution from a partial interval to be considered. The contribution of a whole interval (a_n, b_n) in the range is

$$\int_{a_n}^{b_n} \frac{d\,F_n(x)}{x-t} - \int_{a_n}^{b_n} \frac{d\,G_n(x)}{x-t},$$

which is numerically not greater than

$$\omega_n \left| \frac{1}{a_n - t} - \frac{1}{b_n - t} \right| = \frac{\omega_n (b_n - a_n)}{(t-a_n)(t-b_n)}.$$

It follows from the lemma that the sum of such contributions is an absolutely convergent series for almost all values of t, so that, for almost all the values of t considered,

$$\lim \int_a^{t-\epsilon} \frac{d\{F(x) - G(x)\}}{x-t} \tag{2.4}$$

exists as $\epsilon \to 0$ in the special way.

Suppose now that $t-\epsilon$ lies in (a_n, b_n). Then there is the contribution of $(a_n, t-\epsilon)$ to be considered. Since the distance of t from this partial interval is at least λ_n, the contribution is numerically not greater than

$$\frac{2\omega_n}{\lambda_n} \to 0,$$

by (2.2). It follows that (2.4) exists as $\epsilon \to 0$ in any way.

In the same manner,

$$\lim \int_{t+\epsilon}^{b} \frac{d\{F(x)-G(x)\}}{x-t}$$

exists for almost all values of t outside S_λ, so that

$$\int_a^b \frac{d\{F(x)-G(x)\}}{x-t}$$

exists as a Cauchy integral for almost all values of t in $a < t < b$ outside S_λ.

Now $G(x)$ is continuous and so*

$$\int_a^b \frac{dG(x)}{x-t}$$

exists as a principal value integral for almost all values of t. Thus (2.1) must exist as a principal value integral for almost all t outside S_λ, which, in view of (2.3), gives what is required.

3. THEOREM II. *If $F(x)$ is of bounded variation in (a, b), then*

$$\int_a^b \frac{dF(x)}{x-t}$$

exists as a principal value integral for almost all values of t.

First write $$F(x) = F_2(x) - F_1(x),$$

where $F_2(x)$ and $F_1(x)$ are both increasing functions. Then resolve each of these into the sum of a continuous increasing function and a merely discontinuous increasing function. $F(x)$ is expressed as a linear combination of four functions for each of which the theorem has been proved. This gives what is required.

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* 3.

4. Suppose now that $F(x)$ is a continuous function, not restricted in any way. Then

$$\int_a^{t-\epsilon} \frac{dF(x)}{x-t}, \quad \int_{t+\epsilon}^b \frac{dF(x)}{x-t} \quad (a < t < b)$$

exist as ordinary Stieltjes integrals for all sufficiently small positive values of ϵ.

We shall say that τ is a *singularity* if there is no interval enclosing τ for almost all values of t in which (2.1) exists as a principal value. An application of Borel's covering theorem shows that, if (2.1) does not exist for almost all values of t, then there must be at least one singularity.

It is easily verified that the set of singularities, when it exists, is perfect and such that every portion* of it is of positive measure. In the complementary intervals (2.1) exists as a principal value almost everywhere.

5. THEOREM III. *If $f(x)$ is integrable in (a, b) in the special Denjoy sense, then*

$$\int_a^b \frac{f(x)dx}{x-t} \tag{5.1}$$

exists as a principal value for almost all values of t.

We shall show that the existence of the set of singularities leads to a contradiction.

Write $$F(x) = \int_a^x f(t)dt.$$

If possible, let the set of singularities exist and denote it by Π. Then, since $f(x)$ is integrable in the special Denjoy sense, Π contains a portion π in which $f(x)$ is absolutely integrable and in the complementary intervals† of which the oscillations of $F(x)$ form a convergent series.

Let the complementary intervals of π, arranged in some order, be

$$(a_1, b_1), \ (a_2, b_2), \ \ldots, \ (a_n, b_n), \ \ldots,$$

and let the corresponding oscillations of $F(x)$ be

$$\omega_1, \ \omega_2, \ \ldots, \ \omega_n, \ \ldots.$$

---

* The portion of a perfect set Π determined by an interval I is the common part of Π and the interior of I augmented by the addition of such end points of I as are limit points of the common part just obtained. Every portion of Π is evidently a perfect subset of Π.

† If π is connected, so that there are no complementary intervals, the contradiction is already reached, since $f(x)$ is absolutely integrable in π.

Carry out the procedure of § 2, forming a second set of intervals

$$(a_1, \beta_1),\ (a_2, \beta_2),\ \ldots,\ (a_n, \beta_n),\ \ldots$$

covering a set S_λ as before.

Write $$f(x) = g(x) + \phi(x)$$

where $g(x)$ coincides with $f(x)$ in π and vanishes elsewhere, and $\phi(x)$ coincides with $f(x)$ in the intervals (a_n, b_n) and vanishes elsewhere.

The integral

$$\int_a^b \frac{g(x)\,dx}{x-t} \tag{5.2}$$

has already been discussed*. Since $g(x)$ is absolutely integrable, (5.2) exists as a principal value for almost all values of t.

Since $\phi(x) = f(x) - g(x)$, the integral

$$\int_a^b \frac{\phi(x)\,dx}{x-t} \tag{5.3}$$

exists as a principal value for almost all values of t in the intervals (a_n, b_n) and it remains only to discuss it in relation to π.

Write $$v_n = \int_{a_n}^{b_n} \phi(x)\,dx$$

and define $$\psi(x) = \frac{v_n}{b_n - a_n} \quad (a_n \leqslant x \leqslant b_n),$$

$$= 0 \qquad \text{outside the intervals } (a_n, b_n).$$

Consider now

$$\int_a^{t-\epsilon} \frac{\phi(x) - \psi(x)}{x-t}\,dx, \tag{5.4}$$

where t lies in the interval $a < t < b$ and outside S_λ. Exactly as in § 2 we can show that this tends to a finite limit as $\epsilon \to 0$.

For, since both $\phi(x)$ and $\psi(x)$ vanish outside the intervals (a_n, b_n), (5.4) is made up of contributions from such intervals†, together with, possibly, a contribution from a partial interval.

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* 2.

† The absolute convergence of the series of such contributions is ensured by the convergence of $\Sigma\omega_n$.

By the second mean value theorem, the contribution of (a_n, b_n) can be written in the form

$$\frac{1}{a_n-t}\int_{a_n}^{c_n} + \frac{1}{b_n-t}\int_{c_n}^{b_n} \{\phi(x)-\psi(x)\}\,dx \qquad (a_n \leqslant c_n \leqslant b_n)$$

$$= \left(\frac{1}{a_n-t} - \frac{1}{b_n-t}\right)\int_{a_n}^{c_n} \{\phi(x)-\psi(x)\}\,dx,$$

since $\phi(x)$ and $\psi(x)$ have the same integral in (a, b). The numerical value of this last expression is not greater than

$$\frac{b_n-a_n}{(a_n-t)(b_n-t)}\,(\omega_n+|v_n|) \leqslant \frac{2(b_n-a_n)\omega_n}{(a_n-t)(b_n-t)},$$

and the lemma now shows that, for almost all the values of t considered, (5 . 4) tends to a limit as $\epsilon \to 0$ in the special way.

By the second mean value theorem, the contribution of a partial interval $(a_n, t-\epsilon)$ can be written in the form

$$\frac{1}{\epsilon}\int_{\gamma_n}^{t-\epsilon} \{\phi(x)-\psi(x)\}\,dx \qquad (a_n \leqslant \gamma_n \leqslant t-\epsilon),$$

and this is not greater in absolute value than

$$\frac{\omega_n+|v_n|}{t-b_n} \leqslant \frac{2\omega_n}{\lambda_n} \to 0$$

as before.

Exactly as in § 2 we see that

$$\int_a^b \frac{\phi(x)-\psi(x)}{x-t}\,dx$$

exists as a Cauchy integral for almost all values of t in $a < t < b$ outside S_λ.

Now $$\int_{-\infty}^{\infty} |\psi(x)|\,dx = \sum_{n=1}^{\infty} |v_n| \leqslant \sum_{n=1}^{\infty} \omega_n < \infty,$$

so that, by a case already discussed,

$$\int_a^b \frac{\psi(x)\,dx}{x-t}$$

exists as a principal value for almost all values of t. It follows that the same is true of (5 . 3) outside S_λ.

By making $\lambda \to 0$ we see that (5 . 3), and therefore (5 . 1), exists as a principal value for almost all values of t in π. But (5 . 1) already exists as a principal value for almost all values of t in the complementary

intervals of π. Therefore (5.1) exists as a principal value for almost all values of t in the interval between the extreme points of π. This contradicts the fundamental property of π. It follows that Π cannot exist and the theorem is proved.

6. THEOREM IV. *If $F(x)$ is strictly resoluble in (a, b), then* (2.1) *exists as a principal value for almost all values of t.*

Since a strictly resoluble function is the indefinite integral of a function which is integrable in the special Denjoy sense, this reduces to the theorem just proved.

7. THEOREM V. *If $f(x)$ is either of bounded variation or strictly resoluble in (a, b), then*

$$\int_a^b \frac{f(x)-f(t)}{(x-t)^2}\,dx$$

exists as a principal value for almost all values of t, and it is subject to the formula for integration by parts.

The proof given in a previous paper* can be adapted, the only new fact required being Denjoy's result to the effect that a function which has a special integral is almost everywhere the differential coefficient of any one of its indefinite integrals.

8. THEOREM VI. *If $f^{(n)}(x)$ is either of bounded variation or strictly resoluble in (a, b), then*

$$\int_a^b \frac{f(x)-f(t)-(x-t)f'(t)-\ldots-(1/n!)(x-t)^n f^{(n)}(t)}{(x-t)^{n+2}}\,dx$$

exists as a principal value for almost all values of t.

The remark just made applies also to this theorem.

An interesting special condition is the existence everywhere in (a, b) of a finite $(n+1)$-th order differential coefficient $f^{(n+1)}(x)$. Under this condition it is known that $f^{(n)}(x)$ is strictly resoluble.

References.

1. I. I. Privaloff, *Intégrale de Cauchy* (1919).
2. A. S. Besicovitch, *Journal London Math. Soc.*, 1 (1926), 120–128.
3. S. Poliard, *Journal London Math. Soc.*, 2 (1927), 37–41.

* 3.

NOTES ON THE THEORY OF SERIES (VII): ON YOUNG'S CONVERGENCE CRITERION FOR FOURIER SERIES

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1. The Fourier series of an integrable function $f(t)$ converges to s, for $t = x$, if and only if

$$J(n) = \frac{1}{\pi}\int_0^\pi \phi(t)\,\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt,$$

where

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\},$$

tends to zero when $n \to \infty$.

One of the most interesting of the more modern criteria for convergence is that of Young: *it is sufficient that*

$$\phi(t) \to 0 \tag{1.1}$$

and

$$\chi(t) = \int_0^t |\,d(u\phi)\,| = O(t) \tag{1.2}$$

for small t [*i.e.* that the variation of $u\phi(u)$ in $(0, t)$ is $O(t)$ for small t]. The theorem was proved in this form by Young in the *Proceedings* in 1916†; in an earlier note in the *Comptes rendus* he had stated the criterion a little less generally‡. The condition (1.2), which we shall refer to as Y, may be stated in alternative forms, such as (*a*) that the variation of $t^c\phi$ is $O(t^c)$, for some (or any) positive c, or (*b*) that the variation of ϕ in $(t, 2t)$ is $O(1)$; and it plainly includes (and is more general than) the ordinary (Jordan's) condition that ϕ is of bounded variation in some interval $(0, \delta)$.

† Young, **18**.
‡ Young, **15**.

It is natural to ask whether the condition (1 . 1) cannot be generalized. If we write

$$\phi_1(t) = \int_0^t \phi(u)\,du, \qquad \phi_2(t) = \int_0^t \phi_1(u)\,du, \qquad \ldots, \tag{1 . 3}$$

we may say that

$$\phi(t) \to 0 \quad (C, r) \tag{1 . 4}$$

if

$$\phi_r(t) = o(t^r). \tag{1 . 5}$$

It is stated by Young, in a later note in the *Comptes rendus*†, that (1 . 1) may be replaced by (1 . 4); but apparently no proof of this theorem, in its general form, has been published. The case $r = 1$ has, however, been dealt with recently by Pollard‡, who gives a concise direct proof, and also shows that, in this form, Young's criterion may be deduced from a generalization of the well known criterion of Lebesgue§.

There is another direction in which the criterion may be developed. We may strengthen Y by demanding that (1 . 2) shall hold, not merely for small t, but for all t of $(0, \pi)$: we shall call the more stringent condition Y^*. It is plain that ϕ satisfies Y^* if, and only if (1) it satisfies Y, and (2) it is of bounded variation in any interval $0 < \delta \leqslant t \leqslant \pi$. It was observed by Young‖ that Y^*, together with (1 . 1), implies more than convergence, and in fact summability $(C, -1+\delta)$ for every positive δ, though he gives only a very summary indication of the proof.

All these writings have points of contact with our own recent researches. Our first object here is to prove a theorem (Theorem 1) which includes comprehensively all the results to which we have referred. We also prove (Theorem 3) what may be described as the "transform" of Young's theorem in its most general form, and some other theorems which dispose of points of incidental interest.

2. THEOREM 1. *If $\phi(t)$ satisfies Y, then the Fourier series is convergent whenever it is summable by any Cesàro mean. The necessary and sufficient condition for convergence (to sum s) is*

$$\phi_1(t) = o(t). \tag{2 . 1}$$

† Young, **17**.

‡ Pollard, **13**.

§ It had already been shown by Hardy (**3**) that Young's criterion, in its original form, can be deduced from Lebesgue's.

‖ Young, **18**, 211.

If, further, $\phi(t)$ satisfies Y^*, then the series is summable $(C, -1+\delta)$, for every positive δ, whenever it is summable by any Cesàro mean.

If ϕ satisfies Y, so that (1 . 2) is true (say) in $(0, \xi)$, f^* is equal to f in $(x-\xi, x+\xi)$ and to 0 outside, and ϕ^* is the ϕ of f^*, then ϕ^* satisfies Y^*. The Fourier series of $f-f^*$ converges to 0 for $t=x$. It is therefore sufficient to consider a function subject to Y^*, and to prove that the series is then summable $(C, -1+\delta)$ if summable at all, and that (2 . 1) is the necessary and sufficient condition for summability.

We use four known theorems which we state as lemmas†.

LEMMA 1. *If a series is summable (C) and bounded $(C, k+\delta)$, where $k > -1$, for every positive δ, then it is summable $(C, k+\delta)$ for every positive δ.*

LEMMA 2. *If $\phi(t)$ is bounded and $\phi(t) \to 0\ (C)$‡, then $\phi(t) \to 0\ (C, 1)$.*

LEMMA 3. *The necessary and sufficient condition that the Fourier series should be summable (C) is that $\phi(t) \to 0\ (C)$.*

LEMMA 4. *If $-1 < \gamma < 1$, $\phi(t) \sim \frac{1}{2}a_0 + \Sigma a_n \cos nt$, and $s_n^{(\gamma)}$ is the (C, γ) mean of Σa_n, then*

$$s_n^{(\gamma)} - \frac{1}{\pi}\int_0^\pi \phi(t)\,\Omega(t)\,dt, \tag{2 . 2}$$

where

$$\Omega(t) = \Omega_1(t) + \Omega_2(t), \tag{2 . 3}$$

$$\Omega_1(t) = 2^{-1-\gamma}\,\frac{\Gamma(n+1)\,\Gamma(1+\gamma)}{\Gamma(n+1+\gamma)}\,\frac{\sin\{(n+\frac{1}{2}+\frac{1}{2}\gamma)\,t - \frac{1}{2}\gamma\pi\}}{(\sin\frac{1}{2}t)^{1+\gamma}}, \tag{2 . 4}$$

$$|\,\Omega(t)\,| \leqslant An, \qquad |\,\Omega_2(t)\,| \leqslant \frac{A}{nt^2}, \tag{2 . 5}$$

and the A's are independent of n and t§.

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† Lemmas 1 and 2 are propositions of a well-known type. Lemma 1 is proved, in the form required here, by Andersen, **1**, 56. For Lemma 2, see Hardy and Littlewood, **8**, 83. The result is proved there for means of $\phi(t)$ of "Hölder" type, but the equivalence of these to those of Cesàro type is familiar. See Landau, **11** : in any case the proof is easily adapted.

Lemma 3 is the main result of our memoir **8**. It is not actually necessary here to use the full force of this theorem; we only need Lebesgue's theorem that $\phi_1 = o(t)$ involves summability $(C, 2)$, and Riemann's theorem that convergence implies $\phi_2 = o(t^2)$.

Finally Lemma 4, which we quote from Zygmund, **19**, is due substantially to Kogbetliantz, **10**. The most natural proof is that of Kogbetliantz, by Cauchy's theorem; but a proof by real analysis is given by Szegö, **14**.

‡ That is to say, $\phi \to 0\ (C, r)$ for some r.

§ It should be observed that we must not replace $\sin\frac{1}{2}t$ by $\frac{1}{2}t$ in the denominator of $\Omega_1(t)$.

3. We prove first that $s_n^{(\gamma)}$ is bounded for any $\gamma \geq -1$. We write

$$(3.1) \qquad s_n^{(\gamma)} = \frac{1}{\pi}\int_0^{1/n} \phi\Omega\, dt + \frac{1}{\pi}\int_{1/n}^{\pi} \phi\Omega_1\, dt + \frac{1}{\pi}\int_{1/n}^{\pi} \phi\Omega_2\, dt = J_1 + J_2 + J_3.$$

Since ϕ satisfies Y^* it is bounded, and so, by (2.5),

$$(3.2) \qquad J_1 = O\Big(n\int_0^{1/n} dt\Big) = O(1), \qquad J_3 = O\Big(\frac{1}{n}\int_{1/n}^{\pi} \frac{dt}{t^2}\Big) = O(1).$$

Hence we have only to prove that J_2 is bounded.

We observe first that Y^* is equivalent to

$$\int_0^t \Big| d\Big(\sin \tfrac{1}{2}u\, \phi(u)\Big)\Big| = O(t).$$

If now we write

$$\Lambda = \Lambda(n, t) = \int_t^{\pi} \frac{\sin\{(n+\frac{1}{2}+\frac{1}{2}\gamma)u - \frac{1}{2}\gamma\pi\}}{(\sin \frac{1}{2}u)^{2+\gamma}}\, du,$$

we have

$$\Lambda = O(n^{-1} t^{-2-\gamma}),$$

by the second mean-value theorem. Hence

$$(3.3) \qquad J_2 = O\Big(n^{-\gamma}\Big|\int_{1/n}^{\pi} \sin \tfrac{1}{2}t\, \phi(t)\, d\Lambda\Big|\Big) = O(n^{-\gamma}|K|),$$

say, where

$$K = [\sin \tfrac{1}{2}t\, \phi(t)\, \Lambda]_{1/n}^{\pi} + \int_{1/n}^{\pi} \Lambda\, d(\sin \tfrac{1}{2}t\, \phi) = K' + K''.$$

Here

$$(3.4) \qquad K' = O(n^{-1}) + O(n^{-1} \cdot n^{-1} n^{2+\gamma}) = O(n^{\gamma}).$$

Also, if we write

$$\sin \tfrac{1}{2}t\, \phi = \psi, \quad \Psi = \int_0^t |\, d\psi\,| = O(t),$$

we have

$$(3.5) \qquad |K''| \leqslant \int_{1/n}^{\pi} |\Lambda|\, |d\psi| \leqslant \frac{A}{n}\int_{1/n}^{\pi} \frac{d\Psi}{t^{2+\gamma}} = \frac{A}{n}\Big[\frac{\Psi}{t^{2+\gamma}}\Big]_{1/n}^{\pi} + \frac{A}{n}\int_{1/n}^{\pi} \frac{\Psi\, dt}{t^{3+\gamma}}$$

$$\leqslant \frac{A}{n}\frac{\Psi(\pi)}{\pi^{2+\gamma}} + O\Big(\frac{1}{n}\int_{1/n}^{\pi} \frac{dt}{t^{2+\gamma}}\Big) = O(n^{\gamma}).$$

From (3.3), (3.4), and (3.5) it follows that J_2, and so $s_n^{(\gamma)}$, is bounded.

The proof of Theorem 1 is now immediate. The series is bounded (C, γ) for all $\gamma \geq -1$, and therefore, by Lemma 1, summable (C, γ) if summable at all. The necessary and sufficient condition for summability (C, γ) is therefore the same as for summability (C). This condition is, by Lemma 3, that $\phi \to 0$ (C). But ϕ is bounded, and therefore, by Lemma 2, it is necessary and sufficient that (2.1) be true.

4. It is of some interest to observe that (2.1) is a consequence of the summability of the series by some negative mean, whether Young's condition be satisfied or not. It is known that it is not a consequence of mere convergence†.

THEOREM 2. *If* Σa_n *is summable* $(C, -k)$, *where* $0 < k < 1$, *to sum* s, *then*

$$S(t) = \Sigma a_n \frac{\sin nt}{nt} \to s$$

when $t \to 0$.

This is naturally equivalent to (2.1) when $\phi(t)$ and a_n are defined as in Theorem 1. We suppose that $s = 0$, $a_0 = 0$, and write

$$c_n = s_n^{(-k)} = o(n^{-k}),$$

$$\gamma_n = \frac{\Gamma(n+k)}{\Gamma(k)\,\Gamma(n+1)} = O(n^{k-1}),$$

$$\Delta u_n = u_n - u_{n+1}, \qquad \phi_n(t) = \Delta \frac{\sin nt}{nt}.$$

Then‡
$$s_n = a_1 + a_2 + \ldots + a_n = \sum_{\nu=1}^{n} \gamma_{n-\nu} c_\nu$$

and

$$S(t) = \sum_{n=1}^{\infty} s_n \phi_n(t) = \sum_{n=1}^{\infty} \phi_n(t) \sum_{\nu=1}^{n} \gamma_{n-\nu} c_\nu \tag{4.1}$$

$$= \sum_{\nu=1}^{\infty} c_\nu \sum_{n=\nu}^{\infty} \gamma_{n-\nu} \phi_n(t) = \sum_{\nu=1}^{\infty} c_\nu T_\nu,$$

say, if we may invert the order of summation. This inversion is not entirely trivial, since the double series is not absolutely convergent. We have, however,

$$S(t) = \lim_{N\to\infty} \sum_{n=1}^{N} \sum_{\nu=1}^{n} = \lim_{N\to\infty} \sum_{\nu=1}^{N} \sum_{n=\nu}^{N}.$$

Now, for a fixed $t > 0$, we have

$$\sum_{n=N+1}^{\infty} \gamma_{n-\nu} \phi_n(t) = O\left\{(N-\nu+1)^{k-1} \operatorname{Max}_{m>N} \left| \sum_{N+1}^{m} \phi_n(t) \right| \right\}$$

$$= O\left\{\frac{(N-\nu+1)^{k-1}}{N}\right\}$$

† See Hardy and Littlewood, **4**, 224–227, and **5**, 255–261.

‡ Andersen, **1**, 6.

and

$$\frac{1}{N}\sum_{\nu=1}^{N}|c_\nu|(N-\nu+1)^{k-1}=O\left\{\frac{1}{N}\sum_{\nu=1}^{N}\nu^{-k}(N-\nu+1)^{k-1}\right\}=O\left(\frac{1}{N}\right).$$

Hence
$$\sum_{\nu=1}^{N}\sum_{n=N+1}^{\infty}$$
exists and tends to zero when $N\to\infty$, so that

$$S(t)=\lim_{N\to\infty}\sum_{\nu=1}^{N}\sum_{n=\nu}^{\infty}=\sum_{\nu=1}^{\infty}\sum_{n=\nu}^{\infty}.$$

We have thus proved (4 . 1).

We shall now prove that

(4 . 2) $$T_\nu=O(t^{1-k})\ \ (\nu t\leqslant 1),\quad T_\nu=O(\nu^{-1}t^{-k})\ \ (\nu t>1),$$

uniformly in ν and t. If this be granted, the theorem follows, since

$$S(t)=O(t^{1-k})\sum_{\nu\leqslant 1/t}o(\nu^{-k})+O(t^{-k})\sum_{\nu>1/t}o(\nu^{-k-1})=o(1).$$

To prove (4 . 2) we write

$$\rho=\left[\frac{1}{t}\right],\quad T_\nu=\sum_{\nu}^{\nu+\rho}+\sum_{\nu+\rho+1}^{\infty}=U_\nu+V_\nu.$$

Then, first,

$$V_\nu=\sum_{\nu+\rho+1}^{\infty}\gamma_{n-\nu}\phi_n(t)=\gamma_{\rho+1}\frac{\sin(\nu+\rho+1)t}{(\nu+\rho+1)t}-\sum_{\nu+\rho+1}^{\infty}\frac{\sin(n+1)t}{(n+1)t}\Delta\gamma_{n-\nu}$$

$$=O\left\{\frac{1}{1+\nu t}\left[\rho^{k-1}+\sum_{\nu+\rho+1}^{\infty}(n-\nu+1)^{k-2}\right]\right\}=O\left(\frac{\rho^{k-1}}{1+\nu t}\right)=O\left(\frac{t^{1-k}}{1+\nu t}\right),$$

and so satisfies inequalities like those in (4 . 2). Secondly, in U_ν, we use the inequalities

$$\phi_n(t)=O(n^{-1})\ \ (nt\geqslant 1),\quad \phi_n(t)=O(nt^2)\ \ (nt\leqslant 2),$$

according as $\nu t>1$ or $\nu t\leqslant 1$. If $\nu t\geqslant 1$, then

$$U_\nu=\sum_{\nu}^{\nu+\rho}\gamma_{n-\nu}\phi_n(t)=O\left\{\frac{1}{\nu}\sum_{\nu}^{\nu+\rho}(n-\nu+1)^{k-1}\right\}=O(\nu^{-1}\rho^k)=O(\nu^{-1}t^{-k}).$$

If $\nu t\leqslant 1$, then $(\nu+\rho)t\leqslant 2$ and

$$U_\nu=O\left\{t^2\sum_{\nu}^{\nu+\rho}n(n-\nu+1)^{k-1}\right\}=O\left\{t^2\left(\nu+\frac{1}{t}\right)\rho^k\right\}=O(t^{1-k}).$$

Thus U_ν satisfies (4 . 2), and this completes the proof.

5. We now apply to Young's theorem the heuristic process of "transformation" explained in § 4 of our fourth note†. The theorem to which we are ultimately led is one concerning the summability of the allied series of a Fourier series, viz.

THEOREM 3. *Suppose that the Fourier constants of $f(t)$ satisfy*

$$\sum_1^n \nu(|a_\nu|+|b_\nu|) = O(n). \tag{5.1}$$

Then the series $\Sigma B_n = \Sigma(b_n \cos nx - a_n \sin nx)$, the allied series of the Fourier series of $f(t)$ for $t = x$, is summable (C, δ) for every positive δ or not summable (C), and the necessary and sufficient condition for summability is that

$$\int_0^\pi \frac{f(x+t)-f(x-t)}{t}\,dt \tag{5.2}$$

should exist as a Cauchy integral.

It is sufficient to prove this in the "reduced"‡ form

THEOREM 3A. *If $\psi(t)$ is an odd function whose Fourier series is $\Sigma a_n \sin nt$, and*

$$\sum_1^n \nu|a_\nu| = O(n), \tag{5.3}$$

then Σa_n is summable (C, δ) for every positive δ or not summable (C); and the necessary and sufficient condition for summability is the existence of

$$\int_0^\pi \frac{f(t)}{t}\,dt \tag{5.4}$$

as a Cauchy integral.

We again require certain known theorems, which we state as lemmas.

LEMMA 5. *If Σa_n is summable (C), and satisfies (5.3), then it is bounded, and summable (C, δ)§.*

Let $t_n^{(r)}$ be the r-th Cesàro sum formed from $\Sigma n a_n$. Then (5.3) involves *a fortiori*

$$t_n^{(0)} = \sum_1^n \nu a_\nu = O(n),$$

† Hardy and Littlewood, **7**.

‡ Cf. Hardy and Littlewood, **9**.

§ But not necessarily convergent, as may be shown by an example due to Neder, **12**, 180. On the other hand (as may be seen from the proof which follows) the hypothesis (5.3) of Lemma 5 may be replaced by

$$\sum_1^n \nu a_\nu = O(n).$$

and so $t_n^{(r)} = O(n^{r+1})$ for $r \geqslant 0$. But†

$$\frac{t_n^r}{n^{r+1}} = \frac{n+r+1}{n}\,\frac{s_n^r}{n^r} - (r+1)\,\frac{s_n^{r+1}}{n^{r+1}}.$$

Hence Σa_n, if bounded $(C, r+1)$, is bounded (C, r). It is therefore bounded $(C, 0)$, and therefore, by Lemma 1, summable (C, δ).

LEMMA 6. *The necessary and sufficient condition that Σa_n should be summable (C) is that the integral (5 . 4) should be summable (C)*‡.

LEMMA 7. *If*

$$\int_0^t f(u)\,du = o(t), \tag{5 . 5}$$

then the integral (5 . 4), if summable, is convergent.

We prove first that, if (5 . 4) is summable $(C, 1)$, it is convergent. We write

$$\int_t^\pi \frac{f}{u}\,du = \chi(t).$$

Then, if δ is any number between t and π, we have

$$t\chi = t\int_t^\delta \frac{f}{u}\,du + t\int_\delta^\pi \frac{f}{u}\,du = \int_t^\tau f\,du + t\int_\delta^\pi \frac{f}{u}\,du,$$

where $t \leqslant \tau \leqslant \delta$, and we can make this as small as we please by choice first of δ and then of t. Hence $t\chi \to 0$. It then follows by partial integration that

$$\frac{1}{t}\int_0^t \chi\,du = \frac{1}{t}\,[u\chi]_0^t + \frac{1}{t}\int_0^t f\,du = \chi + \frac{1}{t}\int_0^t f\,du$$

or

$$\psi = \frac{1}{t}\int_0^t f\,du = \frac{1}{t}\int_0^t \chi\,du - \chi. \tag{5 . 6}$$

Since $\psi \to 0$, by (5 . 5), χ tends to a limit in the ordinary sense if it tends to a limit $(C, 1)$.

It is plain, moreover, that (5 . 6) contains a proof of the lemma in its general form. For from (5 . 6) it follows that

$$\frac{1}{t}\int_0^t \psi\,du = \frac{1}{t}\int_0^t \frac{du}{u}\int_0^u \chi\,dw - \frac{1}{t}\int_0^t \chi\,du,$$

so that χ, if it tends to a limit $(C, 2)$, leads to one $(C, 1)$; and so generally.

† Hardy, 2, 304.

‡ This is substantially the principal theorem of our paper 6. See p. 214 for the definition of summability of the integral.

6. Passing to the proof of the theorem, we observe first that the *sufficiency* of the condition results immediately from Lemmas 5 and 6, since Σa_n is summable by Lemma 6 and summable (C, δ) by Lemma 5. We have thus only to establish its *necessity*. If Σa_n is summable, the integral is summable, by Lemma 6, and its convergence will follow from Lemma 7 if we can prove (5 . 5).

Now (5 . 3) and the summability (C) of Σa_n imply (by Lemma 5) the boundedness of s_n and the summability of the series $(C, 1)$; and

$$\int_0^t f\,du = \Sigma a_n \frac{1-\cos nt}{n}.$$

Hence everything is reduced to the proof of the following lemma.

LEMMA 8. *If* $s_n = O(1)$ *and* $s_1+s_2+\ldots+s_n = o(n)$, *then*

$$S(t) = \Sigma a_n \frac{1-\cos nt}{nt} \to 0.$$

7. We choose positive numbers h (small) and H (large). O's are uniform in t, h, H. We write

$$S = \sum_{n \leqq H/t} + \sum_{n > H/t}$$

$$= \sum_{n \leqq h/t} s_n \Delta \frac{1-\cos nt}{nt} + \sum_{h/t < n < \nu} s_n \Delta \frac{1-\cos nt}{nt}$$

$$+ s_\nu \frac{1-\cos \nu t}{\nu t} + \sum_{n > H/t} a_n \frac{1-\cos nt}{nt} = S_1+S_2+S_3+S_4,$$

say, where $\nu = [H/t]$. Since (5 . 3) involves

$$\sum_n^\infty \frac{|a_\nu|}{\nu} = O\left(\frac{1}{n}\right)\dagger,$$

we have $$S_4 = O\left(\sum_{n > H/t} \frac{|a_n|}{nt}\right) = O\left(\frac{1}{t}\,\frac{t}{H}\right) = O\left(\frac{1}{H}\right);$$

and $$S_3 = O\left(\frac{1}{H}\right)$$

because s_n is bounded. In S_1

$$\Delta \frac{1-\cos nt}{nt} = O(t),$$

and so $$S_1 = O\left(t \sum_{n \leqq h/t} 1\right) = O\left(t \cdot \frac{h}{t}\right) = O(h).$$

† See Hardy and Littlewood, **9** (Lemma α.)

We can therefore choose h and H so that $S_1+S_3+S_4$ is as small as we please independently of t, and we have only to show that S_2 tends to zero with t when h and H are fixed.

Now

$$\Delta \frac{1-\cos nt}{nt} = \frac{1}{n+1}\frac{1-\cos nt}{nt} - \frac{n}{n+1}(1-\cos t)\frac{\cos nt}{nt} - \frac{n}{n+1}\sin t \frac{\sin nt}{nt}.$$

This equation decomposes S_2 into three parts, of which the last is

$$-\sin t \sum_{h/t<n<v} \frac{n}{n+1} s_n \frac{\sin nt}{nt} = O\left(t \operatorname{Max}\left|\sum_l^m s_n \frac{\sin nt}{nt}\right|\right),$$

where l and m vary in the range $(h/t, H/t)$. We can split up the range of summation (l, m) into $O(H)$ pieces in each of which the coefficient of s_n is monotonic. Applying the second mean value theorem to each of them, we obtain

$$O\left(Ht \operatorname{Max}\left|\sum_l^m s_n\right|\right) = o\left(Ht \cdot \frac{1}{t}\right) = o(1).$$

The same argument is applicable with trivial variations to the other sums. Hence $S_2 = o(1)$, which proves Lemma 8, and so Theorem 3A.

Theorems 3 and 3A should be compared with Theorems 2 and 2A of our note **9**. There more is assumed, and the conclusion (the convergence of the allied series) is correspondingly stronger.

8. We conclude with two remarks of a negative character. It is easy to prove that the Fourier constants of a function subject to Y^* are $O(\log n/n)$. It is natural to ask whether they are not, like those of a function of bounded variation, $O(1/n)$, since in this event Theorem 1 would become much more trivial. A *Gegenbeispiel* is given by the function $f(t)$ which is equal to 1 in the intervals $(\pm 10^{-2k-1}\alpha, \pm 10^{-2k}\alpha)$, where $k = 0, 1, 2, \ldots$ and α is chosen appropriately, and otherwise is equal to 0. The Fourier constant a_n of this function is a multiple of

$$\frac{1}{n}\left(\sin n\alpha - \sin\frac{n\alpha}{10} + \sin\frac{n\alpha}{10^2} - \ldots\right),$$

and it is easily proved that, for appropriate α, the series in brackets is not bounded in n.

It is also natural to ask whether the condition (2 . 1) cannot be replaced by the *continuity of* $\phi(t)$. The answer is easily seen to be negative. An affirmative answer would imply that Y^* and (2 . 1) necessarily involve the continuity of $\phi(t)$.

But suppose, for example, that δ_ν is the interval

$$(2^{-\nu}-2^{-2\nu},\ 2^{-\nu}) \quad (\nu = 1, 2, \ldots);$$

that $f(x)$ is 0 outside the intervals δ_ν, and that in each interval it rises symmetrically and linearly from 0 at the ends to 1 in the middle. Consider now the point $x = 0$, and suppose that $s = 0$. Then for $t > 0$ we have $\phi(t) = \frac{1}{2}f(t)$. The variation of $\phi(t)$ in $(2^{-\nu-1}, 2^{-\nu})$ is 1, and its variation in any interval $(t, 2t)$ is less than 2. Hence $\phi(t)$ satisfies Y^*. Also

$$\phi_1(t) \leqslant \sum_{2^{-\nu} \leqslant 2t} \tfrac{1}{2} \cdot 2^{-2\nu} = O(t^2) = o(t),$$

so that $\phi(t)$ satisfies (2 . 1). But $\phi(t)$ is plainly not continuous at the origin. This *Gegenbeispiel* is adapted, by the transformation process, from Neder's, quoted above†.

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† See p. 307, f.n. §.

SOME CONFIGURATIONS BASED ON FIVE GENERAL PLANES IN SPACE OF TEN DIMENSIONS (AND THE DOUBLE-TEN OF PLANES AND LINES IN SPACE OF FOUR DIMENSIONS)

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It has long been known that there is in [4]* a double-ten of planes and lines, analogous to the double-six of lines in ordinary space†. The original object of this paper was to find a construction of this configuration from given planes and lines, either in the [4] itself, or using elements in a space of higher dimension containing the [4]. The direct construction in the [4] is hampered by the fact that in the double-ten, only four of the five lines transversal to any six of the planes are used, a difficulty which does not arise when dealing with the double-six. The use of space of higher dimensions is complicated on account of the different dimensions of the spaces comprising the double-ten. Corresponding to the double-four of lines in [4] from which the double-six of lines in [3] may be derived‡, there is a double-five of lines and planes in

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* *Notation*:

"Space" = flat manifold,

$[n]$ = space of n dimensions—plural "$[n]$'s".

If A and B are the symbols for any two spaces, then

$[A, B]$ = space of lowest dimension containing A and B,

(A, B) = space of intersection of A and B,

$[(A, B), C]$ = space of lowest dimension containing C and the intersection of A and B, &c.

"Double r of $[p]$'s and $[q]$'s in $[n]$" = the figure in $[n]$ consisting of r $[p]$'s and r $[q]$'s, $p < n$, and $q < n$, in which, if $p+q < n$, each $[p]$ meets in points all but one of the $[q]$'s, and each $[q]$ meets in points all but one of the $[p]$'s, or if $p+q \geqslant n$, the spaces have the property dual to this, viz. each $[p]$ lies with all but one of the $[q]$'s in $[n-1]$'s, and therefore meets them in $[p+q-n+1]$'s, &c.

† The references are not definite, but the existence of the configuration is clearly implied in Bordiga, *Atti R. Accad. Linc.* (Mem.), 4, 4a (1887), 182; White, *Proc. Camb. Phil. Soc.*, 21 (1922), 220. The existence of configurations of which the double-ten is a special case was established in the *Proc. Royal Soc.* (A), 111 (1926), 386.

‡ Baker, *Proc. Camb. Phil. Soc.*, 20 (1920), 133, also *Principles of geometry*, 4 (1925), 116.

[6], whose projection on to a [4] gives the double-ten. But this double figure in [6] is not so simple as that in [4], because, whereas the five lines in it are general, the five planes are considerably specialized. A more symmetrical figure of which it is the projection has to be sought, and this is found in the double-five of planes in [10]. By now the double-ten of lines and planes in [4] is relatively unimportant, the properties of the figures in [10] and [6] required for its construction being few, and usually occurring quite incidentally in the development of the theory. And, anyway, there is a double-twenty of planes in [5], easily derivable from the figure in [10], which has as much claim to detailed consideration as the double-ten.

There are, meeting the five planes in [10], ∞^3 [3]'s, which form a variety analogous to the dual of Segre's cubic locus in [4], and it is to finding properties of this variety, and of its projection from one of the [3]'s, and of the duals of these, that the paper is mainly devoted. From certain aspects it might have been easier to start with the variety of order twenty in [15], whose equations are

$$x_{\alpha\beta} = R_\alpha S_\beta \quad (\alpha, \beta = 1, 2, 3, 4)$$

(sixteen), in which the $x_{\alpha\beta}$ are homogeneous coordinates in the [15], and the R_α and S_β are parameters, as this kind of figure is fairly well known, at least in a general way*. The projection of this locus from five of its points is the figure derived from five planes in [10], and some of the properties of the latter, as, for instance, its order, are obtained in this way in the course of the paper. But the symmetry of the figure in [10], its direct analogy with the Segre locus in [4], and the number of its properties, which bear no simple relation to properties of the locus in [15], justify its being treated from the beginning independently.

I. *The double-five of planes in* [10].

1. *The symbols of the twenty points of incidence.* Any four planes in [10] are met by a unique plane. For suppose that K_1, K_2, K_3, K_4 are four planes; if the plane H is to meet K_1, K_2, K_3 it must lie in the [8], $[K_1, K_2, K_3]$, defined by them. This [8] meets K_4 in a point, through which H must pass. The four [8]'s defined by sets of three of the planes have a common plane, which meets each of the planes in a point, and this is the required plane H.

* For a discussion of loci of which this is a particular case, see Richmond and Bath, *Proc. Camb. Phil. Soc.*, 22 (1924), 319; Segre, *Encyk. der Math. Wiss.*, III, C7, §10, 828.

Take in [10] five general planes $K_1, \ldots, K_5$. Each four of these has a unique transversal plane, so that the five define five planes $H_1, \ldots, H_5$, with the property that H_ρ meets in points all K_σ except K_ρ, the twenty points of incidence forming the following scheme :

	H_1	H_2	H_3	H_4	H_5
K_1		B_1	C_1	D_1	E_1
K_2	A_2		C_2	D_2	E_2
K_3	A_3	B_3		D_3	E_3
K_4	A_4	B_4	C_4		E_4
K_5	A_5	B_5	C_5	D_5	.

The twelve points $A_2, A_3, A_4, B_1, B_3, B_4, C_1, C_2, C_4, D_1, D_2, D_3$ lie in a [10], and must therefore be connected by a single linear relation*. This may be taken to be

$$I \equiv B_1+C_1+D_1+A_2+C_2+D_2+A_3+B_3+D_3+A_4+B_4+C_4 = 0. \qquad (1.1)$$

E_1 lies in a plane with B_1, C_1, D_1, so that, for some multiples b_1, c_1, d_1,

$$E_1+b_1B_1+c_1C_1+d_1D_1 = 0,$$

and there are similar relations for E_2, E_3, E_4. But $E_1, \ldots, E_4$ are in syzygy, and, expressed in terms of A, B, C, D, the relation can only be (1.1), so that $b_1 = c_1 = d_1$, etc. Further, we may choose the symbols $E_1, \ldots, E_4$, so that all the coefficients are unity, and we have then the five relations

$$\left.\begin{aligned} B_1+C_1+D_1+E_1 &= 0 \\ A_2+\quad C_2+D_2+E_2 &= 0 \\ A_3+B_3+\quad D_3+E_3 &= 0 \\ A_4+B_4+C_4+\quad E_4 &= 0 \\ A_5+B_5+C_5+D_5\quad &= 0 \end{aligned}\right\}. \qquad (1.2)$$

Similarly, we get five relations

$$A_2+A_3+A_4+A_5 = 0, \quad \&c. \qquad (1.3)$$

* The use of symbols for points is as in Baker's *Principles of geometry*, 1 (1922), 71.

2. *The* [3]*'s which meet the five planes* K. Five conditions are imposed on a [3] in [10] by making it meet a plane, so that, there being ∞^{28} [3]'s in [10], ∞^3 [3]/s meet the five planes K. Suppose that the [3] through the points

$$\alpha_2 B_1+\alpha_3 C_1+\alpha_4 D_1, \quad \beta_1 A_2+\beta_3 C_2+\beta_4 D_2,$$

$$\gamma_1 A_3+\gamma_2 B_3+\gamma_4 D_3, \quad \delta_1 A_4+\delta_2 B_4+\delta_3 C_4,$$

of $K_1, \ldots, K_4$, meets the plane K_5, then, assuming that these symbols already contain suitable multiples, for some set of values of $\epsilon_1, \ldots, \epsilon_4$*,

$$\alpha_2 B_1+\alpha_3 C_1+\alpha_4 D_1+\beta_1 A_2+\beta_3 C_2+\beta_4 D_2+\gamma_1 A_3+\gamma_2 B_3+\gamma_4 D_3$$
$$+\delta_1 A_4+\delta_2 B_4+\delta_3 C_4+\epsilon_1(A_2+A_3+A_4)+\epsilon_2(B_1+B_3+B_4)$$
$$+\epsilon_3(C_1+C_2+C_4)+\epsilon_4(D_1+D_2+D_3) \equiv \lambda I.$$

Equating the multiples of the twelve points we obtain the result that all the [3]'s, S, which meet the planes $K_1, \ldots, K_5$, do so in the points

$$\left.\begin{aligned} X_1 &= \qquad\qquad \alpha_2 B_1+\alpha_3 C_1+\alpha_4 D_1 \\ \cdots &\quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ X_4 &= \alpha_1 A_4+\alpha_2 B_4+\alpha_3 C_4 \\ X_5 &= \alpha_1 A_5+\alpha_2 B_5+\alpha_3 C_5+\alpha_4 D_5 \end{aligned}\right\}. \qquad (2.1)$$

These can be written in a more symmetrical form by introducing the points E. Thus X_1 is the same as

$$(\alpha_2+\alpha_5) B_1+(\alpha_3+\alpha_5) C_1+(\alpha_4+\alpha_5) D_1+\alpha_5 E_1,$$

and then, putting α_ρ for the old $\alpha_\rho+\alpha_5$, the five points are

$$\begin{aligned} X_1 &= \qquad +\alpha_2 B_1+\alpha_3 C_1+\alpha_4 D_1+\alpha_5 E_1 \\ \cdots &\quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ X_5 &= \alpha_1 A_5+\alpha_2 B_5+\alpha_3 C_5+\alpha_4 D_5, \end{aligned}$$

where now only the differences of the five parameters α_ρ are significant. The form (2.1) is the more convenient, since it gives the points in terms of four independent parameters, and, moreover, the four points $X_1, \ldots, X_4$, which are all that are wanted to define a [3], are in symmetrical form.

* These four quantities ϵ are not independent, their differences only being significant, in consequence of the relation between the points A, B, C, D, but this method is more symmetrical than that of taking multiples of only three points in K_5.

There is a similar family of ∞^3 [3]'s, T, which meet the planes H. The symbols for the points of incidence of any member are

$$\left.\begin{aligned} Y_1 &= \beta_2 A_2+\beta_3 A_3+\beta_4 A_4 \\ &\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ Y_4 &= \beta_1 D_1+\beta_2 D_2+\beta_3 D_3 \\ Y_5 &= \beta_1 E_1+\beta_2 E_2+\beta_3 E_3+\beta_4 E_4 \end{aligned}\right\}. \tag{2.2}$$

The planes H and K are included algebraically in the families of [3]'s, S and T respectively, H_1 being, for example $(\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4) = (1, 0, 0, 0)$, and H_5 $(\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4) = (1, 1, 1, 1)$. In fact, the join of H_1 to any point of K_1 is a [3] of the system S, and all these [3]'s have $(\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4) = (1, 0, 0, 0)$. They can be distinguished from one another by assigning $\alpha_2 : \alpha_3 : \alpha_4$, these fixing the point of K_1.

Through any point of K_1 there pass ∞^1 [3]'s S, namely those having a given $\alpha_2 : \alpha_3 : \alpha_4$, but an arbitrary α_1. The family includes the join of the point to H_1. It is clear from the symbols for their points—as well as from the geometry of the figure—that no two [3]'s S intersect, unless either they do so in a point of a plane K, or they pass through the same plane H.

Between the symbols for points of a [3], S, and a [3], T, we have the identity

$$\beta_1 X_1+\beta_2 X_2+\beta_3 X_3+\beta_4 X_4 = \alpha_1 Y_1+\alpha_2 Y_2+\alpha_3 Y_3+\alpha_4 Y_4,$$

so that every [3]S meets every [3] T in a point, and through every point of a [3] S one [3] T at least can be drawn. No values of α and β can be found giving a S and a T which meet in a line.

3. *The variety formed by the* [3]*'s* S *and* T. Any point of the variety Ω, generated by the [3]'s S, is $\beta_1 X_1+\beta_2 X_2+\beta_3 X_3+\beta_4 X_4$, where the β are parameters. To get its equations we may suppose that a general point of the [10] is

$$x_{12}A_2+x_{13}A_3+x_{14}A_4+x_{21}B_1+x_{23}B_3+x_{24}B_4+x_{31}C_1+x_{32}C_2+x_{34}C_4 \\ +x_{41}D_1+x_{42}D_2+x_{43}D_3,$$

where, since the sum of the A, B, C, D is zero, only the differences of the $x_{\rho\sigma}$ are significant. The parametric equations of Ω are then

$$x_{\rho\sigma} = \beta_\rho \alpha_\sigma+\epsilon, \quad \rho \neq \sigma; \quad \rho, \sigma = 1, 2, 3, 4,$$

ϵ being an additional parameter. To remove this additional parameter

we may consider Ω as the projection from the unit point in [11] of the variety Ω',

$$x_{\rho\sigma} = \beta_\rho \alpha_\sigma.$$

Now add four independent points A_{11}, B_{22}, C_{33}, D_{44} to the [11], obtaining a [15], whose general point is

$$x_{11}A_1 + x_{12}A_2 + x_{13}A_3 + x_{14}A_4 + x_{21}B_1 + \ldots + x_{31}C_1 + \ldots + x_{41}D_1 + \ldots.$$

Ω' is the projection from the four points A_{11}, B_{22}, C_{33}, D_{44} of the variety Ω'' in [15],

$$x_{\rho\sigma} = \beta_\rho \alpha_\sigma, \quad \rho, \sigma = 1, 2, 3, 4 \text{ (all possible pairs).}$$

Eliminating the parameters, the equations of Ω'' are given by the vanishing of all determinants of two rows and columns in

$$\begin{Vmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{Vmatrix}.$$

Ω'' is therefore of order twenty*. It consists of two ∞^3 families of [3]'s, given by $\alpha_\sigma =$ constant, and by $\beta_\rho =$ constant, in which any member of one family meets in points all members of the other. It is the locus of [3]'s which meet five general [3]'s in [15], and it has no singular points or specialities of any kind†.

Ω is the projection of Ω'' from the four points A_{11}, B_{22}, C_{33}, D_{44}, and from the unit point, all of which lie on Ω'', and is therefore of order fifteen‡. The only singular elements on Ω are the planes H and K, which are the projections of the [3]'s of the two families on Ω'' through the points of projection; Ω contains the five [5]'s $[H_\rho, K_\rho]$.

4. *The* [4] U, *which meets all the* [3]*'s* S *and* T. The six points

$$A_2+B_1, \quad A_3+C_1, \quad A_4+D_1, \quad B_3+C_2, \quad B_4+D_2, \quad C_4+D_3$$

are connected by the relation $I = 0$ (1..1), and therefore lie in a [4]. This [4] also contains the points A_5+E_1, B_5+E_2, C_5+E_3, D_5+E_4. It

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* Segre, *Encyk. der Math. Wiss.*, III, C 7, 828.

† This is a special case of the figure considered by Richmond and Bath, *Proc. Camb. Phil. Soc.*, 22 (1924), 319.

‡ In Schubert's notation the order of Ω is given by $(2, 8, 9, 10)^5(4, 8, 9, 10)$, and this can be seen to be fifteen by his degeneration method, cf. Segre, *loc. cit.*, 813.

meets in a point each [3] S and T, for

$$a_1X_1+a_2X_2+a_3X_3+a_4X_4 = a_1a_2(A_2+B_1)+a_1a_3(A_3+C_1)+a_1a_4(A_4+D_1)$$
$$+a_2a_3(B_3+C_2)+a_2a_4(B_4+D_2)+a_3a_4(C_4+D_3).$$

It does not meet the planes H and K, but it does meet every [3] S, through a plane H, that is it meets the [5]'s $[H_\rho, K_\rho]$ in planes. $[H_1, K_1]$, *c.g.*, contains the points A_2+B_1, A_3+C_1, A_4+D_1, A_5+E_1, which are coplanar, and all lie in U.

Take $A_2+B_1 = P_{12} = P_{21}$, etc. We have ten points in the [4] U, connecting which are the five relations

$$P_{12}+P_{13}+P_{14}+P_{15} = 0, \text{ etc.}$$

From these we can deduce relations of the type

$$P_{23}+P_{34}+P_{42} = P_{15},$$

and there are thus ten other planes which contain sets of four of the points. Each of the four planes

$$[P_{12}, P_{13}, P_{14}], \quad [P_{21}, P_{23}, P_{24}], \quad [P_{31}, P_{32}, P_{34}], \quad [P_{41}, P_{42}, P_{43}]$$

meets a set of three of the planes

$$[P_{23}, P_{34}, P_{42}], \quad [P_{34}, P_{41}, P_{13}], \quad [P_{41}, P_{12}, P_{24}], \quad [P_{12}, P_{23}, P_{31}]$$

in lines. The point of intersection of $[P_{12}, P_{13}, P_{14}]$ with $[P_{23}, P_{34}, P_{42}]$, *i.e.* with the plane which it does not meet in a line, is P_{15}; so that the four points $P_{15}, P_{25}, P_{35}, P_{45}$ are the intersections of pairs of corresponding planes of a double-four. They therefore lie in the fifth associated plane of either half of the double-four.

The five planes $([H_\rho, K_\rho], U)$ *are associated.*

The locus of points in which the [3]'s S or T meet U is

$$a_2a_3P_{23}+a_3a_1P_{31}+a_1a_2P_{12}+a_1a_4P_{14}+a_2a_4P_{24}+a_3a_4P_{34}.$$

Taking the general point of U to be $\sum_{\rho\sigma} x_{\rho\sigma}P_{\rho\sigma}$ $(\rho \neq \sigma,\ \rho, \sigma = 1, 2, 3, 4)$, where, since $\sum_{\rho\sigma} P_{\rho\sigma} = 0$, only the differences of $x_{\rho\sigma}$ matter, we obtain as the parametric equation of the locus

$$x_{\rho\sigma} = a_\rho a_\sigma + \epsilon,$$

and then, eliminating the parameters, we get

$$(x_{23}-x_{24})(x_{31}-x_{34})(x_{12}-x_{14}) = (x_{23}-x_{34})(x_{31}-x_{14})(x_{12}-x_{24}).$$

The sums of the factors of the two sides of the equation are equal, so

that the locus is a Segre cubic variety Γ. It consists of six ∞^2 families of lines, each comprising all the lines which meet one of the sets of five associated planes defined by the ten points $P_{12}, \ldots, P_{45}$. Of particular importance among these is the family which meets the planes

$$[P_{12}, P_{13}, P_{14}, P_{15}] = ([H_1, K_1], U), \text{ etc.}$$

The lines of this family will be called u.

The point of incidence of a [3] S and a [3] T lies on U if, and only if, the ratios of their parameters are equal, *i.e.* if

$$(\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4) = (\beta_1 : \beta_2 : \beta_3 : \beta_4).$$

Since the S meets U in a point, $[S, U]$ is a [7]. It can easily be shown that this [7] also contains the [3] T which meets U in the same point, and further that the [3], in which the [6] through these two [3]'s cuts U, is the tangent to Γ at the point, *i.e.* contains all six lines of the locus through the point, and among them the line u there.

5. *The* [3]*'s S and T which meet U along a line u.* Any line u of the family which meets the planes $([H_\rho, K_\rho] U)$ joins points in those planes whose symbols are

$$\left.\begin{array}{cccc} & \frac{\lambda_2}{\lambda_2-\lambda_1}P_{12}+ & \frac{\lambda_3}{\lambda_3-\lambda_1}P_{13}+ & P_{14} \\ \frac{\lambda_1}{\lambda_1-\lambda_2}P_{21}+ & & \frac{\lambda_3}{\lambda_3-\lambda_2}P_{23}+ & P_{24} \\ \frac{\lambda_1}{\lambda_1-\lambda_3}P_{31}+ & \frac{\lambda_2}{\lambda_2-\lambda_3}P_{32}+ & & P_{34} \\ \lambda_1 P_{41}+ & \lambda_2 P_{42}+ & \lambda_3 P_{43} & \end{array}\right\}. \qquad (5.1)$$

Comparing the symbols for the points concerned it is easily seen that the line u of Γ through the point where the [3], S, (a_1, a_2, a_3, a_4), meets u is given by

$$\lambda_1 : \lambda_2 : \lambda_3 = \frac{a_1}{a_1-a_4} : \frac{a_2}{a_2-a_4} : \frac{a_3}{a_3-a_4}, \qquad (5.2)$$

and that, on the other hand, the [3]'s, S, which meet U along the line u $(\lambda_1, \lambda_2, \lambda_3)$, have parameters

$$\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 = \frac{\lambda_1}{\lambda_1-\mu} : \frac{\lambda_2}{\lambda_2-\mu} : \frac{\lambda_3}{\lambda_3-\mu} : 1, \qquad (5.3)$$

μ varying from point to point of the line.

From algebra similar to that at the beginning of §§ 3 and 4, where the orders of Ω and Γ were found, it appears that the locus L, of the ∞^1 [3]'s, S, which meet a line u, is a normal rational 7-ic fourfold. Such a 7-ic has on it generally a unique line (not lying in any of the [3]'s on it, similar to the axis of the cubic surface in [4]). For a variety, L, this line is the line u which defines it. There are ∞^4 planes which meet all the [3]'s, S, of L, and each of them does so in points of a conic. Among them are the planes K; K_1, for example, meets L in the points

$$\frac{\lambda_2}{\lambda_2-\mu}B_1+\frac{\lambda_3}{\lambda_3-\mu}C_1+D_1,$$

whose locus is $\lambda_2 x_{31}(x_{21}-x_{41}) = \lambda_3 x_{21}(x_{31}-x_{41})$.

The points B_1, C_1, D_1, E_1 lie on this conic, as clearly they ought to do, since the planes H lie on L, the join of H_1 to the point where u meets the plane $([H_1, K_1], U)$ being one of the [3]'s, S, on the locus.

Associated with the rational 7-ic fourfold in [10] there are ∞^8 "secant" [3]'s, which cut it in cubic curves. These are the only [3]'s which meet all the ∞^1 [3]'s on the variety, so that the [3]'s T must belong to the family secant to any of the loci L, and therefore cut each of these loci in cubic curves. Among the ∞^8 secant [3]'s there are ∞^2 "secant" [4]'s, Z, which are of importance later when the projection of Ω from a [3] T is being considered. A Z cuts each [3] S of the fourfold in a line, these lines forming a ruled cubic surface (in [4]) whose axis is the line u. All the secant [4]'s, in fact, pass through u, and if a secant [3] meets u, its join to u is a secant [4]. Exactly one Z passes through any point of L not on u; in particular, one [4] secant to a given L can be drawn from any point of a plane H.

The point of intersection of a [3] S, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, with a [3] T, $(\beta_1, \beta_2, \beta_3, \beta_4)$, lies on U if, and only if, $\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 = \beta_1 : \beta_2 : \beta_3 : \beta_4$, so that precisely similar results hold for the locus M of [3]'s T meeting U along a line u.

6. *The figure in the* [7] *defined by two* [3]*'s* S. Let S' and S''—(a_1', a_2', a_3', a_4'), $(a_1'', a_2'', a_3'', a_4'')$, defined by points X_1', X_2', X_3', X_4' and X_1'', X_2'', X'', X_4''—be two general [3]'s of the system meeting the planes K. They have no common point, and therefore define a [7] $[S', S'']$. This [7] meets each of the planes K in the line joining the points where S' and S'' meet it, and will therefore contain all the ∞^1 [3]'s S which meet the five lines so obtained, and no others. Further, since any five lines in [7] are met by ∞^1 [3]'s, there can be no [7]'s meeting the

five planes K in lines other than those defined by pairs of [3]'s S. The parameters of any [3] S, in the [7] $[S', S'']$, are

$$a_\rho = \lambda' a'_\rho + \lambda'' a''_\rho \quad (\rho = 1, 2, 3, 4).$$

Any point on the locus Q of the [3]'s S, in $[S', S'']$ is therefore $\sum_\rho \mu_\rho(\lambda' X'_\rho + \lambda'' X''_\rho)$. That is, if X'_ρ, X''_ρ be taken as the simplex of reference in the [7], the parametric equations of Q are

$$x'_\rho = \lambda' \mu_\rho,$$
$$x''_\rho = \lambda'' \mu_\rho,$$

and its explicit equations are therefore

$$\begin{Vmatrix} x'_1 & x'_2 & x'_3 & x'_4 \\ x''_1 & x'' & x''_3 & x''_4 \end{Vmatrix} = 0.$$

Q is a normal quartic fourfold in [7]. Any one of its points may also be written as $\{\lambda' \Sigma \mu_\rho X'_\rho + \lambda'' \Sigma \mu_\rho X''_\rho\}$, which shows that it consists of ∞^3 lines, defined by the sets of μ. If we express X' and X'' in terms of the A, B, C, D, it is clear that one of these lines lies in each [3] T; actually the line $(\mu_1, \mu_2, \mu_3, \mu_4)$ lies in the T whose parameters are

$$(\beta_1 : \beta_2 : \beta_3 : \beta_4) = (\mu_1 : \mu_2 : \mu_3 : \mu_4).$$

Thus the points in which a T meets a Q lie on a line.

Q meets U in the points whose symbols are obtained by the variation of λ', λ'' in $\sum_{\rho\sigma} (\lambda' a'_\rho + \lambda'' a''_\rho)(\lambda' a'_\sigma + \lambda'' a''_\sigma) P_{\rho\sigma}$ $(\rho, \sigma = 1, 2, 3, 4;\ \rho \neq \sigma)$, say $\lambda'^2 Q_{11} + \lambda' \lambda'' Q_{12} + \lambda''^2 Q_{22}$, where Q_{11} is the point $\Sigma a'_\rho a'_\sigma P_{\rho\sigma}$, etc. These points lie on a conic. The plane $[Q_{11}, Q_{12}, Q_{22}]$ therefore cuts the Segre locus Γ in a conic on Q, and further in a line. From general reasoning this must be a line u, because, if it were not, some of the conics would degenerate into pairs of lines, one of which was a u. Then we should have the impossibility that part of the locus of [3]'s S, which meets U along one of these conics, is of order seven, whereas the whole locus is of order four. More definitely it can be shown that

$$\{(a''^2_1 p_{23} + a''^2_2 p_{31} + a''^2 p_{12}) Q_{11} - (a'_1 a''_1 p_{23} + a'_2 a''_2 p_{31} + a'_3 a''_3 p_{12}) Q_{12} + (a'^2_1 p_{23} + a'^2_2 p_{31} + a'^2_3 p_{12}) Q_{22}\}$$
$$\equiv -(q_1 p_{12} p_{13} P_{14} + q_2 p_{21} p_{23} P_{24} + q_3 p_{31} p_{32} P_{34}),$$

where
$$p_{\rho\sigma} = \begin{vmatrix} a'_\rho & a'_\sigma \\ a''_\rho & a''_\sigma \end{vmatrix} = -p_{\sigma\rho},$$

and
$$q_1 = p_{23} + p_{34} + p_{42}, \text{ etc.},$$

the signs of these being arranged so that

$$q_1+q_2+q_3+q_4=0.$$

This identity shows that the plane $[Q_{11}, Q_{12}, Q_{22}]$, *i.e.* $([S', S''], U)$, meets the planes $([K_1, H_1], U)$, $([K_2, H_2], U)$, $([K_3, H_3], U)$ in the points

$$\begin{aligned} Q_1' = {} & q_2p_{23}p_{24}P_{12}+q_3p_{32}p_{34}P_{13}+q_4p_{42}p_{43}P_{14} \\ Q_2' = {} & q_1p_{13}p_{14}P_{21} \qquad +q_3p_{31}p_{34}P_{23}+q_4p_{41}p_{43}P_{24} \\ Q_3' = {} & q_1p_{12}p_{14}P_{13}+q_2p_{21}p_{24}P_{32} \qquad +q_4p_{41}p_{42}P_{34}. \end{aligned}$$

Then we have $p_{41}Q_1'+p_{42}Q_2'+p_{43}Q_3' \equiv q_4p_{41}p_{42}p_{43}\Sigma P_{\rho\sigma}$; so that these points lie on a line. Similarly, the points in which the plane meets $([K_4, H_4], U)$ and $([K_5, H_5], U)$ lie on this line, and the line is therefore one of the family u.

Thus associated with each quartic fourfold Q there is a 7-ic fourfold L, the conic and line of intersection of the two respectively with U being coplanar. The two varieties have twelve common points, two in each of the planes K, and two in U. More interesting is the relation between the fourfold Q of [3]'s S and the fourfold M of [3]'s T, whose intersections with U are coplanar. This will be considered in the next section.

In the special case of S' and S'' meeting in a point of one of the planes K, say K_1, *i.e.* when $a_2' : a_3' : a_4' = a_2'' : a_3'' : a_4''$, the space defined by them is of dimension six only. This cuts the remaining four planes K in lines, and contains the plane H_1. The [3]'s S in this [6] are those through the point of K_1, which meet the lines in K_2, K_3, K_4, K_5; they form a cubic cone.

7. *The* [3]*'s* T *which meet the* [7]*'s* $[S', S'']$ *in planes.* If S is not one of the [3]'s which lie in the [7] $[S', S'']$, it meets that [7] in a point. This point cannot lie on Q. $[S', S'']$ has therefore ∞^3 points common with Ω which do not lie on Q. Through any of these there passes a single [3] T. This [3] already contains a line of Q, and therefore on the whole must meet the [7] in a plane. There are ∞^1 of these planes in the [7].

The three [3]'s S, S', S'', define an ∞^2 family, whose parameters are $\lambda a_\rho+\lambda' a_\rho'+\lambda'' a_\rho''$, one of which passes through any point of a plane K. They form a fivefold F, the symbol for whose general point is $\Sigma_\rho \mu_\rho(\lambda X_\rho+\lambda' X_\rho'+\lambda'' X_\rho'')$. The twelve points X_ρ, X_ρ', X_ρ'' are in syzygy, so that F is best regarded as the projection of a figure F' in [11], whose

point equations are

$$x_\rho = \mu\lambda_\rho, \quad x' = \mu'\lambda_\rho, \quad x'' = \mu''\lambda_\rho,$$

which, on elimination of the parameters λ and μ, result in the equations obtained from the vanishing of determinants of two rows and columns in

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1' & x_2' & x_3' & x_4' \\ x_1'' & x_2'' & x_3'' & x_4'' \end{vmatrix}.$$

F' is therefore of order ten. It consists of the ∞^2 [3]'s meeting five general planes in [11], or of the ∞^3 planes meeting four general [3]'s. The point of projection, although possibly specially related to F', does not lie on it, and the projected figure F, in [10], is therefore of order ten also. The projections of the [3]'s are the [3]'s S, and the projections of the planes are the planes in which the [3]'s T meet F.

We want now to find those planes of this last family which lie in the [7] $[S', S'']$. Suppose that $(\beta_1, \beta_2, \beta_3, \beta_4)$ are the parameters of a [3] T, which meets $[S', S'']$ in a plane. Expressing the condition that any two points of T are in syzygy with $X_1', \ldots, X_4', X_1'', \ldots, X_4''$, we find

$$\frac{\beta_1}{\beta_1-\beta_4} : \frac{\beta_2}{\beta_2-\beta_4} : \frac{\beta_3}{\beta_3-\beta_4} = \frac{q_1}{p_{23}} : \frac{q_2}{p_{31}} : \frac{q_3}{p_{12}},$$

where p and q have the meanings assigned in the last section. These equations are exactly the same as (5 . 2), with q_1/p_{23}, q_2/p_{31}, q_3/p_{12} written for λ_1, λ_2, λ_3, and the α replaced by β. This family of [3]'s T is therefore that which meets the line u in which the plane $([S', S''], U)$ cuts the Segre locus Γ.

Summarizing the position so far we have : two general [3]'s, S', S'', of the system S, define a [7], which contains ∞^1 [3]'s S, forming a rational quartic fourfold Q. Q meets the [4] U in a conic. The plane of this conic cuts the Segre locus Γ in U further in a line u. The [3]'s T, which meet U along this line form a 7-ic fourfold M, and each of them meets $[S', S'']$ in a plane. They are the only [3]'s T doing this. The fact that both Q and M are rational fourfolds, loci of the simplest projectively generated type, enables us to write down at once some of the intersection properties :

$[S', S'']$ is a [7] meeting each of the [3]'s of the 7-ic M in a plane. These planes form a rational 5-ic threefold. This 5-ic contains a unique line, actually the u. The [3]'s S of Q meet each of these planes in a

point, and therefore are among the secants to the locus, meeting it in cubic curves.

Now consider the quartic fourfolds Q, of [3]'s S, and R of [3]'s T, meeting U in the same conic γ. Each [3] of Q meets the [3]'s T of R in points of a line. The [3] S at a point where γ meets the line u in its plane meets the [7] containing R in a plane. The planes at the two points (u, γ) cannot have a common point, so that [7]'s containing Q and R have a common [5]. Part of the intersection of Q with this [5] is the two planes. The rest must be a quadric surface, and this will be common to Q and R.

8. *The dual configuration.* Although the algebra for the dual configuration proceeds on exactly the same lines as that for the original one, except that symbols of points are replaced by equations of [9]'s, it frequently happens that the geometric picture of the dual is more easily imaginable, possibly because the intersection of two spaces is conceived more readily than the join of their duals. For example, there is nothing in the original figure to suggest that the [7]'s $[S, U]$—which are the same as $[T, U]$—are the generators of one system of a quadric [4]-cone, so that the projection of Ω from U is one system of generators of an ordinary quadric fourfold in [5], but the theorem corresponding to this occurs quite naturally in the course of the consideration of the dual figure.

The dual double-five consists of two sets of [7]'s, $h_1, \ldots, h_5$ and $k_1, \ldots, k_5$, in which h_ρ meets in [5]'s all k_σ except k_ρ, but has no special intersection—*i.e.* intersection of dimension greater than four—with the other spaces. (h_1, h_2) is a [4] and (h_1, h_2, h_3) is a line. This line lies in the [4], (k_4, k_5); for (h_1, k_5) and (h_2, k_5) are [5]'s lying in the [7] k_5, so that (h_1, h_2, k_5) is a [3]. (h_1, h_2, k_5) and (h_1, h_3, k_5) are [3]'s lying in the [5] (h_1, k_5), and therefore (h_1, h_2, h_3, k_5) is a line, and can only be the line (h_1, h_2, h_3). That is, (h_1, h_2, h_3) lies in k_5, and similarly in k_4. Further, the lines (h_1, h_2, h_3) and (k_3, k_4, k_5) meet in a point, for both lie in the plane (h_1, h_2, k_4, k_5). Denoting the line (h_1, h_2, h_3) by λ_{45}, and the line (k_1, k_2, k_3) by μ_{45}, etc., we see that the configuration can be constructed from the two sets of ten lines in [10], $\lambda_{\rho\sigma}$ and $\mu_{\rho\sigma}$ $(\rho, \sigma = 1, 2, 3, 4, 5; \rho \neq \sigma)$, where $\lambda_{\rho\sigma}$ meets the three lines $\mu_{\tau\nu}$ having no common suffix, and no others. h_1, for example, is

$$[\mu_{12}, \mu_{13}, \mu_{14}, \mu_{15}],$$

and contains, further, the lines

$$\lambda_{34}, \lambda_{42}, \lambda_{23}, \lambda_{25}, \lambda_{35}, \lambda_{45}.$$

(h_1, k_2) and (h_2, k_1) are two independent [5]'s in the [10], and therefore have a common point. The point $\big((h_1, h_2), (k_1, k_2)\big)$ will be denoted by $F_{12} = F_{21}$.

By reference to the symbols of the points of incidence of the planes of a double-five, we see that we may take the equations of the [7]'s of a double-five to be

$$\begin{aligned}
&k_1 \quad && y_1 = z_1 = w_1 = 0,\\
&k_2 \quad x_2 && = z_2 = w_2 = 0,\\
&k_3 \quad x_3 = y_3 && = w_3 = 0,\\
&k_4 \quad x_4 = y_4 = z_4 && = 0,\\
&k_5 \quad x_2+x_3+x_4 = y_1+y_3+y_4 = z_1+z_2+z_4 = w_1+w_2+w_3 &&= 0,
\end{aligned}$$

and

$$\begin{aligned}
&h_1 \quad && x_2 = x_3 = x_4 = 0,\\
&h_2 \quad y_1 && = y_3 = y_4 = 0,\\
&h_3 \quad z_1 = z_2 && = z_4 = 0,\\
&h_4 \quad w_1 = w_2 = w_3 && = 0,\\
&h_5 \quad y_1+z_1+w_1 = x_2+z_2+w_2 = x_3+y_3+w_3 = x_4+y_4+z_4 &&= 0,
\end{aligned}$$

where $$\Sigma x+\Sigma y+\Sigma z+\Sigma w \equiv 0,$$

the x, y, z, w being treated as coordinates in the [10].

The most important points in the dual figure, the correspondents of which in the original figure were not considered, are the points $F_{\rho\sigma}$. F_{12} is

$$y_1 = z_1 = w_1 = x_2 = x_3 = x_4 = y_3 = y_4 = z_2 = w_2 = w_2+z_4 = 0.$$

Taking the non-zero coordinates to be ± 1 in every case, the signs can be arranged so that there are the syzygies

$$\begin{aligned}
F_{12}-F_{13}+F_{14}+F_{15} &= 0,\\
-F_{21}+F_{23}+F_{24}+F_{25} &= 0,\\
F_{31}-F_{32}+F_{34}+F_{35} &= 0,\\
-F_{41}-F_{42}-F_{43}+F_{45} &= 0,\\
F_{51}+F_{52}+F_{53}+F_{54} &= 0.
\end{aligned}$$

Four of these relations are independent, so that the ten points lie in

a [5], W. The equations of the [5] are

$$x_2+y_1=x_3+z_1=x_4+w_1=y_3+z_2=y_4+w_2=z_4+w_3=0,$$

i.e. it is the dual of the [4], U, in the original figure. The four points $F_{\rho\sigma}$, having a common suffix ρ, all lie in the [4] (h_ρ, k_ρ), so that (h_ρ, k_ρ, w) is a plane. This is better expressed thus : the [7] h_ρ passes through the plane (k_ρ, w).

Corresponding to the family of [3]'s S, which meet the planes K, there is a family of [6]'s s, which meet the [7]'s k in [4]'s. The equations of a general member of this family are

$$\begin{aligned} \qquad\qquad a_2y_1+a_3z_1+a_4w_1&=0,\\ a_1x_2\qquad\quad +a_3z_2+a_4w_2&=0,\\ a_1x_3+a_2y_3\qquad\quad +a_4w_3&=0,\\ a_1x_4+a_2y_4+a_3z_4\qquad\quad &=0. \end{aligned}$$

There will also be a similar family of [6]'s t, corresponding to the [3]'s T. Two [6]'s, s or t, meet in a plane, and an s and a t meet in a [3]. s meets k in a [4], and the ∞^3 [4]'s (s, k), just fill the [7], *i.e.* one [6] s passes through any general point of k. The two families of [6]'s generate the same quartic ninefold, ω, whose equation is

$$\begin{vmatrix} 0 & x_2 & x_3 & x_4 \\ y_1 & 0 & y_3 & y_4 \\ z_1 & z_2 & 0 & z_4 \\ w_1 & w_2 & w_3 & 0 \end{vmatrix} = 0.$$

Of the ten equations obtained by adding those of the [5] W to those of a [6] s, only eight are independent, so that the s and W meet in a plane. This plane meets the planes (h_ρ, k_ρ, W) in the points

$$\begin{aligned} \qquad\qquad a_2F_{12}-a_3F_{13}+a_4F_{14},&\\ -a_1F_{21}\qquad\quad +a_3F_{23}+a_4F_{24},&\\ a_1F_{31}-a_2F_{32}\qquad\quad +a_4F_{34},&\\ a_1F_{41}+a_2F_{42}+a_3F_{43},&\\ (a_1F_{51}+a_2F_{52}+a_3F_{53}+a_4F_{54}).& \end{aligned}$$

By comparing the coefficients of a point of any of these planes with a general point

$$x_{23}F_{23}+x_{31}F_{31}+x_{12}F_{12}+x_{14}F_{14}+x_{24}F_{24}+x_{34}F_{34}$$

of the [5], it is found that the equation of the locus of these planes is

$$x_{23}x_{14}+x_{31}x_{24}+x_{12}x_{34}=0.$$

The locus is therefore a general quadric fourfold. This result could also be obtained directly from the equation of ω and the equations of W.

The locus ω has on it double and triple loci given by the vanishing of determinants of three and two rows respectively in the four-row determinant. The double locus is of order twenty and dimension six. It contains the twenty [4]'s, (h_ρ, a_σ) and (k_ρ, k_σ), and the quadric fourfold in W. It is composed of two ∞^4 families of planes, one the intersections of all pairs of [6]'s s, and the other of [6]'s t, and it is cut by any of the [6]'s, s or t, in a sextic fourfold consisting of ∞^2 planes. The triple locus is also of order twenty, being actually the twenty lines $\lambda_{\rho\sigma}$ and $\mu_{\rho\sigma}$.

II. *The double-five of lines and planes in* [6].

9. *The projection of the double-five of planes in* [10], *from a* [3] T. *The point defined by five lines in* [6]. Any four lines in [6] are met by a unique plane; for such a plane will lie in each of the [5]'s defined by sets of three of the lines, and the intersection of these four [5]'s is a plane; whilst the intersection of three of them is a [3], and contains the plane and one of the lines. This plane, therefore, meets all four lines, and is the only plane which does so. There is thus in [6] a double-five of lines and planes, formed from five arbitrary lines $h_1, \ldots, h_5$, and the five planes $K_1, \ldots, K_5$, which meet sets of four of them. This figure cannot be determined so simply in terms of its incidences as that in [10] for a variety of reasons: (i) two points on each of the lines are not sufficient to define the planes; (ii) the freedom of five lines in [6] is fifty, and there is no number of generally situated points in [6] which has this freedom; (iii) although the five lines are general, the five planes are not, any four of them, in fact, being subjected to the twofold condition that they shall have a transversal line, or, from the point of view of freedoms, whereas the freedom of five general planes is sixty, the freedom of these is only fifty.

The figure in [10] provides the most symmetrical scheme, but needs twelve reference points, between which there will have to be five syzygies. If the double-five of planes H and K in [10] is projected from a [3], T_0, which meets all the planes H, we get, in [6], five general lines* h from the planes H, and the five planes K meeting four of

* The question of the generality of the lines h is discussed at the end of this section.

them from the planes K. Suppose that T_0 has parameters

$$(\beta_1, \beta_2, \beta_3, \beta_4) = (b_1, b_2, b_3, b_4);$$

then the intersections of the planes and lines in [6] give rise to the following scheme :

	h_1	h_2	h_3	h_4	h_5
K_1		B_1	C_1	D_1	$B_1+C_1+D_1$
K_2	A_2		C_2	D_2	$A_2+C_2+D_2$
K_3	A_3	B_3		D_3	$A_3+B_3+D_3$
K_4	A_4	B_4	C_4		$A_4+B_4+C_4$
K_5	$A_2+A_3+A_4$	$B_1+B_3+B_4$	$C_1+C_2+C_4$	$D_1+D_2+D_3$	

where the five syzygies connecting the twelve points are

$$\left.\begin{aligned} \Sigma A + \Sigma B + \Sigma C + \Sigma D &= 0 \\ b_2A_2+b_3A_3+b_4A_4 &= 0 \\ b_1B_1 + b_3B_3+b_4B_4 &= 0 \\ b_1C_1+b_2C_2 + b_4C_4 &= 0 \\ b_1D_1+b_2D_2+b_3D_3 &= 0 \end{aligned}\right\} \quad (9.1)$$

We shall frequently need to work without the points A_4, B_4, C_4, and the syzygies connecting the rest are

$$\left.\begin{aligned} b_1D_1+b_2D_2+b_3D_3 &= 0 \\ (b_1-b_4)(B_1+C_1+D_1)+(b_2-b_4)(A_2+C_2+D_2)+(b_3-b_4)(A_3+B_3+D_3) &= 0 \end{aligned}\right\} \quad (9.2)$$

Associated with the double-five of planes in [10] there is [4] U, containing the ten points such as A_2+B_1, which meets in points all the [3]'s, S and T. Since, in particular, it meets T_0 in a point, its projection in the [6] is a [3], which may also be called U. The [5]'s $[H_\rho, K_\rho]$ in [10] meet U in planes, so that the [4]'s $[h_\rho, K_\rho]$, in [6] meet the projection of U in planes. Further, in the figure in [10] there is a line u_0 of the Segre variety Γ, which meets all the five planes $([H_\rho, K_\rho], U)$, and passes through the point (T_0, U), so that the five planes, $([h_\rho, K_\rho], U)$, in [6], have a common point.

If $h_1, \ldots, h_5$ are any five lines in [6], *and K_σ is the plane which meets each h_ρ except h_σ, then the five* [4]'s $[h_\rho, K_\rho]$ *have a common point.*

The point will be denoted by G; its symbol is

$$b_1^2(b_2B_1+b_3C_1+b_4D_1)+b_2^2(b_1A_2+b_3C_2+b_4D_2)+b_3^2(b_1A_3+b_2B_3+b_4D_3)$$
$$+b_4^2(b_1A_4+b_2B_4+b_3C_4).$$

Expressed as a point of $[h_1, K_1]$ this is

$$b_2(b_1-b_3)(b_1-b_4)\,B_1+b_3(b_1-b_4)(b_1-b_2)\,C_1+b_4(b_1-b_2)(b_1-b_3)\,D_1$$
$$+b_2(b_1b_2-b_3b_4)\,A_2+b_3(b_1b_3-b_4b_2)\,A_3+b_4(b_1b_4-b_2b_3)\,A_4.$$

This theorem appears in so accidental a manner that further justification seems advisable. Two proofs of the existence of the point are given above, one depending on the figure in [10], and the other on the possible symbols for the point [the algebra leading to the latter is given in detail for the dual configuration in § 12 (ii)]. Both depend on the truth of the statement: "If five general planes in [10] are projected from a [3] which meets them all, we obtain five *general* lines in [6]", the contrary of which, "five *general* planes projected from a [3] which meets them, give five *special* lines", involves that five *general* lines in [6] each joined to a different point of a [3] outside the [6], give five *special* planes in [10]; so that five planes would project into five general lines, only when restricted in some way.

But a purely geometrical proof of the existence of the point can be easily constructed. Take five general lines, $h_1, \ldots, h_5$, in [6], and let $K_1, \ldots, K_5$ be the planes meeting the sets of four of them. The three [4]'s, $[h_1, K_1]$, $[h_2, K_2]$, $[h_3, K_3]$, have a common point, say G. Project from this point on to a [5]; the resulting lines h'_ρ, and planes K'_ρ, have these incidences: h'_1, h'_2, h'_3 meet all of $K'_1, \ldots, K'_5$, and h'_4 meets K'_1, K'_2, K'_3, K'_5, and h'_5 meets K'_1, K'_2, K'_3, K'_4. We show that h'_4 meets K'_4, and that h'_5 meets K'_5. There are ∞^2 lines meeting the three planes K'_1, K'_2, K'_3, forming a cubic threefold J. Any plane which meets a general four of these lines* meets all of them. There are ∞^1 of these planes, and they also generate the cubic threefold. $h'_1, \ldots, h_5$ all meet K'_1, K'_2, K'_3, and therefore lie on J. Also K'_4 lies on J, since it meets h'_1, h'_2, h'_3, h'_5, and therefore it meets h'_4. That is, K'_4, h'_4 are the projections from G of a plane and a line which lie together in a [4] which passes through G, so that $[h_4, K_4]$, and similarly $[h_5, K_5]$, pass through G.

The [3] U, in the [6], passes through G, and meets the five [4]'s $[h_\rho, K_\rho]$ in lines. It therefore projects from G into a plane which meets the five [3]'s $[h'_\rho, K'_\rho]$ in lines. This appears to be a restriction on the

* *I.e.*, four which do not lie in a [4].

five [3]'s, but it can be shown that any five [3]'s, each of which is defined by a plane and a line of a cubic threefold in [5], are met by exactly one plane in lines*.

10. *The* [3]*'s* T, *which meet the five lines* h. The [3]'s T, which meet the planes H, in [10], since they have in general no intersection with T_0, project into [3]'s T, meeting the lines h. But the [7] $[T, T_0]$ in [10] contains ∞^1 other [3]'s T, and itself projects into a space of only three dimensions, so that all the [3]'s in it project into the same [3] in [6]. There are therefore only ∞^2 [3]'s meeting the lines h, which arise in this way. Any [3] in [6] which meets the lines h must be the section by the [6] of a [7] in the [10] which meets the planes H in lines and passes through T_0, and it has been shown that the only such [7]'s are those defined by T_0 and some other [3] T. The [3]'s found are therefore all that meet the lines h. That there should be ∞^2 is confirmed by appeal to freedom: [3]'s in [6] have freedom twelve, and each intersection with a line imposes two conditions.

There are some apparently exceptional projections, namely the projections of the [3]'s T, which meet T_0 in a point of one of the planes H, say H_1. But if T_0 meets T_1 on H_1, then the [6] $[T_0, T_1]$ contains the plane K_1, so that the cubic cone of [3]'s T in the [6] $[T_0, T_1]$ projects into the plane into which K_1 projects.

The points in which the [3]'s T, in [6], meet the lines $h_1, \ldots, h_4$ are

$$\left.\begin{aligned} &\beta_2 A_2+\beta_3 A_3+\beta_4 A_4 \\ \beta_1 B_1+\quad &\beta_3 B_3+\beta_4 B_4 \\ \beta_1 C_1+\beta_2 C_2+\quad &\beta_4 C_4 \\ \beta_1 D_1+\beta_2 D_2+\beta_3 D_3\quad & \end{aligned}\right\},$$

giving an apparently ∞^3 family, but, in view of the syzygies connecting the points in [6], the points

$$\beta_2 A_2+\beta_3 A_3+\beta_4 A_4 \quad \text{and} \quad (\beta_2+\lambda b_2) A_2+(\beta_3+\lambda b_3) A_3+(\beta_4+\lambda b_4) A_4$$

are the same, and consequently the [3]'s whose parameters are

$$(\beta_1, \beta_2, \beta_3, \beta_4) \quad \text{and} \quad (\beta_1+\lambda b_1, \beta_2+\lambda b_2, \beta_3+\lambda b_3, \beta_4+\lambda b_4)$$

coincide. This ∞^2 family of [3]'s may be considered as the projection from T_0 of any ∞^2 family of [3]'s T, in [10], whose parameters are $\lambda\beta_\rho+\lambda'\beta'_\rho+\lambda''\beta''_\rho$, which does not include T_0. Such a family generates a locus of order ten, on which the planes H lie and are simple. T_0 meets

* This problem will be discussed further in the *Journal* of the Society.

the locus in five simple points, and the projected locus in [6] is therefore of order five*.

In [10] the [7]'s $[T, T_0]$ cut the Segre variety Γ, in U, in conics, the residual intersections of whose planes with Γ are the lines u. The [3]'s T in [6], therefore cut the [3] U in lines which are the projections of the generators of one system of a Segre locus, *i.e.* in lines of a congruence Γ of order and class two.

The [3]'s T, in [10], which meet U along the line u_0, project into [3]'s T, which pass through the point G, and there can be no other [3]'s T, through G. Then, since that family of [3]'s in [10] forms a 7-ic fourfold, on which T_0 lies, its projection from T_0 is a cubic cone. The projections of the other 7-ic fourfolds M have no particular property except that parts of their intersections with U are lines of the congruence Γ.

11. *The planes s which meet the five planes K.* The [3]'s S, in [10], which meet the planes K, since they all meet T_0 in points, project into planes s, which meet the planes K in the [6]. Moreover, since a [6] $[T_0, S]$ contains no other [3]'s S, there are ∞^3 planes obtainable in this way. Among them are included the joins of the lines h to points of the corresponding planes K. The points in which the planes s meet the planes $K_1, \ldots, K_4$ are

$$\left.\begin{aligned} &a_2B_1+a_3C_1+a_4D_1\\ a_1A_2+\quad &a_3C_2+a_4D_2\\ a_1A_3+a_2B_3+\quad &a_4D_3\\ a_1A_4+a_2B_4+a_3C_4\quad &\end{aligned}\right\}. \tag{11.1}$$

Among these planes there is one particular one, s_0, the projection of the [3] S_0, in [10], which passes through the point (U, T_0), given by $a_1 : a_2 : a_3 : a_4 = b_1 : b_2 : b_3 : b_4$. It was stated at the end of § 4 that the [7] $[U, T_0]$ contains S_0, and further that the [3] in which the [6] $[S_0, T_0]$ meets U is tangent to the Segre locus Γ. The plane s_0, in [6], therefore lies in the [3] U, and passes through G. It is actually the sixteenth singular plane of the congruence Γ of lines in U, the other fifteen being the projections of the fifteen planes on the Segre locus associated with the figure in [10].

Each point of U is the projection of two points of the Segre locus, so that two planes s pass through it. The only exceptional points are the

* In Schubert's notation the order is given by $(1, 4, 5, 6)^6$, and this is equal to five.

sixteen double points of the congruence Γ; these are the projections of the ten points $P_{\rho\sigma}$, which are double on the Segre locus, and the lines of the six systems on the locus through the point (T, U)*.

The [3]'s S, in [10], were divided into two main sets of ∞^1 families: those which lie in [7]'s, and formed quartic fourfolds Q, and those which meet U along lines u, and formed 7-ic fourfolds L. The planes s, in [6], split into two similar families. Corresponding to the first we have: in the [5] defined by two of the planes, s' and s'', there are ∞^1 planes, forming a cubic locus which meets U in a conic. There are two classes of exceptions. (i) The original [3]'s, S' and S'', may define a [7] which contains S_0; the locus of the planes defined by s' and s'' is not different, but its intersection with U is the plane s_0 together with a line of the congruence Γ. (ii) The original [3]'s, S' and S'', may define a [7] which meets T_0 in a plane; s' and s'' then intersect. This case will be considered at the end of the section.

A 7-ic fourfold L is met by T_0 in a cubic curve, so that its projection is a quartic threefold. The planes of this threefold meet U along a line of the congruence Γ, but are not the only planes which do so. The others are the projections of the [3]'s S, which meet the [4] U along a conic which is the residual intersection of the Segre locus with the plane through the line u concerned and the point (T_0, U), *i.e.* of one of the exceptional families (i) above. We have then: the planes which meet U along a line u fall into two families, one generating a cubic threefold and the other a quartic threefold. The only 7-ic L, in [10], specially related to T_0 is that, L_0, which meets U in u_0. This projects into a quartic cone of planes—the join of a ruled quartic surface in [5] to a point outside the [5].

The planes s are not the only planes meeting the planes K, but are the dominant family, the other having only ∞^2 members. It can easily be seen that any plane which meets $K_1, \ldots, K_4$ does so in points:

$$\left.\begin{array}{llll} & & (a+b_1a_2)\,B_1+ & (a+b_1a_3)\,C_1+(a+b_1a_4)\,D_1 \\ (a+b_2a_1)\,A_2+ & & & (a+b_2a_3)\,C_2+(a+b_2a_4)\,D_2 \\ (a+b_3a_1)\,A_3+ & (a+b_3a_2)\,B_2+ & & (a+b_3a_4)\,D_3 \\ (a+b_4a_1)\,A_4+ & (a+b_4a_2)\,B_4+ & (a+b_4a_3)\,C_4 & \end{array}\right\}. \quad (11.2)$$

This will meet the fifth plane, K_5, if the following four conditions, of

* Baker, *Principles of geometry*, 4 (1925), 154 *et seq.*

which two only are independent, are satisfied:

$$(a+a_\rho b_\sigma)(a+a_\sigma b_\tau)(a+a_\tau b_\rho) = (a+a_\rho b_\tau)(a+a_\tau b_\sigma)(a+a_\sigma b_\rho),$$

where $\rho, \sigma, \tau = 1, 2, 3, 4$, and no two are equal. $a = 0$ is a solution of all these equations, and gives the family of planes s. The other solution is

$$a = \frac{b_\rho(b_\sigma - b_\tau)\, a_\sigma a_\tau + b_\sigma(b_\tau - b_\rho)\, a_\tau a_\rho + b_\tau(b_\rho - b_\sigma)\, a_\rho a_\sigma}{(b_\sigma - b_\tau)\, a_\rho + (b_\tau - b_\rho)\, a_\sigma + (b_\rho - b_\sigma)\, a_\tau}, \qquad (11.3)$$

two of these four equations implying the others. There are two independent relations between the five parameters a, and this family of planes, which will be called z, therefore contains ∞^2 members.

All these planes z pass through the point G. In fact, the condition that a plane of the ∞^4 family (11.2) should pass through G is exactly the condition that it should meet the plane K_5, except that $a = 0$ is not admissible. They form a cubic [0]-cone—the join of the lines of a rational cubic threefold in [5] to a point outside the [5]—which contains the planes K simply. They are actually the other system of generators of the cubic cone of [3]'s T, which pass through G.

The planes z may also be obtained directly by projection from the figure in [10]. At the end of § 5 it was seen that, associated with the 7-ic fourfolds M, there are some [4]'s Z which meet them in ruled cubic surfaces. In particular, ∞^2 [4]'s Z_0 are associated with M_0. Each of these meets T_0 in a line, and therefore projects into a plane. Further, it meets the planes K and contains the line u_0, so that it projects into a plane which meets the planes K and passes through G. The final identification of the projections of these [4]'s Z_0 with the planes z can be obtained easily by algebra. The ruled cubic surface in which one of these meets M_0, composed of the lines in which it meets the [3]'s T, of M_0, has u_0 as its axis, so that the planes z meet the [3]'s T, which pass through G in the lines of plane pencils. This indicates that the planes z and the [3]'s are the generators of the two systems on a cubic cone, as has already been stated.

Among the [4]'s Z_0, there are the joins of the [3]'s S, of L_0 to u_0, so that the planes s, which pass through G, may also be considered as planes z. Algebraically this is clear, since to obtain the planes common to the two families we have merely to put $a = 0$ in the equations (11.3). The resulting equations are exactly those which express the condition that a plane s passes through G.

If two planes, s' and s'', meet in a point which does not lie on a plane K, they lie in a [4] which meets the five planes K in lines. The [4], $[s', s'']$ therefore contains ∞^1 of the planes which meet the five planes

K, these forming altogether a threefold of order five. s' and s'' are the projections of [3]'s, S' and S'', in [10], whose joining [7], $[S', S'']$, lies with T_0 in an [8], *i.e.* meets T_0 in a plane. In § 7, towards the end, it was shown that this happens only if the plane in which $[S', S'']$ meets U contains the line u_0. The [4] $[s', s'']$ therefore passes through G^*. Part of the 5-ic locus is therefore the projection of the quartic locus Q, in the [7], from a plane which meets Q in a line, and lies in the [7]. This is a cubic threefold—the dual of the ruled cubic surface in [4]. The other planes in $[s', s'']$ must be planes z. These form a quadric [0]-cone, with vertex G.

12. *The dual configuration.*

(i) *The set of ten points which defines the dual configuration.* The configuration dual to that that has so far been considered in this chapter is based on five general [4]'s in [6]. Any three [4]'s in [6] have one common point, so that these five will define ten points. In [4], from many points of view, it is easier to deal with the double-four of planes than with the dual double-four of lines, because the first can be constructed directly from six points. Similarly, because of its direct dependence on ten points, the dual double-five in [6] has many advantages over that already treated. The dual of the plane meeting four lines in [6] is the [3] which meets in planes four [4]'s. The [3] is actually just the join of the four points of intersection of sets of three of the [4]'s. Call the five [4]'s $H_1, \ldots, H_5$, and the five [3]'s $k_1, \ldots, k_5$, and the ten points $A_{12}, \ldots, A_{45}$ $(A_{\rho\sigma} = A_{\sigma\rho})$, where $(H_1, H_2, H_3) = A_{45}$. The typical [4] and [3] are

$$H_1 = [A_{34}, A_{42}, A_{23}, A_{25}, A_{35}, A_{45}]$$

and

$$k_1 = [A_{12}, A_{13}, A_{14}, A_{15}].$$

The ten points are arranged in five sets of six which are in syzygy. There can, however, be only three independent syzygies connecting them, and we have to find a form for the five syzygies such that any three of them imply the other two. A simple form can be found by considering the figure as the section of the double-five of [7]'s, h and k, in

* It follows that any [4], which meets the planes K in lines, passes through the point G, and thence, in [10], that if an [8] contains T_0 and meets the planes K in lines, it contains ∞^1 [3]'s S, which meet those five lines, and form a quartic fourfold, Q, and therefore that the lines actually lie in a [7]. A direct algebraic proof of the corresponding result for the dual figure in [6] is given later [§ 12 (iii)].

[10], by a [6] t_0, meeting all the [7]'s h in [4]'s. Suppose that the parameters of t_0 are (b_1, b_2, b_3, b_4), then the equations of the [4]'s $H_\rho = (h_\rho, t_0)$, are

$$\left.\begin{array}{ll} H_1, & x_2 = x_3 = x_4 = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots & \\ H_4, & w_1 = w_2 = w_3 \quad = 0 \\ H_5, & y_1+z_1+w_1 = x_2+z_2+w_2 = x_3+y_3+w_3 = x_4+y_4+z_4 = 0 \end{array}\right\}, \quad (12.1)$$

where the identical relations between the coordinates are

$$\left.\begin{array}{rrrr} \Sigma x+\Sigma y+\Sigma z+\Sigma w = 0 & & & \\ b_2x_2+b_3x_3+b_4x_4 = 0 & & & \\ \cdots \quad \cdots \quad \cdots \quad \cdots & & & \\ b_1w_1+b_2w_2+b_3w_3 \quad = 0 & & & \end{array}\right\}. \quad (12.2)$$

The coordinates of A_{45} are

$$x_4 : y_4 : z_4 = b_2-b_3 : b_3-b_1 : b_1-b_2$$

with all the rest zero, and the coordinates of the other points A are of a similar nature. If we write

$$p_{\rho\sigma} = b_\rho - b_\sigma$$

and

$$p_{\rho 5} = -p_{5\rho} = b_\rho,$$

the signs of the coordinates of the points A can be arranged so that the syzygies between them are

$$I_\tau \equiv \sum_{\rho\sigma} p_{\tau\rho}p_{\tau\sigma}A_{\rho\sigma} = 0, \quad (12.3)$$

where $\rho, \sigma, \tau = 1, 2, 3, 4, 5$, and no two are equal. I_1, *e.g.*, is

$$p_{13}p_{14}A_{34}+p_{14}p_{12}A_{42}+p_{12}p_{13}A_{23}+p_{12}p_{15}A_{25}+p_{13}p_{15}A_{35}+p_{14}p_{15}A_{45}.$$

There are five relations, of which two are independent, connecting these, namely

$$I_1/p_{12}p_{13}p_{14}+I_2/p_{21}p_{23}p_{24}+I_3/p_{31}p_{32}p_{34}+I_4/p_{41}p_{42}p_{43} \equiv 0, \text{ etc.}$$

There is a set of twenty points $B_{\rho\sigma}$ $(\neq B_{\sigma\rho})$, which is of considerable importance in the configuration, given by

$$B_{\rho\sigma} = p_{\sigma\lambda}A_{\rho\lambda}+p_{\sigma\mu}A_{\rho\mu}+p_{\sigma\nu}A_{\rho\nu}, \quad (12.4)$$

where $(\rho, \sigma, \lambda, \mu, \nu)$ is some permutation of (1, 2, 3, 4, 5). We have

$$I_2 = (p_{23}p_{24}A_{34}+p_{24}p_{25}A_{45}+p_{23}p_{25}A_{35})+p_{21}B_{12} = 0,$$

and the point whose symbol is in the round brackets lies in H_1, while B_{12} lies in k_1, so that B_{12}, and similarly B_{13}, B_{14}, B_{15}, lie on the line $l_1 = (H_1, k_1)$. Any three $B_{\rho\sigma}$ points having a common first suffix are therefore in syzygy. The typical syzygy is

$$p_{\mu\nu}B_{\rho\lambda}+p_{\nu\lambda}B_{\rho\mu}+p_{\lambda\mu}B_{\rho\nu} = 0. \qquad (12.5)$$

(ii) *The fifth associated plane in the* [4]*'s* H*, and the* [5] *containing them.* The [4] H_1, is met by the other four [4]'s H in the planes

$$[A_{45}, A_{53}, A_{34}], \quad [A_{45}, A_{52}, A_{24}], \quad [A_{25}, A_{53}, A_{32}], \quad [A_{23}, A_{34}, A_{42}],$$

and by the [3]'s $k_2, \ldots, k_5$ in the planes

$$[A_{23}, A_{24}, A_{25}], \quad [A_{32}, A_{34}, A_{35}], \quad [A_{42}, A_{43}, A_{45}], \quad [A_{52}, A_{53}, A_{54}].$$

These two sets of planes form a double-four, *i.e.* any member of one set meets three members of the other in lines. They have a common fifth associated plane, namely that through the four points of intersection of pairs of corresponding planes from the two sets. Call this ϵ_1. The point B_{21} lies in the plane $[A_{23}, A_{24}, A_{25}]$, and, since

$$I_1 \equiv p_{14}p_{15}A_{45}+p_{15}p_{13}A_{53}+p_{13}p_{14}A_{34}+p_{12}B_{21} = 0,$$

it also lies in the plane $[A_{45}, A_{53}, A_{34}]$, which is the corresponding plane of the other half of the double-four. The fifth associated plane, ϵ_1, therefore contains the points B_{21}, B_{31}, B_{41}, B_{51}. The relation connecting these four is

$$p_{21}B_{21}+p_{31}B_{31}+p_{41}B_{41}+p_{51}B_{51} \equiv -2I_1 = 0.$$

Now the line $l_1 = (H_1, k_1)$ meets the four planes in which H_1 is met by $H_2, \ldots, H_5$, which form one-half of the double-four, and it must therefore meet the associated plane ϵ_1. The point of incidence is actually

$$B_{11} = p_{12}A_{12}+p_{13}A_{13}+p_{14}A_{14}+p_{15}A_{15}.$$

We have now enough information to show that the twenty-five points $B_{\rho\sigma}$ (including $B_{\rho\rho}$) all lie in a [5]. There are five planes ϵ_ρ containing fives of them (ϵ_ρ containing those with second suffix ρ), and five lines l_ρ containing fives of them (l_ρ containing those with first suffix ρ). No three of the five points in any of the planes are collinear, so that the [5] containing three of the lines contains all the planes, and therefore the other two lines. This [5], g, is the dual of the point G found in

§ 9. Four general lines of [5] have ∞^1 transversal planes, which form a cubic variety. Among the planes transversal to l_1, ..., l_4 are the five ϵ_ρ, so that l_5, which meets those five planes, contains five points of the cubic variety, and therefore lies entirely on it. There are ∞^2 lines meeting the five planes ϵ_ρ, and ∞^1 planes meeting the five lines l_ρ.

In the figure in [10] there is a [5] W, which meets all the [6]'s, s and t, in planes. In particular there is a plane (t_0, W). This plane can be shown to contain the points $A_{12}+A_{13}+A_{14}+A_{15}$, etc., of the lines l_1, etc., and it therefore lies on the cubic locus.

(iii) *The lines which meet the five* [3]*'s* k. The lines which meet four [3]'s in [6] generally form a sextic threefold* (∞^2 lines), which meets each of the [3]'s in a quadric surface. The locus is considerably specialized in the case of the lines meeting the four [3]'s, k_1, ..., k_4, because these have a [4] which meets them all in planes, and the lines which lie in this [4] and meet the planes form part of the locus. Part of the quadric in the [3] k_ρ is therefore the plane (k_ρ, H_5). Since there are still only ∞^2 lines in the locus, the rest of its intersections with the [3]'s must be planes. These planes will lie in a [5], since there are ∞^2 lines meeting them. The locus consists therefore of two parts, both cubic (so that the order of the whole variety is six as usual), one lying in a [5] and the other in the [4] H_5. The [5] has a [3] common with H_5, and the two parts of the locus have a quadric common in this [3].

In the same way the lines meeting k_2, ..., k_5 split up into two families, one lying entirely in the [4] H_1, and the other in a [5]. Symmetry indicates that the [5]'s obtained from k_1, ..., k_4, and from k_2, ..., k_5, coincide, and that all the lines which meet k_1, ..., k_4, and which do not lie in H_5, also meet k_5. This is confirmed by the algebra, to which we now proceed.

Suppose that the points

$$\alpha_2 A_{12}+\alpha_3 A_{13}+\alpha_4 A_{14}+\alpha_5 A_{15} \text{ in } k_1,$$

and

$$\beta_1 A_{21}+\beta_3 A_{23}+\beta_4 A_{24}+\beta_5 A_{25} \text{ in } k_2,$$

define a line which meets k_3 and k_4. By referring to the syzygies, $I_\tau = 0$, which connect the points $A_{\rho\sigma}$, it can be shown that

$$\beta_1 p_{45} = \lambda(\alpha_2 p_{54}+\alpha_4 p_{25}+\alpha_5 p_{42}), \quad \beta_4 p_{15} = \lambda\alpha_4 p_{25}, \quad \beta_5 p_{14} = \lambda\alpha_5 p_{24},$$

$$\beta_1 p_{35} = \mu(\alpha_2 p_{53}+\alpha_3 p_{25}+\alpha_5 p_{32}), \quad \beta_3 p_{15} = \mu\alpha_3 p_{25}, \quad \beta_5 p_{13} = \mu\alpha_5 p_{23}.$$

The β are determined uniquely in terms of the α, and, further, there is

* The order of the variety is given in Schubert's notation as $(3, 6)^5$, and this is equal to six.

one relation between the a : if $a_5 \neq 0$, then

$$\frac{a_2 p_{21}}{p_{23} p_{24} p_{25}} + \frac{a_3 p_{31}}{p_{32} p_{34} p_{35}} + \frac{a_4 p_{41}}{p_{42} p_{43} p_{45}} + \frac{a_5 p_{51}}{p_{52} p_{53} p_{54}} = 0,$$

i.e. one family of lines meets k_1 in points of a plane κ_1, and meets k_2, k_3, k_4 in similar planes κ_2, κ_3, κ_4, and the other family, given by $a_5 = 0$, which implies $\beta_1 = 0$, lies in the [4] H_5, being actually the family which meets the four planes (H_5, k_ρ).

The equation of the plane κ_1 (treating a_2, a_3, a_4, a_5 as coordinates in the [3] k_1) is completely symmetrical in the suffixes 2, 3, 4, 5, and therefore the lines which meet the [3]'s, k_1, k_2, k_3, k_5, meet k_1 in the points of the same plane. Similarly they meet k_2 and k_3 in points of κ_2 and κ_3. The two families of lines therefore coincide entirely, and we get the theorem that was indicated by pure geometry.

All lines which meet four of the [3]*'s of a double-five of* [4]*'s and* [3]*'s in* [6]*, and which do not lie in the* [4] *which meets those four* [3]*'s in planes, meet the fifth* [3]*. The lines meet the* [3]*'s in points of planes, and then form a cubic variety in a* [5].

The coordinates of B_{12} referred to A_{12}, A_{13}, A_{14}, A_{15}, are $(0, p_{23}, p_{24}, p_{25})$. These satisfy the equations of the plane κ_1, so that B_{12}, and similarly B_{13}, B_{14}, B_{15}, lie in κ_1. That is, the line l_ρ lies in the plane κ_ρ. From this it can be proved at once that the [5] in which the planes κ_ρ lie is g. For the lines l_ρ lie both in it and in g, and if it were different from g, they, and therefore the planes ϵ_ρ, would have to lie in a [4].

iv. *The* [3]*'s S, which meet in lines the five* [3]*'s k.* The planes t, which meet in lines the five [4]'s H, have no properties noticeably easier to find than their duals, but there is one property of the [3]'s S which meet in lines the five [3]'s k, whose analogue in the dual configuration cannot be expressed at all neatly, namely,

The [3]*'s which meet in lines the five* [3]*'s of a double-five of* [4]*'s and* [3]*'s in* [6] *do so in the lines of tetrahedral complexes.*

It is sufficient to show that the cross ratio of the points in which a line of the system meets the planes (k_1, H_2), ..., (k_1, H_5) is constant; this is a matter of straightforward algebra.

This property makes the duals of the planes s easier to picture; in particular, we can now get a fairly simple idea of the pairs of [3]'s, S', S'', which lie together in a [5] not containing any of the [3]'s k. These are the duals of pairs of planes s, which have a common point not on a plane, K. Such a pair of [3]'s has a common line (S', S''), which

meets all the [3]'s k. This line must be one of the family f, considered in 12 (iii), which lies in the [5] g, and meets the [3]'s k, in points of the planes $\kappa_\rho = (k_\rho, g)$. S' meets k_1 in a line, and therefore meets κ_1 in a point, and the line f through this point, lies in it. Thus all the [3]'s S, which meet S' in lines, do so in the same line, one of the family f, because only one line of that family passes through a point of any of the planes κ_1. The [3]'s S through the point (κ_ρ, f) meet k_ρ in the lines of the tetrahedral complex through the point, *i.e.* k_ρ meets the [1]-cone of [3]'s S, through a line f, in a quadric [0]-cone. From the dual figure we know that the [1]-cone is cubic. Two lines of the tetrahedral complex pass through (κ_ρ, f), and lie in k_ρ, so that the [5] g contains two of the [3]'s S through the line f. Its complete intersection with the cubic cone is these two [3]'s S, and the [3] through f which meets all the generators of the cone in planes.

The second family of [3]'s Z, which meets the [3]'s k in lines consists of the [3]'s which contain pairs of the lines f. If f' and f'' are the lines in any one of these, it cuts k_ρ in the line $[(k_\rho, f'), (k_\rho, f'')]$, *i.e.* a line of κ_ρ. It contains ∞^1 lines f, these forming a quadric surface. There are ∞^2 of these [3]'s Z, forming the family of secant [3]'s of the cubic threefold of lines f. They all lie in the [5] g, and one of them passes through any point of it.

The [3]'s common to the two families, S and Z, cut the [3] k_ρ, in the lines common to the tetrahedral complex, and the plane κ_ρ, *i.e.* in lines which envelop a conic; they form, therefore, a fourfold dual to the ruled quartic surface in [5]. The [3]'s Z, which meet a given [3] S in a line, must do so in the line f in the S. They are therefore the secant [3]'s of the cubic threefold which pass through one of the lines, and form a quadric [1]-cone.

Note on the construction of some known double-configurations.

The double-fives that have been considered in this paper can be constructed directly from incidence relations, and have therefore, intrinsically, not much interest, but there can be obtained from them by projection two other double-configurations, in which comparatively few of the elements can be taken arbitrarily, and in which the intersections of the remaining elements depend on axioms more limiting to the geometrical systems possible than the simple incidence relations. They are the double-twenty of planes in [5], and the double-ten of lines and planes in [4]*.

* These are special cases of figures considered in *Proc. Royal Soc.* (A), 111 (1926), 388.

I. *The double-twenty of planes in* [5].

(i) *The projection of the variety of* [3]*'s meeting five planes in* [10] *from a general* [4]. The variety Ω of [3]'s, S and T, meeting the sets of five planes, K and H respectively, of a double-five in [10], was shown to be of order fifteen (§ 3), that is, any [4] meets fifteen of the [3]'s S, and fifteen of the [3]'s T. The projection of the variety from a general [4] on to a general [5] consists of two ∞^3 families of [3]'s, in each of which there are twenty planes, namely the projections, K' and H', of the planes, K and H; and S' and T', of the [3]'s S and T, which meet the projecting [4]. K' meets four of the projected planes H', and all the planes, S', *i.e.* nineteen of the planes of the other family. A plane, T', meets all five planes H', and all the planes S', except that which is the projection of the S which passes through the same point of the projecting [4], as does the T of which T' is the projection. The two sets of twenty planes are therefore such that each member of either set meets all but one of those of the other, *i.e.* they form a "double-twenty".

The theorem for the dual configuration in [10], corresponding to "any [4] meets fifteen of the [3]'s S" is "any [5] Σ meets fifteen of the [6]'s s in planes". Σ also meets each of the [7]'s k in a plane. The section by Σ of the quartic variety ω consists therefore of two ∞^3 families of lines, in each of which a set of twenty planes is included, these two sets, by reasoning similar to that above, forming a double-twenty. The section is actually a general quartic determinantal fourfold in the [5], since the equation of one of these can always be reduced to the form

$$\begin{vmatrix} 0 & x_2 & x_3 & x_4 \\ y_1 & 0 & y_3 & y_4 \\ z_1 & z_2 & 0 & z_4 \\ w_1 & w_2 & w_3 & 0 \end{vmatrix} = 0,$$

where the x, y, z, w are linear functions of the coordinates*.

(ii) *The construction of the double-twenty from a certain nine planes.*

It is now to be shown that certain planes can be taken arbitrarily in Σ, and a double-twenty constructed that contains them; that is, in consequence of the above, that a double-five of [7]'s in a [10] containing Σ can be found, whose section by Σ includes the given planes. Such planes are the nine $a_1, \ldots, a_5$, and $b_1, \ldots, b_4$, of which pairs a_ρ and b_σ have

* Cf. *Proc. Royal Soc., ibid.*, 391.

a common point if $\rho \neq \sigma$, but no other pairs meet*. In any [10] through Σ draw a [7] a_5, which passes through the plane a_5, and in it take four [5]'s, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, which pass respectively through the points (a_5, b_1), (a_5, b_2), (a_5, b_3), (a_5, b_4). $(\lambda_\rho, \lambda_\sigma)$ is a [3], and $(\lambda_\rho, \lambda_\sigma, \lambda_\tau)$ is a line. Call the line $(\lambda_1, \lambda_2, \lambda_3)$, c_4, etc., and denote by β_ρ the [7] $[\lambda_\rho, b_\rho]$. Through c_4 draw the [9]'s, $[\beta_1, a_4]$, $[\beta_2, a_4]$, $[\beta_3, a_4]$. These meet in a [7] a_4, which meets $\beta_1, \beta_2, \beta_3$ in [5]'s, and contains the plane a_4. Construct similarly the [7]'s, a_1, a_2, a_3. Then a_1, a_2, a_3, a_4, a_5 are five [7]'s having no special intersection, and $\beta_1, \beta_2, \beta_3, \beta_4$ are [7]'s meeting sets of four of these in [5]'s. There will be a [7] β_5, which meets the last set of four, a_1, a_2, a_3, a_4, in [5]'s. The ten [7]'s, α and β, form a double-five, whose section by Σ includes the nine given planes, and a double-twenty of planes can therefore be found which includes these. The double-twenty is unique for a given [7] a_5, through the plane a_5, and given [5]'s λ_ρ in it; further, by an application of an extension of Desargues' Theorem, it is independent of the [7] and [5]'s chosen. For, if a_5 and a_5' are two [7]'s having a common plane, and $\lambda_\rho, \lambda_\rho'$ are sets of four [5]'s in them, such that λ_ρ meets the plane (a_5, a_5') in the same point as λ_ρ' does, then the four [3]'s, $[(\lambda_1, \lambda_2, \lambda_3), (\lambda_1', \lambda_2', \lambda_3')]$, etc., are concurrent, say in O, and thus the figure in the [10] $[\Sigma, a']$, is the projection from O, of the figure in the [10] $[\Sigma, a]$.

II. *The double-ten of lines and planes in* [4].

(i) *The section of the double-five of* [4]*'s and* [3]*'s in* [6] *by a general* [4]*, and the construction of the double-ten using this figure.*

By exactly the same method as in the last section [1(i)] it can be shown that the section of the planes which meet in lines the five [4]'s of a double-five in [6], by a [4], consists of ∞^2 points, among which there are ten lines (including the sections of the [3]'s of the double-five), and that the section of the [3]'s which meet in lines the five [3]'s consists of ∞^3 lines, among which there are ten planes (including the sections of the [4]'s of the double five), and that these ten lines and ten planes form a double-ten. The points, in fact, form a general determinantal sextic surface in [4], and the ∞^3 lines are its trisecants†.

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* The freedom of this set of planes (nine, with sixteen simple incidences) is sixty-five, and this is the freedom of the general quartic determinantal fourfold in [5], on which the double-twenty is known to lie. Further, two quartic varieties cannot contain the same double-twenty, so that it is reasonable to suppose that we can construct a finite number of double-twenties containing the nine planes.

† Cf. Bordiga, *Atti R. Accad. Linc.* (Mem.) IV, 4a (1887), 182; and White, *Proc. Camb. Phil. Soc.*, 21 (1922), 220.

It has to be shown that five arbitrary planes, and three lines which meet different sets of four of them, can be taken in any [4] Σ, and a double-five of [4]'s and [3]'s in a [6] constructed, whose intersection with Σ includes these planes and lines. Then at least one double-ten contains these planes and lines, and, by an argument similar to that used at the end of 1(ii), there is only one.

The double-five of [4]'s and [3]'s in [6] is most easily constructed on the ten points, $A_{\rho\sigma}$, in which sets of three of the [4]'s and pairs of the [3]'s intersect [§ 12(i)], and it is to determine these that we first proceed. Let the five planes be $b_1, \ldots, b_5$, and the three lines, a_1, a_2, a_3, where a_ρ meets all b_σ except b_ρ. Through the point (b_4, b_5) draw any plane b, having no further intersection with Σ. $[\Sigma, b]$ is a [6]. In b take any three points A_{23}, A_{31}, A_{12}. Then, in the [4] $[b, b_4] = \beta_4$, we have the three sets of three points A_{23}, A_{31}, A_{12}; $(b_4, a_1), (b_4, a_2), (b_4, a_3)$; and $(b_4, b_1), (b_4, b_2), (b_4, b_3)$. We want to find points A_{15}, A_{25}, A_{34}, in β_4, lying in the planes

$$[(a_1, b_4), A_{12}, A_{13}], \quad [(a_2, b_4), A_{21}, A_{23}], \quad \text{and} \quad [(a_3, b_4), A_{31}, A_{32}]$$

respectively and such that the lines $[A_{25}, A_{35}]$, $[A_{35}, A_{15}]$, and $[A_{15}, A_{25}]$ meet the lines $[(b_1, b_4), A_{23}]$, $[(b_2, b_4), A_{31}]$, and $[(b_3, b_4), A_{12}]$ respectively. The line $[A_{25}, A_{35}]$ lies in the [3] $[(a_2, b_4), (a_1, b_4), A_{21}, A_{23}]$, and A_{35} therefore lies on its intersection with the plane, $[(a_3, b_4), A_{31}, A_{32}]$; that is, on a definite line through A_{23}. Similarly it lies on a definite line through A_{13} in the plane. These two lines do not coincide, so that the point A_{35} is uniquely determined. Similarly A_{15} and A_{25} are determinate points, and we can find a similar set of three points A_{14}, A_{24}, A_{34} in $[b, b_4] = \beta_4$.

These nine points A, which have been found determine a double-five of [4]'s and [3]'s, part of whose intersection with Σ is the given planes and lines. Take

$$a_1 = [A_{12}, A_{13}, A_{14}, A_{15}], \quad a_2 = [A_{21}, A_{23}, A_{24}, A_{25}], \quad a_3 = [A_{31}, A_{32}, A_{34}, A_{45}].$$

a_1 contains the points (b_4, a_1) and (b_5, a_1), and therefore contains the line a_1. Similarly, a_3 and a_3 contain a_2 and a_3.

$$\beta_1 = [A_{23}, A_{24}, A_{25}, A_{34}, A_{35}], \quad \beta_2 = [A_{13}, A_{14}, A_{15}, A_{34}, A_{35}],$$

and

$$\beta_3 = [A_{12}, A_{14}, A_{15}, A_{24}, A_{25}].$$

β_4 and β_5 we already have: they contain the sets of six points not having suffixes "4" and "5" respectively. β_1 contains the points (b_1, b_4) and (b_1, b_5), and, since a_2 and the plane $[A_{23}, A_{24}, A_{25}]$ both lie in the [3] a_2, it meets the line a_2. $[\beta_1, a_2]$ is therefore a [5]. It contains three points of

b_1, and, these not being collinear, it contains b_1. Similarly $[\beta_1, a_2]$ contains b_1. These two [5]'s are distinct, and therefore their intersection, which is β_1, contains b_1. Similarly β_2 and β_3 contain b_2 and b_3, while β_4 and β_5 were constructed to contain b_4 and b_5. Take $(\beta_1, \beta_2, \beta_3) = A_{45}$, and $a_4 = [A_{41}, A_{42}, A_{43}, A_{45}]$, and $a_5 = [A_{51}, A_{52}, A_{53}, A_{54}]$, and we then have a complete double-five of [4]'s and [3]'s in [6], which is cut by Σ in a figure which includes the given planes and lines.

II (ii) *The construction of the double-ten of planes and lines in* [4] *from certain planes and lines without going outside the* [4].

The following construction has no necessary connection with the rest of the paper, but it involves several new properties of the double-ten, and is historically the first way in which a double-ten was constructed from given elements. We require two preliminary theorems:

1. *If five planes in* [4] *are met by five lines in points such that those in one of them form, with its points of intersection with the other four planes, a set of nine associated points, then the same is true of the nine similar points in any of the planes.*

Five general planes in [4] are met by ∞^1 lines. The lines meet any of the planes in points of a cubic curve, and form a quintic surface F^*. There is an ∞^1 family of planes, of which the given five are ordinary members, meeting all these lines in points of cubic curves, and these form a quintic threefold V. Two planes of V pass through any point of F, so that F is a double locus on V. No two lines of F intersect, and there are no intersections of pairs of planes of V which do not lie on F.

Take five planes of V, which are not associated, to be K_0, K_1, K_2, K_3, K_4, and five non-associated lines of F to be h_5, h_6, h_7, h_8, h_9. To establish the theorem above, we require two lemmas:

LEMMA A. *If K_a is the plane associated with the planes K_0, K_1, K_2, K_3 and h_a is the line associated with h_5, h_6, h_7, h_8, then the nine points of intersection of K_0 with h_5, h_6, h_7, h_8, h_a, and K_1, K_2, K_3, K_a are associated.*

Take any plane K, which meets h_5, h_6, h_7, h_8. Then K also meets h_a; and h_5, h_6, h_7, h_8, h_a, since they meet K_0, K_1, K_2, K_3, also meet K_a. The lines which meet all of K_0, K_1, K_2, K_3, and K, meet K_0 in points of a cubic curve, which passes through its points of intersection with K, K_1, K_2, K_3, K_a, h_5, h_6, h_7, h_8, h_a. But one plane K, to meet h_5, h_6, h_7,

* Segre, *Rend. di Palermo*, 2 (1888), 51.

h_8, can be drawn through any point of K_0. That is, the required nine points lie on a cubic curve with any other point of the plane. They must therefore be associated.

Lemma B. *If K_1, K_2 are any two planes of V, and two of the lines of F meet them in X_1, X_2; Y_1, Y_2 respectively, and the lines $[X_1, Y_1]$, $[X_2, Y_2]$ meet the surface again in P_1, P_2, then the other plane of V through P_1 passes through P_2.*

This is actually equivalent to: a [3] through two lines of F cuts F further in a cubic curve in one of the planes of V, and a [3] through a plane of V cuts F in the cubic curve of that plane, and further in two lines. This theorem is given by Segre.

We can now prove the main theorem. Denote the point (K_i, K_j) by ij, and the point (K_i, h_j) by ij', where i, j may be 0, 1, ..., 9, or a, and let γ_i be the cubic curve in which K_i meets the surface F defined by the given planes. Suppose that 01, 02, 03, 04, 05′, 06′, 07′, 08′, 09′, are associated. Then also

$$01, 02, 03, 0a, 05', 06', 07', 08', 0a', \text{ and } 10, 12, 13, 1a, 15', 16', 17', 18', 1a'$$

are associated sets. Since $0a$, $0a'$, and 04, 09′, with the same set of seven points of γ_0, complete sets of nine associated points, their joins $[0a, 0a']$ and $[04, 09']$ must meet in a point of γ_0, say, $0p$. Let the residual points of intersection of $[1a, 1a']$, and $[14, 19']$ with γ_1, be $1p$ and $1p'$. Then, by Lemma B, both $[0p, 1p]$ and $[0p, 1p']$ must be lines of F. But no two lines of F intersect, and therefore these two lines must coincide, and $1p = 1p'$. That is, $[1a, 1a']$ meets $[14, 19']$ in a point of γ_1. The required nine points of K_1 are therefore associated.

2. *The eighteen points in which one of the planes of a double-ten is met by the remaining planes and the nine secant lines, lie on a cubic curve, and the nine points in which it is met by any four of the planes, and the five lines which meet all of these and it, are associated.*

The first part of the theorem follows at once from the connection of the double-ten with the determinantal sextic surface*. For the second we may proceed thus: take the planes of the double-ten to be K_0, K_1, ..., K_9, and the lines to be h_0, h_1, ..., h_9. Denote the point (K_i, K_j) by ij, and the point (K_i, h_j) by ij', and the cubic curve in K_i by c_i. We have to show that a set of points such as 01, 02, 03, 04, 05′, 06′, 07′, 08′, 09′ is associated. Suppose, then, that this set is not

* Bordiga, and White, *loc. cit.*

associated, that is, that the cubic curve through them is unique: it must be c_0. c_0 is therefore the cubic curve in which the quintic surface F, of lines meeting K_0, K_1, K_2, K_3, K_4, meets K_0. K_5 meets h_6, h_7, h_8, h_9, and also meets K_0 in a point of c_0, namely 05, *i.e.* meets five lines of the surface F. It can be assumed that the five lines are not associated, for, if they were, we could use K_6 instead of K_5. K_5, then, meets five general lines of the surface F, and therefore meets all of them. But K_5 does not meet h_5 in the double-ten configuration. Thus the required set of nine points must be associated, and, by Theorem 1, all such sets are associated.

Take now u_i to be the elliptic parameter of the point $0i$ on the curve c_i, and v_i that of $0i'$. The result about the nine associated points above can then be expressed

$$u_1+u_2+u_3+u_4+v_5+v_6+v_7+v_8+v_9 \equiv 0 \pmod{\text{period}}.$$

We have also

$$u_1+u_2+u_3+u_5+v_4+v_6+v_7+v_8+v_9 \equiv 0,$$

and therefore

$$u_4-v_4 \equiv u_5-v_5,$$

and generally

$$u_i-v_i \equiv \text{constant}.$$

We can now proceed with the actual construction:

Suppose that we are given K_0, K_1, K_2, K_3, K_4, five general planes of [4], and h_1, h_2, h_3, three lines meeting different sets of four of them, and that we wish to construct the double-ten containing these*. We have first to construct the cubic curve c_0 in K_0. c_0 passes through the points 01, 02, 03, 04, 01′, 02′, 03′, and is such that the elliptic parameters of these points satisfy

$$u_1-v_1 \equiv u_2-v_2 \equiv u_3-v_3, \quad i.e.$$

$$u_1+v_2 \equiv u_2+v_1, \quad \text{etc.},$$

which means that [01, 02′] and [01′, 02] meet on the curve. c_0, therefore, passes through the points

([02, 03′], [03, 02′]), ([03, 01′], [01, 03′]), and ([01, 02′], [02, 01′]),

that is, altogether through nine associated points, and a tenth point, 04. c_0, and similarly c_4, can be determined from the data. Meeting all five

* The freedom of the figure is thirty-six, the same as that of the determinantal sextic surface, and therefore presumably as that of the double-ten. We can expect to be able to construct a finite number of double-tens containing these planes and lines. The construction given shows that there is actually one.

of the given planes there are five lines h_5, h_6, h_7, h_8, h_9, of the double-ten. These lie on the surface F or lines meeting the given planes, which meets K_0 in a cubic curve different from c_0. Four of the intersections of this curve with c_0 are 01, 02, 03, 04, and the others must therefore be $05'$, $06'$, $07'$, $08'$, $09'$. h_5, h_6, h_7, h_8, h_9 are therefore determined. h_4 meets K_0 in a known point, namely that whose elliptic parameter is given by

$$u_4 - v_4 \equiv u_1 - v_1,$$

and has to meet K_1, K_2, K_3, so that h_4, and similarly h_0 (from K_4), have now been fixed. We know the elliptic parameters of all nine points $0i'$ in K_0, and can therefore find those of $0i$, and, then knowing a point, 0, we can draw through it the plane K_i (different from K_0) to meet four suitable lines. By applications of the two theorems proved, K_i can be shown to meet the other five required lines.

The construction of a double-ten can also be carried out by starting with six arbitrary planes, say K_0, K_1, ..., K_5. There are five associated lines meeting all these; select any four to be h_6, h_7, h_8, h_9. The nine points 01, 02, 03, 04, 05, $06'$, $07'$, $08'$, $09'$ in K_0 are not associated, and determine c_0, *i.e.* the cubic curves c in the six planes are known. Then a set of points such as 01, 02, 03, 04, $05'$, $06'$, $07'$, $08'$ $09'$ is associated, so that the point $05'$, and, similarly, the points $01'$, $02'$, $03'$, $04'$, and $10'$ are known, *i.e.* the remaining six lines are known. The other four planes are then found as before. Thus any six planes are included in five double-ten configurations.

ON THE "LAW OF LARGE NUMBERS"

By E. H. LINFOOT.

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1.1. *Introductory.*—Let p_i be an infinite sequence of numbers such that $0 \leqslant p_i \leqslant 1$ for $i = 1, 2, 3, \ldots, \infty$. Suppose that n trials are made, the result of each trial being either a "hit" (T) or a "miss" (U); let the probability of a hit at the i-th trial be p_i, of a miss q_i. Thus $p_i + q_i = 1$.

The number of hits in the n trials will be denoted by $m(n)$; the mathematical expectation of this number is, of course, $\sum_1^n p_i$. If, then, we write

$$m(n) = \mu(n) + \sum_1^n p_i,$$

we may call the number $\mu(n)$ so defined the "deviation".

With $m(n)$ is associated a number t, defined by

$$m(n) = \sum_1^n p_i + t\sqrt{\left(\sum_1^n 2p_iq_i\right)}\,; \tag{1}$$

evidently

$$t = \frac{\mu(n)}{\sqrt{\left(\sum_1^n 2p_iq_i\right)}}.$$

The probability that in the n trials there will be exactly m hits will be denoted by $P(m, n)$, the probability that the number m of hits will lie between m_1 and m_2 inclusive by $P(m_1 \leqslant m \leqslant m_2)$, or $P(m_1, m_2)$, or $P(t_1, t_2)$, where t_1 and t_2 are given by (1).

The question naturally arises: What can be asserted about the order $\mu(n)$? Certainly $\mu(n)$ may take all values from $-\sum_1^n p_i$ to $\sum_1^n q_i$; but it is

well known, for instance, that

$$P\left\{t_1\sqrt{\left(\sum_1^n 2pq\right)} \leqslant \mu(n) \leqslant t_2\sqrt{\left(\sum_1^n 2pq\right)}\right\} \sim \frac{1}{\sqrt{\pi}}\int_{t_1}^{t_2} e^{-t^2}\,dt,$$

as $n\to\infty$, where $t_1 < t_2$ are constants and $\sum_1^n 2pq\to\infty$ as $n\to\infty$ ("Poisson's Formula").

This problem has, of course, been treated many times, notably by Khintchine, *Math. Ann.* 96 (1926), 152, and by Kameda, *Journal Fac. Sci., Tokyo*, 1925. Khintchine proves a very elegant result for the order of $\mu(n)$, on the assumption, however, that for all $i = 1, 2, \ldots$, we have $p_i, q_i \geqslant a > 0$; in a paper to appear shortly* I extend his results to the most general case $\sum_1^n 2pq\to\infty$. Kameda, using the method of generating functions, obtains the first five terms of the expansions of $P(m, n)$ and of $P(m_1, m_2)$ in a series of Hermite polynomials. He discusses the convergence of such an expansion and lays down sufficient conditions for its validity. The present paper embodies results of a rather different type. In the generalization of Khintchine's results referred to above it is necessary to have estimations of $P(m, n)$, $P(m_1, m_2)$ which shall be effective—

(1) uniformly in the p_i, provided only that $\Sigma 2p_iq_i\to\infty$ with n;

(2) uniformly in t_1, t_2, subject to a condition of the form

$$|t_1|,\ |t_2| \leqslant \left(\sum_1^n 2pq\right)^{\frac{1}{12}}.$$

These estimations the present paper supplies. They are obtained by the direct investigation of a Cauchy integral, and the same method will give the Hermite expansions to any number of terms and with a definite error term. In this respect and in simplicity it appears to have advantages over those previously employed.

My thanks are due to Prof. Besicovitch for much valuable assistance and advice.

1.2. Suppose that the probabilities p_i are restricted only by the condition

$$\sum_1^n 2p_iq_i\to\infty \text{ as } n\to\infty.$$

* *Phil. Trans. Royal Soc.* (1928).

The probability $P(m, n)$ that exactly m hits will occur in n trials is seen to be the coefficient of z^m in the expansion of

$$\prod_{i=1}^{n} (p_i z + q_i).$$

We begin by finding an approximate expression for this coefficient. It will certainly be the case that, t being fixed,

$$P(m, n) \sim \frac{e^{-t^2}}{\sqrt{\left(\sum_1^n 2pq\right)}} \quad \text{as} \quad \Sigma 2pq \to \infty ;$$

we shall prove that

$$P(m, n) = \frac{1}{\sqrt{\pi}} \cdot \frac{e^{-t^2}}{\sqrt{(\Sigma 2pq)}} + \frac{1}{\sqrt{\pi}} \cdot \frac{\Sigma pq(p-q)}{3(\Sigma 2pq)^2} e^{-t^2}(3t - 2t^3)$$

plus a certain error term.

By Cauchy's theorem,

$$P(m, n) = \frac{1}{2\pi i} \int_C \frac{\prod_i^n (p_i z + q_i)}{z^{m+1}} dz, \tag{1.21}$$

where C is the circle $|z| = 1$.

In approximating to this integral, we notice that the modulus of the integrand is greatest at the point 1 (where it is 1), and falls off very rapidly on both sides of this point. We therefore divide the path of integration into two parts. Writing $z = e^{i\phi}$, we have

$$\int_C = \left(\int_{-\phi_0}^{\phi_0} + \int_{\phi_0}^{2\pi - \phi_0}\right) \frac{\Pi(pz+q)}{z^{m+1}} e^{i\theta} i d\theta,$$

where we choose $$\phi_0 = \left(\sum_1^n p_i q_i\right)^{-\frac{1}{2}+\epsilon} \qquad (0 < \epsilon < \tfrac{1}{2}).$$

It is next necessary to expand the integrand; we have

$$pe^{i\phi} + q = e^{p\phi i}(pe^{q\phi i} + qe^{-p\phi i})$$

$$= e^{p\phi i}\left[1 - \tfrac{1}{2} pq\phi^2 + \tfrac{1}{6} pq(p-q)\, i\phi^3 + \theta_1 \phi^4 \frac{pq(p^2+q^2)}{4!(1-|\phi|/5)}\right], \tag{1.211}$$

for $|\phi| \leqslant \frac{1}{2}$. Here and henceforth θ denotes a quantity such that $|\theta| < 1$.

Since $$\left|\frac{\phi^4 pq(p^2+q^2)}{4!\,(1-|\phi|/5)}\Big/\{1-\tfrac{1}{2}pq\phi^2+\tfrac{1}{6}pq(p-q)\,i\phi^3\}\right|$$

$$<\frac{\phi^4 pq}{24\cdot\frac{9}{10}}\Big/(1-\tfrac{1}{2}\cdot\tfrac{1}{4}\cdot\tfrac{1}{4})$$

$$<\tfrac{1}{20}\phi^4 pq,$$

it follows that

$$pe^{i\phi}+q=e^{p\phi i}\{1-\tfrac{1}{2}pq\phi^2+\tfrac{1}{6}pq(p-q)\,i\phi^3\}\left[1+\frac{\theta_2}{20}pq\phi^4\right].$$

Now, if $|x|<1$,

$$\log(1-x)=-x+\theta_3\frac{|x^2|}{2(1-|x|)}.$$

When $x=\tfrac{1}{2}pq\phi^2-\tfrac{1}{6}pq(p-q)\,i\phi^3$, we have

$$|x|<\tfrac{7}{12}pq^2<\tfrac{7}{192};$$

it follows that

$$\log\{1-\tfrac{1}{2}pq\phi^2+\tfrac{1}{6}pq(p-q)\,i\phi^3\}=-\tfrac{1}{2}pq\phi^2+\tfrac{1}{6}pq(p-q)\,i\phi^3+\frac{\theta_3}{1400}pq\phi^4.$$

Again, if $|x|<1$, $\quad\log(1+x)=\theta_4\dfrac{|x|}{1-|x|}$.

Thus $$\log\left(1+\frac{\theta_2}{20}pq\phi^4\right)=\frac{\theta_5}{20}pq\phi^4\cdot\frac{1280}{1279},$$

whence

$$pe^{i\phi}+q=\exp(0{\cdot}051\theta_6\phi^4pq)\exp\{pi\phi-\tfrac{1}{2}pq\phi^2+\tfrac{1}{6}pq(p-q)\,i\phi^3\},$$

$$\Pi(pe^{i\phi}+q)=\exp(0{\cdot}051\theta_6\phi^4\Sigma pq)$$

$$\times\exp\{i\phi\Sigma p-\tfrac{1}{2}\phi^2\Sigma pq+\tfrac{1}{6}i\phi^3\Sigma pq(p-q)\}. \quad (1.212)$$

Writing $m=\Sigma p+t\sqrt{(\Sigma 2pq)}$, we obtain

$$\frac{1}{2\pi i}\int_{-\phi_0}^{\phi_0}\frac{\Pi(pe^{i\phi}+q)}{e^{mi\phi}}\,id\phi$$

$$=\frac{1}{2\pi}\exp(0{\cdot}051\theta_6\phi_0^4\Sigma pq)$$

$$\times\int_{-\phi_0}^{\phi_0}\exp\{-i\phi t\sqrt{(\Sigma 2pq)}-\tfrac{1}{2}\phi^2\Sigma pq+\tfrac{1}{6}i\phi^3\Sigma pq(p-q)\}\,d\phi$$

$$=\frac{1}{2\pi}\exp(0{\cdot}051\theta_6\phi_0^4\Sigma pq)\exp[\tfrac{1}{36}\theta_9\phi_0^6\{\Sigma pq(p-q)\}^2]$$

$$\times\int_{-\phi_0}^{\phi_0}\exp\{-it\phi\sqrt{(\Sigma 2pq)}-\tfrac{1}{2}\phi^2\Sigma pq\}.\{1+\tfrac{1}{6}i\phi^3\Sigma pq(p-q)\}\,d\phi.$$

[since for $|x| \leqslant |x_0| < 1$, $1-|x_0| \geqslant 1/\sqrt{2}$,

$$e^x = 1+x+\frac{\theta_7|x_0|^2}{2(1-|x_0|)}$$

$$= (1+x)\left(1+\frac{\theta_8|x_0|^2}{2(1-|x_0|)^2}\right)$$

$$= (1+x)\,e^{\theta_9|x_0|^2}]$$

$$= \frac{1}{2\pi}\exp\{0.051\theta_6(\Sigma pq)^{-1+4\epsilon}+\tfrac{1}{36}\theta_9(\Sigma pq)^{-1+6\epsilon}\}$$

$$\times\int_{-\phi_0}^{\phi_0}\exp\{-it\phi\sqrt{(\Sigma 2pq)}-\tfrac{1}{2}\phi^2\Sigma pq\}\,\{1+\tfrac{1}{6}i\phi^3\Sigma pq(p-q)\}\,d\phi. \quad (1.213)$$

Suppose that $0 < \epsilon < \frac{1}{6}$. Then the exponent in the first factor can be written

$$\theta_{10}(\Sigma pq)^{-1+6\epsilon}(0{\cdot}051+\tfrac{1}{36})$$

$$= 0{\cdot}08\theta_{11}(\Sigma pq)^{-1+6\epsilon}.$$

Therefore

$$\exp\{0{\cdot}051(\Sigma pq)^{-1+4\epsilon}+\tfrac{1}{36}\theta_9(\Sigma pq)^{-1+6\epsilon}\} = 1+\tfrac{1}{10}\theta_{12}(\Sigma pq)^{-1+6\epsilon}. \quad (1.2131)$$

We have supposed that $\phi_0 \leqslant \frac{1}{2}$; our results therefore hold if $\Sigma pq \geqslant 8$.

Now
$$\int_{-\phi_0}^{\phi_0} e^{-it\phi\sqrt{(\Sigma 2pq)}}\,e^{-\frac{1}{2}\phi^2\Sigma pq}\,d\phi$$

$$= \int_{-(\Sigma pq)^\epsilon\sqrt{2}}^{(\Sigma pq)^\epsilon\sqrt{2}} e^{-itx}\,e^{-\frac{1}{4}x^2}\frac{dx}{\sqrt{(\Sigma 2pq)}}, \qquad \text{writing } \phi\sqrt{(\Sigma 2pq)} = x,$$

$$= \frac{1}{\sqrt{(\Sigma 2pq)}}\left(\int_{-\infty}^{\infty} e^{-itx}\,e^{-\frac{1}{4}x^2}\,dx - 2\int_{(\Sigma pq)^\epsilon\sqrt{2}}^{\infty} e^{-\frac{1}{4}x^2}\cos tx\,dx\right)$$

$$= \frac{1}{\sqrt{(\Sigma 2pq)}}\,(2\pi^{\frac{1}{2}}e^{-t^2}+\theta_{13}e^{-\frac{1}{2}(\Sigma pq)^{2\epsilon}}), \quad (1.214)$$

and
$$\int_{-\phi_0}^{\phi_0}\tfrac{1}{6}i\phi^3\Sigma pq(p-q)\,e^{-it\phi\sqrt{(\Sigma 2pq)}}\,e^{-\frac{1}{2}\phi^2\Sigma pq}\,d\phi$$

$$= \frac{\Sigma pq(p-q)}{6(\Sigma 2pq)^2}\int_{-(\Sigma pq)^\epsilon\sqrt{2}}^{(\Sigma pq)^\epsilon\sqrt{2}} ix^3\,e^{-itx}\,e^{-\frac{1}{4}x^2}\,dx$$

$$= \frac{\Sigma pq(p-q)}{6(\Sigma 2pq)^2}\left(\int_{-\infty}^{\infty} ix^3\,e^{-itx}\,e^{-\frac{1}{4}x^2}\,dx - 2\int_{(\Sigma pq)^\epsilon\sqrt{2}}^{\infty} x^3\sin tx\,e^{-\frac{1}{4}x^2}\,dx\right)$$

$$= \frac{4\Sigma pq(p-q)}{(\Sigma 2pq)^2}\left(\tfrac{1}{6}t(3-2t^2)\,\pi^{\frac{1}{2}}e^{-t^2}+\frac{\theta_{14}}{6\sqrt{2}}(\Sigma pq)^{3\epsilon}\,e^{-\frac{1}{2}(\Sigma pq)^{2\epsilon}}\right), \quad (1.215)$$

provided that $(\Sigma pq)^\epsilon \geqslant \sqrt{3}$.

Thus we have an approximate expression for $\int_{-\phi_0}^{\phi_0}$.

The integral over the remainder of the contour is absolutely less than

$$\frac{1}{2\pi}\int_{\phi=\phi_0}^{\phi=2\pi-\phi_0}\frac{|\Pi(pz+q)|}{|z^{m+1}|}|dz|$$

$$\leqslant |\Pi(pe^{i\phi_0}+q)|,$$

since the modulus of the integrand is greatest at ϕ_0.

$$= \exp(0{\cdot}051\theta_6\phi_0^4\Sigma pq-\tfrac{1}{2}\phi_0^2\Sigma pq), \quad \text{by (1.212)},$$

$$= \exp\{0{\cdot}051\theta_6(\Sigma pq)^{-1+4\epsilon}-\tfrac{1}{2}(\Sigma 2pq)^{2\epsilon}\}$$

$$< e^{0{\cdot}003}e^{-\frac{1}{2}(\Sigma pq)^{2\epsilon}}. \tag{1.216}$$

Thus finally

$$P(m,n) = \{1+\tfrac{1}{10}\theta_{12}(\Sigma pq)^{-1+6\epsilon}\}$$

$$\times\left\{\frac{1}{\sqrt{\pi}}\frac{e^{-t^2}}{\sqrt{(\Sigma 2pq)}}+\frac{1}{\sqrt{\pi}}\frac{\Sigma pq(p-q)}{3(\Sigma 2pq)^2}e^{-t^2}t(3-2t^2)\right.$$

$$\left.+\frac{\theta_{13}}{2\sqrt{\pi}.\sqrt{(\Sigma 2pq)}}e^{-\frac{1}{2}(\Sigma pq)^{2\epsilon}}+\frac{\theta_{15}}{12\pi\sqrt{2}}(\Sigma pq)^{-1+3\epsilon}e^{-\frac{1}{2}(\Sigma pq)^{2\epsilon}}\right\}$$

$$+\theta_{16}e^{0{\cdot}003}e^{-\frac{1}{2}(\Sigma pq)^{\epsilon}},$$

provided that $0<2<\frac{1}{6}$, $(\Sigma pq)^{\epsilon}\geqslant\sqrt{3}$, $\Sigma pq\geqslant 8$.

The second condition involves $\Sigma pq\geqslant 27$ and *a fortiori* the third, so that

$$0<\epsilon<\tfrac{1}{6}, \quad (\Sigma pq)^{\epsilon}\geqslant\sqrt{3}$$

are the conditions.

The error term is less than

$$\tfrac{1}{2}5(\Sigma pq)^{-\frac{1}{2}+6\epsilon}e^{-t^2}\left(\frac{|t(3-2t^2)|}{24}+1\right)+\tfrac{9}{8}e^{-\frac{1}{2}(\Sigma pq)^{2\epsilon}},$$

so that our final result is

$$P(m,n) = \frac{1}{\sqrt{\pi}}\frac{e^{-t^2}}{\sqrt{(\Sigma 2pq)}}+\frac{1}{\sqrt{\pi}}\frac{\Sigma pq(p-q)}{3(\Sigma 2pq)^2}e^{-t^2}t(3-2t^2)$$

$$+\theta_{17}\left[\tfrac{1}{2}5(\Sigma pq)^{-\frac{1}{2}+6\epsilon}e^{-t^2}\left(1+\frac{|t(3-2t^2)|}{24}\right)+\tfrac{9}{8}e^{-\frac{1}{2}(\Sigma pq)^{2\epsilon}}\right], \tag{1.22}$$

provided that $0<\epsilon<\frac{1}{6}$, $\Sigma 2pq\geqslant 2.3^{\frac{1}{2}\epsilon}>54$.

Here $|\theta_{17}|<1$ and t is defined by

$$m = \Sigma p+t\sqrt{(\Sigma 2pq)}.$$

1.3. We next consider the probability

$$P\{\Sigma p+t_1\sqrt{(\Sigma 2pq)} \leqslant m \leqslant \Sigma p+t_2\sqrt{(\Sigma 2pq)}\}.$$

It is

$$\frac{1}{\sqrt{\pi}}\sum_{t_1}^{t_2}\frac{e^{-t^2}}{\sqrt{(\Sigma 2pq)}}+\frac{\Sigma pq(p-q)}{3(\Sigma 2pq)^2}\sum_{t_1}^{t_2}e^{-t^2}t(3-2t^2)$$
$$+\theta_{18}\left[\tfrac{1}{2\cdot 5}(\Sigma pq)^{-\frac{3}{2}-6\epsilon}\sum_{t_1}^{t_2}e^{-t^2}\left(1+\frac{|t(3-2t^2)|}{24}\right)+\tfrac{9}{8}e^{-\frac{1}{2}(\Sigma pq)^{2\epsilon}}\sum_{t_1}^{t_2}1\right], \quad (1.31)$$

where the values of t are spaced $1/\sqrt{(\Sigma pq)}$ apart and run from t_1 to t_2.

Write $$h=\frac{1}{\sqrt{(\Sigma 2pq)}}.$$

From Taylor's formulae

$$f(x)=f(a)+(x-a)f'(a)+\tfrac{1}{2}(x-a)^2f''\{a+\theta(x-a)\},$$
$$f(x)=f(a)+(x-a)f'(a)+\tfrac{1}{2}(x-a)^2f''(a)+\tfrac{1}{6}(x-a)^3f'''(a)$$
$$+\tfrac{1}{24}(x-a)^4f^{\mathrm{iv}}\{a+\theta(x-a)\},$$

we have

$$\int_{a-\frac{1}{2}h}^{a+\frac{1}{2}h}f(x)\,dx=hf(a)+\frac{h^3}{24}f''(a+\tfrac{1}{2}\theta h) \quad (1)$$

$$\int_{a-\frac{1}{2}h}^{a+\frac{1}{2}h}f(x)\,dx=hf(a)+\frac{h^3}{24}f''(a)+\frac{h^5}{2^7.3.5}f^{\mathrm{iv}}(a+\tfrac{1}{2}\theta h). \quad (2)$$

Since

$$De^{-t^2}=-2te^{-t^2},$$
$$D^2e^{-t^2}=(4t^2-2)e^{-t^2},$$
$$D^3e^{-t^2}=(-8t^3+12t)e^{-t^2},$$
$$D^4e^{-t^2}=(16t^4-48t^2+12)e^{-t^2},$$

we have, by (2),

$$\frac{1}{\sqrt{\pi}}\sum_{t_1}^{t_2}\frac{e^{-t^2}}{\sqrt{(\Sigma 2pq)}}=\frac{1}{\sqrt{\pi}}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h}e^{-t^2}dt-\frac{1}{24\sqrt{\pi}}\sum_{t_1}^{t_2}\frac{(4t^2-2)e^{-t^2}}{\Sigma 2pq\sqrt{\Sigma 2pq}}$$
$$-\frac{1}{2^7.3.5\sqrt{\pi}}\frac{1}{(\Sigma 2pq)^2}\sum_{t_1}^{t_2}\{16(t+\tfrac{1}{2}\theta h)^4-48(t+\tfrac{1}{2}\theta h)^2+12\}e^{-(t+\frac{1}{2}\theta h)^2}. \quad (1.311)$$

Applying (1) to the second term on the right, we obtain

$$\frac{1}{\sqrt{\pi}}\sum_{t_1}^{t_2}\frac{e^{-t^2}}{\sqrt{(\Sigma 2pq)}}=\frac{1}{\sqrt{\pi}}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h}e^{-t^2}dt-\frac{1}{\sqrt{\pi}}\frac{1}{24\Sigma 2pq}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h}(4t^2-2)e^{-t^2}dt$$
$$-\frac{1}{(\Sigma 2pq)^2}\frac{1}{2^7.3.5.\sqrt{\pi}}\sum_{t_1}^{t_2}\{16(t+\tfrac{1}{2}\theta h)^4-48(t+\tfrac{1}{2}\theta h)^2$$
$$+12\}e^{-(t+\frac{1}{2}\theta h)^2}\frac{1}{\sqrt{(\Sigma 2pq)}}$$
$$+\frac{1}{(\Sigma 2pq)^2}\frac{1}{48.24\sqrt{\pi}}\sum_{t_1}^{t_2}\{16(t+\tfrac{1}{2}\theta h)^4-48(t+\tfrac{1}{2}\theta h)^2$$
$$+12\}e^{-(t+\frac{1}{2}\theta h)^2}\frac{1}{\sqrt{(\Sigma 2pq)}}, \quad (1.312)$$

where h is $1/\sqrt{(\Sigma 2pq)}$ and θ, lying between -1 and 1, is not necessarily the same in different terms.

Now it is easily verified

(1) that $|16t^4-48t^2+12|e^{-t^2}$ has only five maxima,

(2) that it is greatest when $t=0$ and so it never exceeds 12,

(3) that $\int_{-\infty}^{\infty}|16t^4-48t^2+12|e^{-t^2}\,dt<24\sqrt{\pi}$.

For each of the sums on the right of (1.312) we have therefore

$$\begin{aligned}\left|\sum_{t_1}^{t_2}\right| &< \int_{-\infty}^{\infty}|16t^4-48t^2+12|e^{-t^2}\,dt+\frac{60}{\sqrt{(\Sigma 2pq)}}\\ &< 24\sqrt{\pi}+\frac{60}{\sqrt{(\Sigma 2pq)}}\\ &< 30\sqrt{\pi},\end{aligned}$$

since $\Sigma 2pq>54$. The error term is thus less than

$$\frac{1}{(\Sigma 2pq)^2}\left(\frac{27\sqrt{\pi}}{2^7.3.5.\sqrt{\pi}}+\frac{27\sqrt{\pi}}{48.24.\sqrt{\pi}.8}\right)<\tfrac{1}{32}\frac{1}{(\Sigma 2pq)^2}. \qquad (1.313)$$

We have now estimated the value of the first term on the right of (1.31).

Again, $$D^2[(3t-2t^3)e^{-t^2}]=(-8t^5+40t^3-30t)e^{-t^2}.$$

Therefore, using the Taylor formula (1) once more,

$$\begin{aligned}&\frac{1}{\sqrt{\pi}}\frac{\Sigma pq(p-q)}{3(\Sigma 2pq)^{\frac{3}{2}}}\sum_{t_1}^{t_2}\frac{e^{-t^2}t(3-2t^2)}{\sqrt{(\Sigma 2pq)}}\\ &\quad=\frac{1}{\sqrt{\pi}}\frac{\Sigma pq(p-q)}{3(\Sigma 2pq)^{\frac{3}{2}}}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h}e^{-t^2}t(3-2t^2)\,dt-\frac{1}{\sqrt{\pi}}\frac{\Sigma pq(p-q)}{3(\Sigma 2pq)^{\frac{3}{2}}}\frac{1}{24}\frac{1}{\Sigma 2pq}\\ &\qquad\times\frac{1}{\sqrt{(\Sigma 2pq)}}\sum_{t_1}^{t_2}\{-8(t+\tfrac{1}{2}\theta h)^5+40(t+\tfrac{1}{2}\theta h)^3-30(t+\tfrac{1}{2}\theta h)\}e^{-(t+\frac{1}{2}\theta h)^2}.\end{aligned}$$

(1.314)

Now $(-8t^5+40t^3-30t)e^{-t^2}$ has at most six turning points*, as may be seen by differentiation, and $t^ne^{-t^2}$ has turning points when $t=0$, $\pm\sqrt{(\frac{1}{2}n)}$ (the first occurring only when n is even). Therefore, since $t^ne^{-t^2}\to 0$ as $|t|\to\infty$,

$$|t^ne^{-t^2}|\leqslant\left(\frac{n}{2e}\right)^{\frac{1}{2}n}<\tfrac{1}{2},\ \tfrac{10}{27},\ \tfrac{1}{2},\ \tfrac{2}{3},\ 1,$$

* The object of this slightly more elaborate estimation is to improve the appearance of the error term in the final formula (1.33).

for $n = 1, 2, 3, 4, 5$. It follows that

$$\begin{aligned}|-8t^5+40t^3-30t|\,e^{-t^2} &\leqslant \max\left(|8t^5+30t|\,e^{-t^2},\ |40t^3|\,e^{-t^2}\right)\\ &< \max\left(8+30.\tfrac{1}{2},\ 40.\tfrac{1}{2}\right)\\ &= 23.\end{aligned}$$

Also, since, for $|t| \geqslant 2$,

$$\begin{aligned}|8t^5-40t^3+30t| &= |t|.|(8t^4+30)-40t^2|\\ &\leqslant |t|\max\left(8t^4+30,\ 40t^2\right)\\ &\leqslant |t|\max\left(9t^4,\ 10t^4\right)\\ &= 10\,|t|^5,\end{aligned}$$

it follows that

$$\int_{-\infty}^{\infty} |-8t^5+40t^3-30t|\,e^{-t^2}\,dt \leqslant 20\int_0^{\infty} t^5 e^{-t^2}\,dt + 4.23 = 102.$$

We infer finally that the Σ in the second term on the right of (1.314) satisfies

$$\begin{aligned}\left|\sum_{t_1}^{t_2}\right| &< 102+\frac{6.23}{\sqrt{(\Sigma 2pq}}\\ &< 150.\end{aligned}$$

The error term in (1.314) is therefore less than

$$\frac{1}{3.48\sqrt{\pi}}\,\frac{150}{(\Sigma 2pq)^{\frac{3}{2}}} < (\Sigma 2pq)^{-\frac{3}{2}}. \tag{1.315}$$

We have thus estimated the value of the second term on the right of (1.31). This equation can now be written

$$\begin{aligned}P\{\Sigma p+t_1\sqrt{(\Sigma 2pq)} \leqslant m \leqslant \Sigma p+t_2\sqrt{(\Sigma 2pq)}\}\\ = \frac{1}{\sqrt{\pi}}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt + \frac{1}{\sqrt{\pi}}\,\frac{\Sigma pq(p-q)}{3(\Sigma 2pq)^{\frac{3}{2}}}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,t(3-2t^2)\,dt\\ +\theta_{19}\left[\tfrac{1}{32}\,\frac{1}{(\Sigma 2pq)^2}+\frac{1}{(\Sigma 2pq)^{\frac{3}{2}}}+\tfrac{2}{8}\sqrt{(\Sigma 2pq)}.(t_2-t_1+1)\,e^{-\frac{1}{2}(\Sigma pq)^{\frac{1}{2}}}\right.\\ \left.+\tfrac{1}{2}\tfrac{1}{5}(\Sigma pq)^{-\frac{3}{2}+6\epsilon}\sum_{t_1}^{t_2} e^{-t^2}\left(1+\frac{|t(3-2t^2)}{24}\right)\right]\\ -\frac{1}{\sqrt{\pi}}\,\frac{1}{24\,\Sigma 2pq}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h}(4t^2-2)\,e^{-t^2}\,dt. \qquad (1.32)\end{aligned}$$

The last term has to be included in the error term, since actually it is

$$O\left(\frac{1}{\Sigma 2pq}\right).$$

It remains to assess this error term. When $|t| < \sqrt{\frac{3}{2}}$,

$$|t(3-2t^2)| < 6|t^3|,$$

while, when $|t| \leqslant \sqrt{\frac{3}{2}}$,

$$|t(3-2t^2)| \leqslant 3|t|.$$

Thus $$\int_{-\infty}^{\infty} e^{-t^2}\frac{|t(3-2t^2)|}{24} < \int_{-\infty}^{\infty} \tfrac{1}{8}e^{-t^2}|t|\,dt + \int_{-\infty}^{\infty} \tfrac{1}{4}|t^3|e^{-t^2}\,dt = \tfrac{13}{4}.$$

Also $$e^{-t^2}\frac{|t(3-2t^2)|}{24} \leqslant \frac{\max(|6t^3e^{-t^2}|,\ |3te^{-t^2}|)}{24} < 3.$$

Moreover $e^{-t^2}[\{t(3-2t^2)/24\}+1]$ has at most four maxima. Therefore

$$\sum_{t_1}^{t_2} e^{-t^2}\left(\frac{t(3-2t^2)}{24}+1\right) < \sqrt{(\Sigma 2pq)}\left(\tfrac{13}{4}+\sqrt{\pi}+\frac{16}{\sqrt{(\Sigma 2pq)}}\right) < \tfrac{15}{2}\sqrt{(\Sigma 2pq)}.$$

The sum of the corresponding terms in θ_{19} is thus less than

$$\frac{1}{(\Sigma 2pq)^{1-6\epsilon}}\left(\frac{1}{32}\frac{1}{(\Sigma 2pq)^{1+6\epsilon}}+\frac{1}{(\Sigma 2pq)^{\frac{1}{2}+6\epsilon}}+\frac{15\sqrt{2}}{25}\right)$$

$$< \frac{1}{(\Sigma 2pq)^{1-6\epsilon}}\left(\frac{1}{32.2.729}+\frac{1}{3\sqrt{6}.27}+\frac{3\sqrt{2}}{5}\right), \quad (1.321)$$

since $(\Sigma pq)^{2\epsilon} \geqslant 3$.

Finally $$\left|\frac{1}{\sqrt{\pi}}\frac{1}{24\,\Sigma 2pq}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h}(4t^2-2)\,e^{-t^2}\,dt\right|$$

$$< \frac{1}{\sqrt{\pi}}\frac{1}{(\Sigma 2pq)^{1-6\epsilon}}\cdot\frac{1}{24.27}\int_{-\infty}^{\infty}|4t^2-2|\,e^{-t^2}\,dt,$$

which, since the integral is less than

$$4\int_{-\infty}^{\infty} t^2e^{-t^2}+2\int_{-\infty}^{\infty} e^{-t^2}\,dt = 4\sqrt{\pi},$$

is less than $$\frac{1}{6.27}(\Sigma 2pq)^{-1+6\epsilon}. \quad (1.322)$$

The sum of this and (1.321) is less than

$$(\Sigma 2pq)^{-1+6\epsilon}. \quad (1.323)$$

Thus from (1.32) we have

$$P\{\Sigma p+t_1\sqrt{(\Sigma 2pq)} \leqslant m \leqslant \Sigma p+t_2\sqrt{(\Sigma 2pq)}\}$$
$$= \frac{1}{\sqrt{\pi}}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt+\frac{1}{\sqrt{\pi}}\,\frac{\Sigma pq(p-q)}{3(\Sigma 2pq)^{\frac{3}{2}}}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}t(3-2t^2)\,dt$$
$$+\theta_{20}\{(\Sigma 2pq)^{-1+6\epsilon}+\tfrac{9}{8}\sqrt{(\Sigma 2pq)}.(t_2-t_1+1)\,e^{-\frac{1}{2}(\Sigma pq)^{2\epsilon}}\}, \quad (1.33)$$

where h is written for $1/\{\sqrt{(\Sigma 2pq)}\}$ and $|\theta_{20}|<1$.

This result is valid provided that

(i) $0<\epsilon<\frac{1}{6}$,

(ii) $(\Sigma pq)^{2\epsilon}\geqslant 3$,

(iii) $t_1\leqslant t_2$, and $\Sigma p+\dfrac{t_1}{t_2}\sqrt{(\Sigma 2pq)}$ are integers,

but further restrictions on the t's are necessary if it is to be always significant. It is, of course, essential that the first term shall really be the dominant one. Plainly, if t_2 is extremely large, the second part of the error term may dominate the right-hand side. and even if t_1 and t_2 tend to $+\infty$ like a power of $\Sigma 2pq$, while t_2-t_1 remains bounded, the result is valueless, since then $(\Sigma 2pq)^{-1+6\epsilon}$ is dominant. We must restrict t_1 and t_2 so as to prevent this. We must also arrange that the second term is of smaller order than the first.

It is important to observe that to remove the second part of the restriction (iii) on t_1 and t_2 is out of the question, since this would introduce an error term of order $1/\sqrt{(\Sigma 2pq)}$, because a very small change in the real numbers t_1 and t_2 may result in the omission of a term from $P(t_1, t_2)$. The second term in the expansion would then lose its significance. In any case it is significant only when $\epsilon<\frac{1}{12}$.

Condition (i), therefore, had better be replaced by $0<\epsilon<\frac{1}{12}$, conditions (ii) and (iii) must stand, and we have to investigate the restrictions to be placed on t_1 and t_2.

LEMMA I. *As* $x\to\infty$,

$$\frac{1}{x^{\frac{1}{2}}}\int_a^b e^{-t^2}t(3-2t^2)\,dt = O\left(\int_a^b e^{-t^2}\,dt\right)$$

uniformly in $a\leqslant b$, *provided that*

$$|a|,\ |b|\leqslant\phi(x),$$

where

$$\phi(x)=O(x^{\frac{1}{6}}).$$

Proof. (1) It is sufficient to prove this on the hypothesis that $\phi(x)$ is an increasing function. When this is the case, we have for a suitable positive K, and for all $\alpha, \beta > K$ or $\alpha, \beta < -K$,

$$\left| \int_\alpha^\beta e^{-t^2} t(3-2t^2)\, dt \right| < 3 \int_\alpha^\beta |t|^3 e^{-t^2}\, dt$$

$$< 24 \{\phi(x)\}^3 \int_\alpha^\beta e^{-t^2}\, dt$$

$$< \tfrac{1}{3}\epsilon x^{\frac{1}{2}} \int_\alpha^\beta e^{-t^2}\, dt, \qquad \text{for } x \geqslant x_1(\epsilon).$$

(2) For $-K \leqslant \alpha, \beta \leqslant K$,

$$\int_\alpha^\beta e^{-t^2} t(3-2t^2)\, dt \Big| < c \int_\alpha^\beta e^{-t^2}\, dt$$

$$< \tfrac{1}{3}\epsilon x^{\frac{1}{2}} \int_\alpha^\beta e^{-t^2}\, dt, \qquad \text{for } x \geqslant x_2(\epsilon).$$

(3) Dividing up the range of integration into 2 or into 3 parts when necessary, we have in all cases

$$\left| \int_\alpha^b e^{-t^2} t(3-2t^2)\, dt \right| < \epsilon x^{\frac{1}{2}} \int_\alpha^b e^{-t^2}\, dt, \qquad \text{for } x \geqslant x_0(\epsilon).$$

LEMMA II. *With the hypotheses of Lemma I,*

$$\frac{1}{x^{\frac{1}{2}}} \int_{a-\frac{1}{2}x^{-\frac{1}{2}}}^{b+\frac{1}{2}x^{-\frac{1}{2}}} e^{-t^2}(3-2t^2)\, dt = O\left(\int_{a\frac{1}{2}-x^{-\frac{1}{2}}}^{b+\frac{1}{2}x^{-\frac{1}{2}}} e^{-t^2}\, dt \right).$$

Proof. $a-\frac{1}{2}x^{-\frac{1}{2}}$, $b+\frac{1}{2}x^{-\frac{1}{2}}$ also satisfy the hypotheses of Lemma I.

LEMMA III. *If t_1, t_2 are real numbers satisfying*

$$|t_1|, \ |t_2| \leqslant \phi(\Sigma 2pq),$$

where $\phi(x) = O(x^{\frac{1}{2}})$, then the second term on the right of (1.33) *is of lower order than the first as $\Sigma 2pq \to \infty$.**

Proof. The second term is absolutely less than

$$\frac{1}{6\sqrt{\pi}} \frac{1}{\sqrt{(\Sigma 2pq)}} \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2} t(3-2t^2)\, dt\,;$$

writing $\Sigma 2pq = x$ in Lemma II, we obtain the result.

So much for the relationship between the first and second terms. It remains to discuss the error term. Since $\Sigma pq(p-q)$ may be extremely

* This, like all our results, is uniform in the individual p's, q's.

small, we cannot ensure, by restricting the t's, that the second term shall dominate the error term in all cases. (Even in the favourable case $p-q \geqslant a > 0$ we must introduce a stronger restriction than the "natural" one $m_2 - m_1 \to \infty$, *i.e.* $(t_2-t_1)\sqrt{(\Sigma 2pq)} \to \infty$; we must suppose this quantity to tend to infinity like a power of $\Sigma 2pq$.) We therefore demand only that the first term on the right of (1.33) shall dominate the remainder and that the second term shall sometimes dominate the error term.

LEMMA IV. *If* $0 < \epsilon < \frac{1}{12}$ *and* t_1, t_2 $(t_1 \leqslant t_2)$ *are real numbers such that*

$$|t_1|, |t_2| \leqslant c\sqrt{(\log \Sigma 2pq)},$$

where c *is any positive constant less than* $\sqrt{(\frac{1}{2}-6\epsilon)}$, *then, as* $\Sigma 2pq \to \infty$,

$$\frac{(\Sigma 2pq)^{-1+6\epsilon} + \frac{9}{8}\sqrt{(\Sigma 2pq)}.(t_2-t_1+1)\, e^{-\frac{1}{2}(\Sigma 2pq)^{2\epsilon}}}{\frac{1}{\sqrt{\pi}}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\, dt}$$

tends to zero uniformly in t_1, t_2.

Proof. For $\Sigma 2pq > K_1(\epsilon)$ the numerator is less than $2(\Sigma 2pq)^{-1+6\epsilon}$. It is therefore sufficient to prove that

$$(\Sigma 2pq)^{-1+6\epsilon} = O\left(\int_{c\sqrt{(\log \Sigma 2pq)}-\frac{1}{2}(\Sigma 2pq)^{-\frac{1}{2}}}^{c\sqrt{(\log \Sigma 2pq)}+\frac{1}{2}(\Sigma 2pq)^{-\frac{1}{2}}} e^{-t^2}\, dt\right).$$

But the integral is greater than

$$\frac{1}{\sqrt{(\Sigma 2pq)}} \exp\left\{-\left(c\sqrt{(\log \Sigma 2pq)} + \frac{1}{2\sqrt{(\Sigma 2pq)}}\right)^2\right\}$$

$$> \frac{1}{\sqrt{(\Sigma 2pq)}} \exp\left\{-(c+\eta)^2 \log \Sigma 2pq\right\}$$

for $\eta > 0$, $\Sigma 2pq > K_2(\eta)$.

Choose $\eta > 0$, so that $c+\eta < \sqrt{(\frac{1}{2}-6\epsilon)}$. Then

$$(c+\eta)^2 = \tfrac{1}{2} - 6\epsilon - k,$$

where $k > 0$, and our expression is thus greater than

$$\frac{1}{\sqrt{(\Sigma 2pq)}} (\Sigma 2pq)^{-\frac{1}{2}+6\epsilon+k} = (\Sigma 2pq)^{k}(\Sigma 2pq)^{-\frac{1}{2}+6\epsilon},$$

and this proves the lemma.

We therefore take as our final form for the conditions of (1 . 33),

(1) $0 < \epsilon < \frac{1}{12}$,

(2) $(\Sigma 2pq)^{2\epsilon} \geqslant 4$,

(3) $t_1 \leqslant t_2$, and $\Sigma p + \dfrac{t_1}{t_2}\sqrt{(\Sigma 2pq)}$ are integers,

(4) $|t_1|, |t_2| \leqslant \frac{1}{2}\sqrt{(1-12\epsilon)}.\sqrt{(\log \Sigma 2pq)}$.

These results can be extended to a much larger range of value of t. We begin by stating five trivial lemmas. We suppose that t_1 and t_2 (when they occur) are subject to condition (3) above.

LEMMA 1 . 411. *For* $h^2 \leqslant \frac{1}{54}$, $|t| \leqslant h^{-\frac{1}{6}}$,

$$he^{-t^2} < 2\int_{t-\frac{1}{2}h}^{t+\frac{1}{2}h} e^{-u^2}\,du.$$

Proof.

$$\begin{aligned}\int_{t-\frac{1}{2}h}^{t+\frac{1}{2}h} e^{-u^2}\,du &= \int_{-\frac{1}{2}h}^{\frac{1}{2}h} e^{-(t+x)^2}\,dx \\ &> e^{-t^2}e^{-h^{\frac{1}{3}}}\int_{-\frac{1}{2}h}^{\frac{1}{2}h} e^{-x^2}\,dx \\ &> he^{-t^2}e^{-h^{\frac{1}{3}}-h^2/4} \\ &> \tfrac{1}{2}he^{-t^2}.\end{aligned}$$

LEMMA 1 . 412. *For* $0 \leqslant k < \frac{1}{6}$ *and* $1 \leqslant t_1 \leqslant t_2 \leqslant (\Sigma 2pq)^k$ *or* $-(\Sigma 2pq)^k \leqslant t_1 \leqslant t_2 \leqslant -1$,

$$\sum_{t_1}^{t_2} \frac{e^{-t^2}}{\sqrt{(\Sigma 2pq)}} < 2\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt.$$

(In $\sum_{t_1}^{t_2}$ the values of t are, as usual, spaced $1/\sqrt{(\Sigma 2pq)}$ apart and run from t_1 to t_2.)

Proof. Take $h = 1/\sqrt{(\Sigma 2pq)}$ in Lemma 1 . 411.

LEMMA 1 . 413. *For* $0 \leqslant k < \frac{1}{6}$ *and* $1 \leqslant t_1 \leqslant t_2 \leqslant (\Sigma 2pq)^k$ *or* $-(\Sigma 2pq)^k \leqslant t_1 \leqslant t_2 \leqslant -1$,

$$\sum_{t_1}^{t_2} \frac{|(2t^3-3t)\,e^{-t^2}|}{\sqrt{(\Sigma 2pq)}} < 4(\Sigma 2pq)^{3k}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt.$$

Proof. For $|t| \geqslant 1$, $|2t^3-3t| < 2|t|^3$. Thus

$$\sum_{t_1}^{t_2} < \sum_{t_1}^{t_2} \frac{2\ t_1{}^3 e^{-t^2}}{\sqrt{(\Sigma 2pq)}}$$

$$< 2(\Sigma 2pq)^{2k} \sum_{t_1}^{t_2} \frac{e^{-t^2}}{\sqrt{(\Sigma 2pq)}}$$

$$< 4(\Sigma 2pq)^{3k} \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt,$$

by Lemma 1.412.

LEMMA 1.414. *For* $0 \leqslant k < \frac{1}{6}$ *and* $1 \leqslant t_1 \leqslant t_2 \leqslant (\Sigma 2pq)^k$ *or* $-(\Sigma 2pq)^k \leqslant t_1 \leqslant t_2 \leqslant -1$,

$$\left| \sum_{t_1}^{t_2} \frac{(16t'^4-48t'^2+12)\, e^{-t'^2}}{\sqrt{(\Sigma 2pq)}} \right|$$

$$< 48\,(\Sigma 2pq)^{4k} \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt,$$

where t' *denotes any number lying in the interval* $(t-\frac{1}{2}h,\ t+\frac{1}{2}h)$.

Proof. $$\left|\sum_{t_1}^{t_2}\right| < \sum_{t_1}^{t_2} \frac{16t^{14}\, e^{-t^{12}}}{\sqrt{(\Sigma 2pq)}}.$$

Now $$|e^{t^2-t'^2}| \leqslant e^{|t-t'|\,|t+t'|}$$

$$< e^{h^2+\frac{1}{4}h^2}$$

$$< \tfrac{9}{8}.$$

Thus $$\left|\sum_{t_1}^{t_2}\right| < 16(\Sigma 2pq)^{4k} \left(1+\frac{h}{2}\right)^4 \cdot \tfrac{9}{8} \sum_{t_1}^{t_2} \frac{e^{-t^2}}{\sqrt{(\Sigma 2pq)}}$$

$$< 48(\Sigma 2pq)^{4k} \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt.$$

LEMMA 1.415. *For* $0 \leqslant k < \frac{1}{6}$ *and* $1 \leqslant t_1 \leqslant t_2 \leqslant (\Sigma 2pq^k)$ *or* $-(\Sigma 2pq)^k \leqslant t_1 \leqslant t_2 \leqslant -1$,

$$\left| \sum_{t_1}^{t_2} \frac{8t'^5-40t'^3+30t'\rangle\, e^{-t'^2}}{\sqrt{(\Sigma 2pq)}} \right| < 12(\Sigma 2pq)^{5k} \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt.$$

Proof. $$\left|\sum_{t_1}^{t_2}\right| < \sum_{t_1}^{t_2} \frac{8\,|t'|^5\, e^{-t^{12}}}{\Sigma(2pq)}$$

$$< 2(\Sigma 2pq)^{5k} \left(1+\frac{h}{2}\right)^5 \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt$$

$$< 12(\Sigma 2pq)^{5k} \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt.$$

LEMMA 1.416. *For* $0 \leqslant k < \frac{1}{6}$ *and* $1 \leqslant t_1 \leqslant t_2 \leqslant (\Sigma 2pq)^k$ *or* $-(\Sigma 2pq)^k \leqslant t_1 \leqslant t_2 \leqslant -1$,

$$\left| \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} (4t^2-2)\, e^{-t^2}\, dt \right| < 6(\Sigma 2pq)^{2k} \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\, dt.$$

Proof. For $|t| \geqslant 1$, $\quad |4t^2-2| < 4t^2$.

Thus
$$\left| \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} (4t^2-2)\, e^{-t^2}\, dt \right| < 4(\Sigma 2pq)^{2k}(1+\tfrac{1}{2}h)^2 \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\, dt$$
$$< 6(\Sigma 2pq)^{2k} \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\, dt.$$

To apply these lemmas, we observe that, by (1.31), (1.312), and (1.314),

$$P(m_1, m_2) = \frac{1}{\sqrt{\pi}} \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\, dt + \frac{1}{\sqrt{\pi}} \frac{\Sigma pq(p-q)}{3(\Sigma 2pq)^{\frac{3}{2}}} \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\, t(3-2t^2)\, dt$$
$$- \frac{1}{\sqrt{\pi}} \frac{1}{24} \frac{1}{\Sigma 2pq} \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} (4t^2-2)\, e^{-t^2}\, dt$$
$$- \frac{1}{48.24\sqrt{\pi}} \frac{1}{(\Sigma 2pq)^2} \sum_{t_1}^{t_2} \frac{(19t'^4-48t'^2+12)\, e^{-t'^2}}{\sqrt{(\Sigma 2pq)}}$$
$$- \frac{1}{2^7.3.5\sqrt{\pi}} \frac{1}{(\Sigma 2pq)^2} \sum_{t_1}^{t_2} \frac{(16t'^4-48t'^2+12)\, e^{-t'^2}}{\sqrt{(\Sigma 2pq)}}$$
$$- \frac{1}{\sqrt{\pi}} \frac{\Sigma pq(p-q)}{3(\Sigma 2pq)^{\frac{3}{2}}} \frac{1}{24} \frac{1}{\Sigma 2pq} \sum_{t_1}^{t_2} \frac{(-8t'^5+40t'^3-30t')\, e^{-t'^2}}{\sqrt{(\Sigma 2pq)}}$$
$$+ \theta_{18} \left[\tfrac{1}{25} (\Sigma pq)^{-\frac{3}{2}+6\epsilon} \left(\sum_{t_1}^{t_2} e^{-t^2} + \sum_{t_1}^{t_2} \frac{|t(3-2t^2)|\, e^{-t^2}}{24} \right) \right.$$
$$\left. + \tfrac{9}{8} e^{-\frac{1}{2}(\Sigma pq)^{2\epsilon}} (t_2-t_1+1)\sqrt{(\Sigma 2pq)} \right]. \quad (1.42)$$

For

$1 \leqslant t_1 \leqslant t_2 \leqslant (\Sigma 2pq)^k$ or $-(\Sigma 2pq)^k \leqslant t_1 \leqslant t_2 \leqslant -1$ *and* $0 \leqslant k < \frac{1}{6}$,

the sum of the last six terms is now seen to be less than

$$\frac{1}{\sqrt{\pi}} \int_{t_1-\frac{1}{2}h}^{t_1+\frac{1}{2}h} e^{-t^2}\, dt \left(\tfrac{1}{4} h^{2-4k} + \tfrac{1}{24} h^{4-8k} + \tfrac{1}{40} h^{4-8k} + \tfrac{1}{12} h^{3-10k} \right.$$
$$\left. + \frac{4\sqrt{(2\pi)}}{25} h^{2-12\epsilon} + \frac{\sqrt{(2\pi)}}{75} h^{2-6k-12\epsilon} \right)$$
$$+ \tfrac{9}{8} (\Sigma 2pq)^{\frac{1}{2}+k}\, e^{-\frac{1}{2}(\Sigma 2pq)^{2\epsilon}}. \quad (1.43)$$

To ensure that the second term of (1.43) is of smaller order than $\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt$ we must have

$$k < \epsilon, \tag{i}$$

and to make the first term negligible we must have

$$1-3k-6\epsilon > 0;$$

if the second term of our main expansion is to be significant, we must have more than this, however, namely,

$$2-6k-12\epsilon > 1. \tag{ii}$$

The conditions (i) and (ii) imply that $k < \frac{1}{18}$.

Choose $k = \frac{1}{24}$, $\epsilon = \frac{1}{18}$, and we see that for $1 \leqslant t_1 \leqslant t_2 \leqslant (\Sigma 2pq)^{\frac{1}{24}}$ or $-(\Sigma 2pq)^{\frac{1}{24}} \leqslant t_1 \leqslant t_2 \leqslant -1$,

$$|\text{Error term}| < \frac{1}{\sqrt{\pi}}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt \,.\, \gamma h^{\frac{13}{12}} + \tfrac{9}{8}h^{-\frac{13}{12}} \exp\left\{-\tfrac{1}{2}\left(\frac{h}{2}\right)^{-\frac{2}{11}}\right\}, \tag{1.44}$$

where $\gamma < \frac{1}{3}$ is a constant.

Since $$\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt > \tfrac{1}{2}he^{-h^{-\frac{1}{12}}},$$

this is less than $$\tfrac{1}{3}h^{\frac{13}{12}} \cdot \frac{1}{\sqrt{\pi}}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt$$

for all sufficiently large $\Sigma 2pq$*.

Thus, for $1 \leqslant t_1 \leqslant t_2 \leqslant (\Sigma 2pq)^{\frac{1}{24}}$ or $-(\Sigma 2pq)^{\frac{1}{24}} \leqslant t_1 \leqslant t_2 \leqslant -1$ and $\Sigma 2pq \geqslant K_1$,

$$P(m_1, m_2) = \frac{1}{\sqrt{\pi}}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt\left(1+\frac{\theta}{3(\Sigma 2pq)^{\frac{1}{2}+\frac{1}{24}}}\right)$$

$$+\frac{1}{\sqrt{\pi}}\,\frac{\Sigma pq(p-q)}{3(\Sigma 2pq)^{\frac{3}{2}}}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,t(3-2t^2), \tag{1.45}$$

where $|\theta| < 1$, h is $(\Sigma 2pq)^{-\frac{1}{2}}$, and $\Sigma p + \begin{matrix} t_1 \\ t_2 \end{matrix}\sqrt{(\Sigma 2pq)}$ are integers.

Now, for $-\frac{101}{100} \leqslant t_1 \leqslant t_2 \leqslant \frac{101}{100}$ and $\Sigma 2pq \geqslant e^{36}$,

$$\frac{1}{\sqrt{\pi}}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt > \tfrac{1}{2}h,$$

and $$(\Sigma 2pq)^{-\frac{3}{2}} + \tfrac{27}{8}(\Sigma 2pq)^{\frac{1}{2}}\,e^{-\frac{1}{2}(\Sigma pq)^{\frac{1}{12}}} < \tfrac{1}{3}h^{\frac{13}{12}}.$$

* It is easily verified that this holds for $\Sigma 2pq \geqslant e^{36}$.

Taking $\epsilon = \frac{1}{24}$ in (1.33), we have, for $-\frac{101}{100} \leqslant t_1 \leqslant t_2 \leqslant \frac{101}{100}$, and $\Sigma 2pq \geqslant e^{36}$,

$$P(m_1, m_2) = \frac{1}{\sqrt{\pi}} \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\, dt. \left(1 + \frac{\theta_{21}}{3(\Sigma 2pq)^{\frac{1}{3}}}\right) + \frac{1}{\sqrt{\pi}} \frac{\Sigma pq(p-q)}{3(\Sigma 2pq)^{\frac{3}{2}}} \int_{t-\frac{1}{2}h}^{t+\frac{1}{2}h} e^{-t^2}\, t(3-2t^2). \quad (1.46)$$

Now suppose simply that

$$-(\Sigma 2pq)^{\frac{1}{24}} \leqslant t_1 \leqslant t_2 \leqslant (\Sigma 2pq)^{\frac{1}{24}}, \quad \text{(i)}$$

$$\Sigma p + \frac{t_1}{t_2}\sqrt{(\Sigma 2pq)} \text{ are integers,} \quad \text{(ii)}$$

$$\Sigma 2pq \geqslant e^{36}. \quad \text{(iii)}$$

Then the intervals $(-\frac{101}{100}, -1)$ and $(1, \frac{101}{100})$ always contain at least one point t such that $\Sigma p + t\sqrt{(\Sigma 2pq)}$ is an integer. Let t_3 be that point of $(-\frac{101}{100}, -1)$ which is nearest to -1, t_4 that point of $(1, \frac{101}{100})$ which is nearest to 1. Then, dividing up (t_1, t_2) at t_3 when it contains the whole of $(-\frac{101}{100}, -1)$, and also at t_4 when it contains the whole of $(1, \frac{101}{100})$, we have, in all cases,

$$P(m_1, m_2) = \left(\frac{1}{\sqrt{\pi}} \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\, dt\right)\left(1 + \frac{\theta_{22}}{(\Sigma 2pq)^{\frac{13}{24}}}\right) + \frac{1}{\sqrt{\pi}} \frac{\Sigma pq(p-q)}{3(\Sigma 2pq)^{\frac{3}{2}}} \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\, t(3-2t^2)\, dt. \quad (1.47)$$

1.5. We conclude by obtaining an expression for

$$P(\Sigma p + t_1\sqrt{(\Sigma 2pq)} \leqslant m \leqslant \Sigma p + t_2\sqrt{(\Sigma 2pq)},$$

in which it is not necessary to assume that t_1, t_2 are bound by (ii) above. As already remarked, this involves the sacrificing of the second term of our expansion.

LEMMA 1.511. *For* $1 \leqslant t_1 \leqslant t_2 \leqslant (\Sigma 2pq)^k$ *or* $-(\Sigma 2pq)^k \leqslant t_1 \leqslant t_2 \leqslant -1$, $0 \leqslant k < \frac{1}{6}$, $\Sigma 2pq \geqslant 54$,

$$\left| \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\, t(3-2t^2)\, dt \right| < 3(\Sigma 2pq)^{3k} \int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\, dt.$$

Proof. In the intervals

$$1 - \tfrac{1}{2}h \leqslant t \leqslant (\Sigma 2pq)^k + \tfrac{1}{2}h$$

and
$$-(\Sigma 2pq)^k - \tfrac{1}{2}h \leqslant t \leqslant 1+\tfrac{1}{2}h,$$
$$\begin{aligned}|e^{-t^2}t(3-2t^2| &\leqslant |2t^3 e^{-t^2}| \\ &\leqslant 2(\Sigma 2pq)^{3k}(1+\tfrac{1}{2}h)^3 e^{-t^2} \\ &< 3(\Sigma 2pq)^{3k} e^{-t^2}.\end{aligned}$$

For the moment we retain the restriction (ii) on the t's. Using this lemma and (1.43) on (1.42), we have

$$\begin{aligned}P(t_1, t_2) = \frac{1}{\sqrt{\pi}}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt\big[1+\theta_{23}(\tfrac{5}{24}h^{2-4k}+\tfrac{1}{64}h^{4-8k}+\tfrac{3}{320}h^{4-8k} \\ +\tfrac{1}{160}h^{3-10k}+\tfrac{1}{15}h^{2-12\epsilon}+\tfrac{1}{160}h^{2-12\epsilon-6k}+\tfrac{1}{2}h^{1-6k}\big] \\ +\tfrac{9}{8}\theta_{24}\, e^{-\frac{1}{2}(\frac{1}{2}h)^{-4\epsilon}} h^{-1-2k},\end{aligned}$$

or, since for $-(\Sigma 2pq)^k \leqslant t_1 \leqslant t_2 \leqslant (\Sigma 2pq)^k$, $\Sigma 2pq \leqslant 54$,

$$\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt > h\sqrt{\pi}\, e^{-2.h^{-2k}},$$

$$\begin{aligned}P(t_1, t_2) = \frac{1}{\sqrt{\pi}}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt\big[1+\theta_{25}(\tfrac{1}{4}h^{2-4k}+\tfrac{1}{15}h^{2-12\epsilon}+\tfrac{1}{160}h^{2-12\epsilon-6k} \\ +\tfrac{1}{2}h^{1-6k}+\tfrac{9}{8}h^{-2-2k}\, e^{-\frac{1}{2}(\frac{1}{2}h)^{-4\epsilon}+2h^{-2k}})\big]. \quad (1.512)\end{aligned}$$

Let
$$k = \tfrac{1}{6}-\eta, \quad \epsilon = \tfrac{1}{12}-\tfrac{1}{4}\eta \quad (0 < \eta < \tfrac{1}{12}).$$

Then
$$\begin{aligned}-\tfrac{1}{2}(\tfrac{1}{2}h)^{-4\epsilon}+2h^{-2k} &= -2^{-\frac{2}{3}+\eta}\, h^{-\frac{1}{3}+\eta}+2h^{-\frac{1}{3}+2\eta} \\ &< -\tfrac{1}{2}h^{-\frac{1}{3}+\eta} \text{ for } h < h_1(\eta) \\ &< -\tfrac{1}{2}h^{-\frac{1}{4}}.\end{aligned}$$

Thus the term in θ_{25} is less than

$$\begin{aligned}&\tfrac{1}{4}h^{\frac{4}{3}+4\eta}+\tfrac{1}{15}h^{3\eta}+\tfrac{1}{160}h^{0\eta}+\tfrac{1}{2}h^{6\eta}+\tfrac{9}{8}h^{-\frac{7}{3}+2\eta}\, e^{-\frac{1}{2}h^{-\frac{1}{4}}} \\ &< \tfrac{1}{3}\frac{1}{\sqrt{\pi}}h^{\eta} \text{ for } h < h_2(\eta).\end{aligned}$$

(1.512) now becomes:

For $-(\Sigma 2pq)^{\frac{1}{6}-\eta} \leqslant t_1 \leqslant t_2 \leqslant -1$ or $1 \leqslant t_1 \leqslant t_2 \leqslant (\Sigma 2pq)^{\frac{1}{6}-\eta}$, $\eta > 0$ (the condition $\eta < \frac{1}{12}$ can now be dropped), $\Sigma 2pq \geqslant A(\eta)$, $\Sigma p + \begin{matrix}t_1\\t_2\end{matrix}\sqrt{(\Sigma 2pq)}$ integers,

$$P(t_1, t_2) = \left(\frac{1}{\sqrt{\pi}}\int_{t_1-\frac{1}{2}h}^{t_2+\frac{1}{2}h} e^{-t^2}\,dt\right)\left(1+\theta_{26}\frac{h^{\eta}}{3}\right). \quad (1.52)$$

But when $-\frac{101}{100} \leqslant t_1 \leqslant t_2 \leqslant \frac{101}{100}$ we easily have, from (1.47),

$$P(t_1, t_2) = \left(\frac{1}{\sqrt{\pi}} \int_{t_1 - \frac{1}{2}h}^{t_2 + \frac{1}{2}h} e^{-t^2}\, dt\right)(1 + \theta_{27} h^{\frac{1}{2}}), \tag{1.53}$$

for $\Sigma 2pq \geqslant B$.

From this and (1.52) we have, as before: To every $\epsilon > 0$, $\eta > 0$ corresponds a $K(\eta, \epsilon)$ such that, provided that

$$-(\Sigma 2pq)^{\frac{1}{6}-\eta} \leqslant t_1 \leqslant t_2 \leqslant (\Sigma 2pq)^{\frac{1}{6}-\eta}$$

and

$$\Sigma p + \begin{matrix} t_1 \\ t_2 \end{matrix} \sqrt{(\Sigma 2pq)}$$

are integers,

$$(1-\epsilon)\frac{1}{\sqrt{\pi}} \int_{t_1 - \frac{1}{2}h}^{t_2 + \frac{1}{2}h} e^{-t^2} dt < P(t_1, t_2) < (1+\epsilon)\frac{1}{\sqrt{\pi}} \int_{t_1 - \frac{1}{2}h}^{t_2 + \frac{1}{2}h} e^{-t^2}\, dt \tag{1.54}$$

for all $\Sigma 2pq \geqslant K(\eta, \epsilon)$.

It only remains to remove the second restriction on the t's. Plainly we cannot do this unless $(t_2 - t_1)\sqrt{(\Sigma 2pq)} \to \infty$ with $\Sigma 2pq$. For let t_1, t_2 be *real* and let t_1', t_2' be defined by

$$\mu_1 = \Sigma p + t_1 \sqrt{(\Sigma 2pq)}, \quad m_1' = [\mu_1]^*, \qquad m_1' = \Sigma p + t_1' \sqrt{(\Sigma 2pq)},$$

$$\mu_2 = \Sigma p + t_2 \sqrt{(\Sigma 2pq)}, \quad m_2' = -[-\mu_2], \quad m_2' = \Sigma p + t_2' \sqrt{(\Sigma 2pq)}.$$

Then

$$\begin{aligned} P\{\Sigma p + t_1 \sqrt{(\Sigma 2pq)} \leqslant m \leqslant \Sigma p + t_2 \sqrt{(\Sigma 2pq)}\} &= P(\mu_1 \leqslant m \leqslant \mu_2) \\ &= P(m_1' \leqslant m \leqslant m_2') \\ &= P(t_1', t_2'). \end{aligned}$$

But (1.54) holds for $P(t_1', t_2')$; it follows that, for all real t_1, t_2 satisfying

$$-(\Sigma 2pq)^{\frac{1}{6}-\eta} \leqslant t_1 \leqslant t_2 \leqslant (\Sigma 2pq)^{\frac{1}{6}-\eta},$$

$P(t_1, t_2)$ lies between $\quad (1 \pm \epsilon)\frac{1}{\sqrt{\pi}} \int_{t_1' - \frac{1}{2}h}^{t_2' + \frac{1}{2}h} e^{-t^2}\, dt,$

provided that $\Sigma 2pq \geqslant K(\eta, \epsilon)$. Now

LEMMA 1.55. *If $\phi(x) \to \infty$ with x, then, given $\epsilon > 0$, we can find $K_1(\phi, \epsilon)$ so that*

$$\int_{t_1 - \frac{1}{2}\theta'/\sqrt{x}}^{t_2 + \frac{1}{2}\theta''/\sqrt{x}} e^{-t^2}\, dt$$

---

* $[x]$ is the greatest integer not greater than x; $-[-x]$ is thus the least integer not less than x.

lies between $$(1\pm\epsilon)\int_{t_1}^{t_2} e^{-t^2}\,dt$$

for all $|\theta'|,\ |\theta''|\leqslant 1,\ x>K_1(\phi,\epsilon)$ *and* $-\infty<t_1\leqslant t_2-\dfrac{\phi(x)}{\sqrt{x}}<\infty$.

The proof is trivial.

Writing $x=\Sigma 2pq$ and applying the lemma, we have finally—

"If $\phi(x)\to\infty$ with x, $0<\eta\leqslant\frac{1}{6}$, and t_1, t_2 are real numbers satisfying

$$|t_1\ ,\ |t_2|\leqslant K(\Sigma 2pq)^{\frac{1}{6}-\eta};$$

and if $m_1=\Sigma p+t_1\sqrt{(\Sigma 2pq)}$ and $m_2=\Sigma p+t_2\sqrt{(\Sigma 2pq)}$ satisfy

$$m_2-m_1\geqslant\phi(\Sigma 2pq),$$

then, given $\epsilon>0$, we have

$$(1-\epsilon)\frac{1}{\sqrt{\pi}}\int_{t_1}^{t_2}e^{-t^2}\,dt<P(m_1\leqslant m\leqslant m_2)<(1+\epsilon)\frac{1}{\sqrt{\pi}}\int_{t_1}^{t_2}e^{-t^2}\,dt$$

for all $\sum_1^n 2pq\geqslant N(K,\ \phi,\ \eta;\ \epsilon)$." (1.56)

CERTAIN HYPERSPATIAL PARTITIONINGS CONNECTED WITH PREFERENTIAL VOTING

By D. M. Y. SOMMERVILLE.

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1.1. The particular homogeneous divisions of space which are here described arose in the discussion of the Nanson preferential system of voting*. In that system all the candidates are arranged in order of preference by each voter, and the total numbers of preferences of each candidate with respect to every other are enumerated. Let A_{rs} denote the number of times candidate r is preferred to candidate s; then, if N is the total number of voters, $A_{rs}+A_{sr}=N$. Write $A_{rs}-A_{sr}=a_{rs}$, so that $A_{rs}=\frac{1}{2}(N+a_{rs})$, $A_{sr}=\frac{1}{2}(N-a_{rs})=\frac{1}{2}(N+a_{sr})$, and

$$a_{rs}=-a_{sr}. \tag{1.11}$$

Thus, in the case of n candidates, we have to deal with $\frac{1}{2}n(n-1)$ numbers a_{rs} $(r\neq s=1, 2, \ldots, n)$. Of these, however, only the ratios and the signs are significant, and they can be taken as homogeneous point-coordinates in a spherical space of $\frac{1}{2}n(n-1)-1=\frac{1}{2}(n-2)(n+1)$ dimensions. There are $n!$ different orders among the n candidates, and each order, as determined by the voters as a whole, will correspond to a group of inequalities among the a_{rs} and will be represented by a region of the spherical space. Thus the spherical space will be divided homogeneously into $n!$ cells. Different types of division will arise according to the principles which are adopted as determining the different orders. The validity of the principles adopted does not now concern us.

* E. J. Nanson, "Methods of election", *Trans. Royal Soc., Victoria*, 19 (1882), 197–240.

2. *Partitions of spherical space of $n-2$ dimensions into $n!$ cells.*

2.1. There is one simple principle (rather a crude one), enunciated by Borda, on which the order of the candidates could be decided, which leads to a homogeneous division of space of much lower dimensions. We shall consider this first.

Let the order of the candidates be that of the total preferences of each candidate with respect to the others. For three candidates we have then the three numbers

$$a_{12}+a_{13},\ a_{21}+a_{23},\ a_{31}+a_{32}.$$

Denote these by x, y, z. Then $x+y+z=0$ identically, and we have to deal with the ratios of just two independent variables, which may be represented on a "spherical space of one dimension", *i.e.* a circle. The order 123 corresponds to the inequalities $x-y>0$, $x-z>0$, $y-z>0$. The six regions are as indicated in Fig. 1.

Fig. 1.

2.2. For four candidates we have four numbers

$$a_{12}+a_{13}+a_{14},\ a_{21}+a_{23}+a_{24},\ a_{31}+a_{32}+a_{34},\ a_{41}+a_{42}+a_{43},$$

or x, y, z, w, with the identity $x+y+z+w=0$. Thus we have the ratios of three independent variables, and we get a homogeneous division of "spherical space of two dimensions", *i.e.* a sphere, into twenty-four parts. These are obtained either by dividing the four triangles of the tetrahedral network by the medians each into six triangles, or by dividing the six quadrilaterals of the hexahedral network by the diagonals each into four triangles. The sphere is divided into twenty-four tri-

anlges by six great circles $x-y$, $x-z$, $x-w$, $y-z$, $y-w$, $z-w$, and the regions are as indicated in Fig. 2.

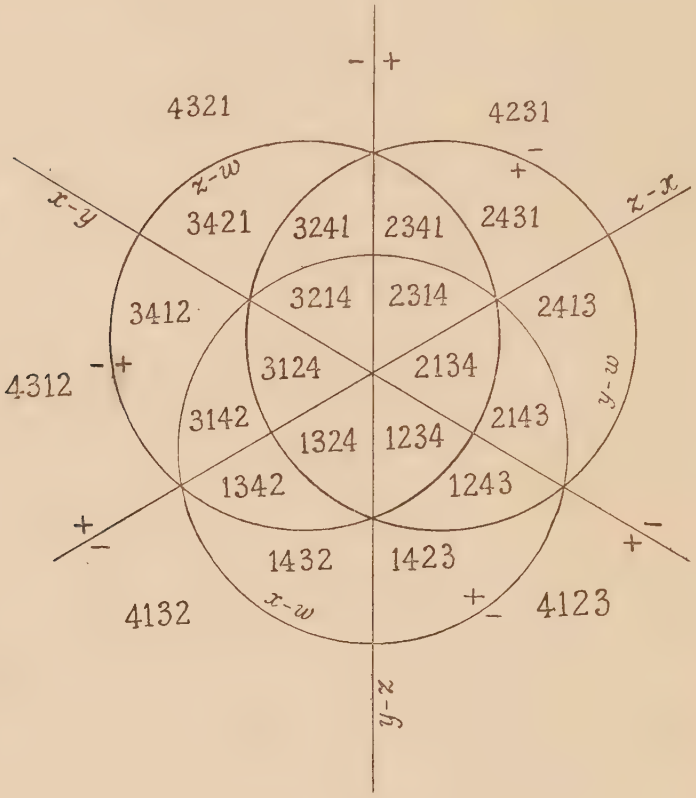

Fig. 2.

2.3. Similarly for five candidates we have a homogeneous division of spherical space of three dimensions into 120 tetrahadra; and generally for n candidates we have a homogeneous division of spherical space of $n-2$ dimensions into $n!$ simplexes.

Each region adjoins $n-1$ others, which correspond to the orders obtained from the given region by interchanging two adjacent numbers.

3. *Partitions of spherical space of* $\frac{1}{2}(n-2)(n+1)$ *dimensions into* $n!$ *cells.*

3.1. We shall consider now a general principle* for deciding the order of the candidates which leads to a homogeneous division of space of the

* This has been described under the designation "the principle of maximum weight" in a paper presented to the Royal Society of Edinburgh entitled "An analysis of preferential voting."

maximum dimensions. Taking first three candidates, if we form for each order, *e.g.* 213, the sum of the preferences $a_{21}+a_{23}+a_{13}$, the suffixes in each case being in the given order, then that order which corresponds to the greatest positive sum is to be regarded as the order determined by the voting. The inequalities which express the preference of the order 123 to the orders 132, 213, 231, 312, 321 respectively are then

$$a_{23}>0,\ \ a_{12}>0,\ \ a_{12}+a_{13}>0,\ \ a_{13}+a_{23}>0,\ \ a_{12}+a_{13}+a_{23}>0.$$

But the last inequality is included in the others. Hence, considering a_{23}, a_{31}, a_{12} as homogeneous point-coordinates on the surface of a sphere, we have a homogeneous division of spherical space of two dimensions into six quadrilaterals, *i.e.* the hexahedral network (Fig. 3).

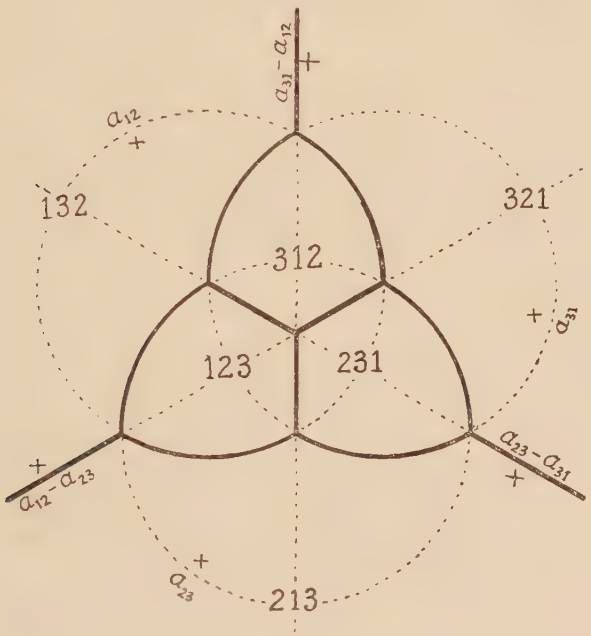

Fig. 3.

3.2. Taking next four candidates, we have six numbers a_{rs}, which may be considered as homogeneous point-coordinates in spherical space of five dimensions. The condition that the order 1234 is preferred to the order 3124, for example, is that

$$a_{12}+a_{13}+a_{14}+a_{23}+a_{24}+a_{34}>a_{31}+a_{32}+a_{34}+a_{12}+a_{14}+a_{24},$$

i.e.

$$a_{13}+a_{23}>0.$$

If (rs), (tu), ... are the transpositions by which the order 1234 is trans-

formed into another order, the condition that the order 1234 is preferred to the other order is $a_{rs}+a_{tu}+\ldots>0$, where $r, s; t, u; \ldots$ are in the order in which they occur in 1234.

Comparing the order 1234 with each of the other orders, we find all the conditions for the order 1234. On inspection it is found that some of the twenty-three conditions are implied by others, and only twelve are independent. These are that the following expressions should be positive. (The order with which 1234 is compared is given in brackets.)

(2134)	a_{12},	(1423)	$a_{24}+a_{34}$,
(1324)	a_{23},	(2341)	$a_{12}+a_{13}+a_{14}$,
(1243)	a_{24},	(2413)	$a_{12}+a_{14}+a_{34}$,
(2314)	$a_{12}+a_{13}$,	(3142)	$a_{13}+a_{23}+a_{24}$,
(3124)	$a_{13}+a_{23}$,	(4123)	$a_{14}+a_{24}+a_{34}$,
(1342)	$a_{23}+a_{24}$,	(3412)	$a_{13}+a_{14}+a_{23}+a_{24}$.

If, then, we consider the a's as homogeneous point-coordinates in spherical space of five dimensions, namely

$$a_{23}=x_1, \quad a_{14}=x_4,$$
$$a_{31}=x_2, \quad a_{24}=x_5,$$
$$a_{12}=x_3, \quad a_{34}=x_6,$$

the twenty-four orders correspond to twenty-four 5-dimensional regions R_5. Each R_5 is in contact with twelve other regions along a 4-dimensional boundary R_4. The above scheme shows the boundaries of the region (1234); the regions in contact with the other regions can be obtained by transforming the numbers 1, 2, 3, 4 by the operations of the substitution-group G on the four numbers. The regions (1234) and (4321) are antipodal and have no common boundary of any dimensions, but the other ten regions, as will appear, touch (1234) in boundaries of dimensions lower than four.

3.3. If the twelve equations or inequalities

$$\begin{array}{lll} x_3\geqslant 0, & x_3-x_2\geqslant 0, & x_3-x_2+x_4\geqslant 0, \\ x_1\geqslant 0, & -x_2+x_1\geqslant 0, & x_4+x_5+x_6\geqslant 0, \\ x_6\geqslant 0, & x_1+x_5\geqslant 0, & -x_2+x_1+x_5\geqslant 0, \\ & x_5+x_6\geqslant 0, & x_3+x_4+x_6\geqslant 0, \\ & & -x_2+x_4+x_1+x_5\geqslant 0, \end{array}$$

which determine the region (1234) and its boundaries, are taken five at a time, we obtain the vertices of this region, namely the ten points

(1, 0, 0, 0, 0, 0), (0, 0, 0, 1, 0, 0), (1, 0, 0, 0, −1, 1),
(0, −1, 0, 0, 0, 0), (0, 0, 0, 0, 1, 0), (0, −1, 0, −1, 0, 1),
(0, 0, 1, 0, 0, 0), (0, 0, 0, 0, 0, 1), (0, 0, 1, −1, 1, 0),
(1, 1, 1, 0, 0, 0),

3 . 4. Taking the following ten points, with the bidigital notation,

14 (1, 0, 0, 0, 0, 0), 23 (0, 0, 0, 1, 0, 0), 15 (−1, 0, 0, 0, 1, −1),
24 (0, 1, 0, 0, 0, 0), 31 (0, 0, 0, 0, 1, 0), 25 (0, −1, 0, −1, 0, 1),
34 (0, 0, 1, 0, 0, 0), 12 (0, 0, 0, 0, 0, 1), 35 (0, 0, −1, 1, −1, 0),
45 (1, 1, 1, 0, 0, 0),

and the ten antipodal points, with coördinates reversed in sign and digits interchanged, we see that every operation of the group G, performed upon the suffixes of the a's, transforms each set of eight points which have a common digit into another set having a common digit; and, moreover, we find that each set of four points with a common initial digit is transformed by any odd substitution into a set of four points having a common final digit.

The whole figure has therefore just these twenty points as vertices.

3 . 5. The vertices of the region (1234) are thus

(1234) 14, 42, 34, 23, 31, 12, 51, 25, 53, 45.

In this set of ten pairs of digits each of the digits 1, 2, 3, 4, 5 occurs four times, twice as initial and twice as final, and, by the above result, every substitution of the group preserves this character. There are thirty tetrads of vertices of the type 12, 31, 14, 51, and each of the twenty-four regions contains five such tetrads. Hence each tetrad belongs to four regions, and is therefore a plane quadrilateral. The eight vertices with a common digit form six quadrilaterals and have the configuration of a cube, opposite vertices of the cube being antipodal vertices of the whole figure. The twenty vertices, taken in sets of eight, each twice, form the vertices of five "cubes", every two cubes having a pair of opposite vertices common.

3 . 6. The adjacent regions (1234) and (3142) are bounded by $a_{13}+a_{23}+a_{24}=0$, *i.e.* $x_1-x_2+x_5=0$, and this locus contains the five

vertices 12, 23, 34, 45, 51, which therefore form a simplex or 5-cell. This has five edges which belong to quadrilaterals (12, 23), (23, 34), (34, 45), (45, 51), (51, 12), "quadrilateral edges"; and five, such as (12, 34), "non-quadrilateral edges". The total number of quadrilateral edges is $5 \times {}_4P_2 = 60$, and the total number of non-quadrilateral edges is $\frac{1}{2} \times {}_5P_4 = 60$. Two vertices with the same initial or the same final digit are opposite corners of a quadrilateral and they do not form an edge. The total number of 5-cells in the figure is the number of cyclic permutations of the five digits, namely twenty-four.

3.7. The adjacent regions (1234) and (1423) are bounded by $a_{24}+a_{34} = 0$, *i.e.* $x_5+x_6 = 0$, and this locus contains the six vertices 23, 34, 42, 14, 45, 51. This figure contains one quadrilateral (34, 42, 14, 45). The vertex 23 is joined to the four corners forming a quadrilateral pyramid, and the remaining vertex 51 is joined to each of the others. The 4-dimensional figure is thus a pyramid of the second order, *i.e.* a pyramid with a 3-dimensional pyramid as base. It is bounded by two quadrilateral pyramids and four tetrahadra, and is represented diagrammatically in Fig. 4, where thick lines represent

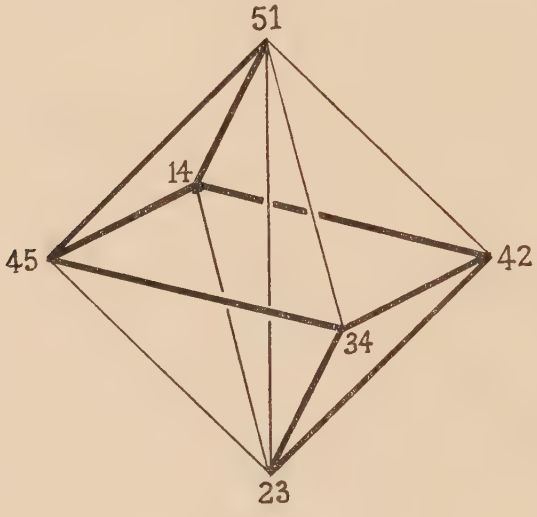

Fig. 4.

quadrilateral edges. We shall refer to this figure simply as a *pyramid*, and denote it by 23 (34, 42, 14, 45) 51. To each of the thirty quadrilaterals correspond two pyramids. If we take the quadrilateral (34, 42, 14, 45), the other two vertices must have as initial either 2 or 5 and as final either 1 or 3, and we have the two pyramids

23 (34, 42, 14, 45) 51 and 54 (34, 45, 14, 42) 21. The total number of pyramids is therefore sixty.

3 . 8. The adjacent regions (1234) and (2134) are bounded by $a_{12} = 0$, *i.e.* $x_3 = 0$, and this locus contains the seven vertices 14, 42, 23, 31, 12, 51, 25. This figure contains two quadrilaterals with a common vertex 12, and it may be written 23, 42, 25 (12) 31, 14, 51. It is bounded by four quadrilateral pyramids and four tetrahedra, and it may be referred to as an 8-cell; it is represented diagrammatically in Fig. 5. The common

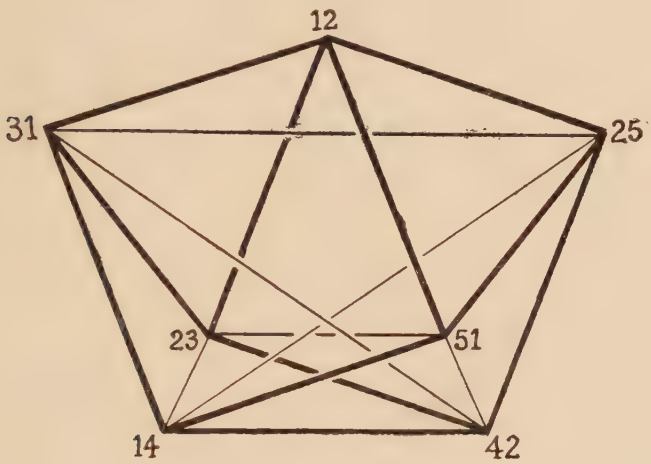

Fig. 5.

vertex of the two quadrilaterals can be chosen in twenty ways. The vertices opposite to this in the two quadrilaterals have a common digit and they can be chosen in three ways. The other vertices are then determined. The total number of 8-cells in the figure is therefore sixty.

3 . 9. These are all the types of 4-dimensional boundaries. The twelve boundaries of the region (1234) are as follows: two 5-cells adjacent to the regions (3142) and (2413); five pyramids adjacent to (1423), (3412), (2314), (1342), and (3124); five 8-cells adjacent to (2134), (1324), (1243), (4123), and (2341).

The following is a summary of all the constituents of the figure. N_r denotes the total number of r-dimensional boundaries, N_{rs} $(r < s)$ denotes the number of r-dimensional boundaries in an s-dimensional boundary, and N_{rs} $(r > s)$ the number of r-dimensional boundaries which pass through an s-dimensional boundary. These have all been checked

by Euler's formulae $N_5-N_4+N_3-N_2+N_1-N_0=0$, etc., and by the relations of the form $N_{rs}.N_s=N_{sr}.N_r$. Quadrilateral edges are denoted by q, non-quadrilateral edges by non-q. In a tetrahedron with three quadrilateral edges these may form a triangle ($3q\Delta$) or they may form an open chain ($3q$ open); in a tetrahedron with four quadrilateral edges these form a closed chain.

$N_5 = 24$,

$N_4 = 144$ (24 5-cells, 60 pyramids, 60 8-cells),

$N_3 = 310$ (190 tetrahedra : 30 with $4q$, 40 with $3q\Delta$, 120 with $3q$ open; 120 quadrilateral pyramids),

$N_2 = 290$ (260 triangles : 20 with $3q$, 120 with $2q$, 120 with $1q$; 30 quadrilaterals),

$N_1 = 120$ ($60q$, 60 non-q),

$N_0 = 20$.

$(N_5-N_4+N_3-N_2+N_1-N_0=0)$.

$N_{45} = 12$ (2 5-cells, 5 pyramids, 5 8-cells),

$N_{35} = 40$ (25 tetrahedra : 5 with $4q$, 5 with $3q\Delta$, 15 with $3q$ open; 15 quadrilateral pyramids),

$N_{25} = 55$ (50 triangles : 5 with $3q$, 25 with $2q$, 20 with $1q$; 5 quadrilaterals),

$N_{15} = 35$ ($20q$, 15 non-q),

$N_{05} = 10$.

$(N_{45}-N_{35}+N_{25}-N_{15}+N_{05}=2)$.

The 4-dimensional boundaries.

5-cell.

$N'_{34} = 5$ (tetrahedra with $3q$ open),

$N'_{24} = 10$ (triangles : with $2q$, 5 with $1q$),

$N'_{14} = 10$ ($5q$ in one cycle, 5 non-q),

$N'_{04} = 5$.

Pyramid.

$N''_{34} = 6$ (4 tetrahedra : 2 with $3q\Delta$, 2 with $3q$ open ; 2 quadrilateral pyramids),

$N''_{24} = 13$ (12 triangles : 2 with $3q$, 4 with $2q$, 6 with $1q$; 1 quadrilateral),

$N''_{14} = 13$ ($8q$, 5 non-q),

$N''_{04} = 6$ (4 vertices with $3q$ and 1 non-q, 2 with $2q$ and 3 non-q).

8-cell.

$N'''_{34} = 8$ (4 tetrahedra : 2 with $4q$, 2 with $3q$ open ; 4 quadrilateral pyramids),

$N'''_{24} = 18$ (16 triangles : 2 with $3q$, 10 with $2q$, 4 with $1q$; 2 quadrilaterals),

$N'''_{14} = 17$ ($11q$, 6 non-q),

$N'''_{04} = 7$ (1 vertex with $4q$, 6 with $3q$, and 2 non-q)

$(N_{34} - N_{24} + N_{14} - N_{04} = 0)$.

$N_{50} = 12$,

$N_{40} = 45$ (6 5-cells, 18 pyramids, 21 8-cells),

$N_{30} = 68$ (38 tetrahedra : 6 with $4q$, 8 with $3q\Delta$, 24 with $3q$ open ; 30 quadrilateral pyramids),

$N_{20} = 45$ (39 triangles : 3 with $3q$, 18 with $2q$, 18 with $1q$; 6 quadrilaterals),

$N_{10} = 12$ ($6q$, 6 non-q)

$(N_{50} - N_{40} + N_{30} - N_{20} + N_{10} = 2)$.

Through a quadrilateral edge.

$N'_{51} = 8$,

$N'_{41} = 21$ (2 5-cells, 8 pyramids, 11 8-cells),

$N'_{31} = 22$ (10 tetrahedra : 2 with $4q$, 2 with $3q\Delta$, 6 with $3q$ open ; 12 quadrilateral pyramids),

$N'_{21} = 9$ (7 triangles : 1 with $3q$, 4 with $2q$, 2 with $1q$; 2 quadrilaterals).

Through a non-quadrilateral edge.

$N''_{51} = 6$,

$N''_{41} = 13$ (2 5-cells, 5 pyramids, 6 8-cells),

$N''_{31} = 13$ (9 tetrahedra : 1 with $4q$, 2 with $3q\Delta$, 6 with $3q$ open; 4 quadrilateral pyramids),

$N''_{21} = 6$ (6 triangles : 2 with $2q$, 4 with $1q$)

$(N_{51} - N_{41} + N_{31} - N_{21} = 0)$.

Through a triangle.

Triangle with $3q$.

$N'''_{52} = 6$,

$N'''_{42} = 12$ (6 pyramids, 6 8-cells),

$N'''_{32} = 8$ (2 tetrahedra with $3q\Delta$, 6 quadrilateral pyramids).

Triangle with $2q$.

$N''_{52} = 5$,

$N''_{42} = 8$ (1 5-cell, 2 pyramids, 5 8-cells),

$N''_{32} = 5$ (3 tetrahedra : 1 with $4q$, 2 with $3q$ open; 2 quadrilateral pyramids).

Triangle with $1q$.

$N'_{52} = 4$,

$N'_{42} = 6$ (1 5-cell, 3 pyramids, 2 8-cells),

$N'_{32} = 4$ (3 tetrahedra : 1 with $3q\Delta$, 2 with $3q$ open).

Through a quadrilateral.

$N^{iv}_{52} = 4$,

$N^{iv}_{42} = 6$ (2 pyramids, 4 8-cells),

$N^{iv}_{32} = 4$ (quadrilateral pyramids)

$(N_{52} - N_{42} + N_{32} = 2)$.

Through a tetrahedron.

Tetrahedron with $4q$.

$N_{53}^{iv} = 4$,

$N_{43}^{iv} = 4$ (8-cells).

Tetrahedron with $3q\Delta$.

$N_{53}''' = 3$,

$N_{43}''' = 3$ (pyramid).

Tetrahedron with $3q$ *open.*

$N_{53}'' = 3$,

$N_{43}'' = 3$ (1 5-cell, 1 pyramid, 1 8-cell).

Through a quadrilateral pyramid.

$N_{53}' = 3$,

$N_{43}' = 3$ (1 pyramid, 2 8-cells).

4. *The polytope corresponding to the homogeneous division of spherical space of five dimensions.*

4.1. If we regard the 5-dimensional space as a hypersphere immersed in a flat space of six dimensions S_6, we obtain a configuration of points, lines, etc., in S_6. This will be a polytope if all the vertices of each of the cells (1234), etc., lie in hyperplanes. The coordinates which we have taken for the vertices may now be regarded as the direction-ratios of the lines from the centre of the hypersphere—the angles between the axes are left as yet undetermined. If the actual coordinates of the vertices are these numbers multiplied by certain factors we easily find, as a condition that the vertices of each region should lie in a hypersphere, that the factors are all equal. Taking, therefore, the coordinates of the vertices as

14 (1, 0, 0, 0, 0, 0),	23 (0, 0, 0, 1, 0, 0),	15 (−1, 0, 0, 0, 1, −1),
24 (0, 1, 0, 0, 0, 0),	31 (0, 0, 0, 0, 1, 0),	25 (0, −1, 0, −1, 0, 1),
34 (0, 0, 1, 0, 0, 0),	12 (0, 0, 0, 0, 0, 1),	35 (0, 0, −1, 1, −1, 0),
		45 (1, 1, 1, 0, 0, 0),

then, for the hyperplane (1234), which contains the vertices 14, 42, 34, 23, 31, 12, 51, 25, 53, 45, we find the equation

$$x_1-x_2+x_3+x_4+x_5+x_6=1;$$

and the tangential coordinates (namely the coefficients in the equation

$$l_1x_1+l_2x_2+l_3x_3+l_4x_4+l_5x_5+l_6x_6=1)$$

of the various regions are as follows :

(3124) (−1, 1, 1, 1, 1, 1), (1243) (1, −1, 1, 1, 1, −1),

(1234) (1, −1, 1, 1, 1, 1), (2341) (1, 1, −1, −1, 1, 1),

(2314) (1, 1, −1, 1, 1, 1), (3142) (−1, 1, 1, 1, −1, 1),

(2134) (1, −1, −1, 1, 1, 1), (1342) (−1, −1, 1, 1, −1, 1),

(3214) (−1, 1, −1, 1, 1, 1), (2143) (1, −1, −1, 1, 1, −1),

(1324) (−1, −1, 1, 1, 1, 1), (3241) (−1, 1, −1, −1, 1, 1),

the twelve antipodal regions having the signs of the tangential coordinates all reversed.

4 . 2. If the axes are rectangular the polytope is circumscribed about a hypersphere, but it is not inscribed in a hypersphere, the vertices 15, 25, 35, 45 being at a distance $\sqrt{3}$ from the centre, while the others are at a distance unity. The condition that the polytope be circumscribed about a hypersphere is clearly a necessary condition in order that its 5-dimensional boundaries should be congruent. Further, the five cubical configurations of the figure are not congruent, and they are not even all cubes. The "cube" 5 is a cube of edge 2, but the other four are parallelepipeds with equal edges $\sqrt{2}$ and angles 60°.

4 . 3. The angles between the axes are, however, capable of arbitrary adjustment, and we shall therefore suppose the axes to be oblique. We require certain general formulae for oblique axes, or the general Cartesian system. Let the angle between the axes x_r, x_s be θ_{rs} and denote $\cos\theta_{rs}$ by c_{rs}. Then the distance of the point P $(x_1, \ldots, x_6)$ from the origin is given by

$$OP^2=\Sigma x_r^2+2\,\Sigma c_{rs}x_rx_s; \qquad (4\,.\,31)$$

and, $x_0=0$ being the equation of the hyperplane at infinity, the equations of the hypersphere at infinity (the absolute) are

$$x_0=0, \quad F(x,x)\equiv\Sigma x_r^2+2\,\Sigma c_{rs}x_rx_s=0. \qquad (4\,.\,32)$$

The angle between the two directions (l) and (l') is given by

$$\cos\theta = \frac{F(l, l')}{\{F(l, l)\,F(l', l')\}^{\frac{1}{2}}}. \tag{4.33}$$

The tangential equation of the absolute is

$$\Phi(l, l) = \begin{vmatrix} 1 & c_{12} \dots c_{16} & l_1 \\ c_{12} & 1 \dots c_{26} & l_2 \\ \dots & \dots \quad \dots & \dots \\ c_{16} & c_{26} \dots 1 & l_6 \\ l_1 & l_2 \dots l_6 & 0 \end{vmatrix} = 0 \tag{4.34}$$

and the angle between the two hyperplanes $(l_1, \dots, l_6)$, $(l_1', \dots, l_6')$ is given by

$$\cos\phi = \frac{\Phi(l, l')}{\{\Phi(l, l)\,\Phi(l', l')\}^{\frac{1}{2}}}. \tag{4.35}$$

The normal to the hyperplane $(l_1, \dots, l_6)$ is determined by the point at infinity which is the pole of the hyperplane with regard to the hypersphere at infinity. Its direction-ratios are

$$\frac{\partial\Phi}{\partial l_1}, \quad \dots, \quad \frac{\partial\Phi}{\partial l_6}.$$

The tangential equation of a hypersphere with centre O and radius r is

$$r^2\Phi(l, l) + \Delta = 0 \tag{4.36}$$

where

$$\Delta \equiv \begin{vmatrix} 1 & c_{12} \dots & c_{16} \\ \dots & \dots & \dots \\ c_{16} & c_{26} \dots & 1 \end{vmatrix}, \tag{4.37}$$

and r is the distance of the hyperplane $l_1 x_1 + \dots + l_6 x_6 = 1$ from the origin.

4.4. We shall find now the conditions that all the five cubical configurations should be cubes; the conditions are that the angles between their diagonals should be equal to $\cos^{-1}\frac{1}{3}$ or the supplement of this angle. The conditions are found to be

$$c_{12} = c_{13} = c_{23} = c_{16} = c_{24} = c_{35} = -\tfrac{1}{3},$$

$$c_{15} = c_{26} = c_{34} = c_{45} = c_{46} = c_{56} = \tfrac{1}{3},$$

$$c_{14} = c_{25} = c_{36}.$$

These conditions also secure that the vertices all lie on a hypersphere, the radius being 1; and the quadrilateral edges, edges of cubes, are each $2/\sqrt{3}$. The squares of the non-quadrilateral edges (12, 34) and (12, 43) are $2-2c_{36}$ and $2+2c_{36}$. These are equal if $c_{36}=0$. With this further condition all the non-quadrilateral edges are equal to $\sqrt{2}$.

4.5. With these values of the c's, the tangential equation of the absolute assumes the simple form

$$\Phi(l, l) \equiv \Sigma l_r^2 - 2\Sigma c_{rs} l_r l_s = 0;$$

and the distance p of the hyperplane $l_1 x_1 + \ldots + l_6 x_6 = 1$ from the origin is given by

$$p^2(\Sigma l_r^2 - 2\Sigma c_{rs} l_r l_s) = \tfrac{5}{9}.$$

We find that all the 5-dimensional regions are now equidistant from the origin, and the polytope is circumscribed about a hypersphere of radius $1/\sqrt{6}$. Hence the vertices of each of these regions lie on a 5-dimensional hypersphere of radius $\sqrt{\frac{5}{6}}$. For the region (3124) $(-1, 1, 1, 1, 1, 1)$ we find the direction-ratios of the normal $(-1, 3, 3, 1, 3, -1)$, and hence the point of contact with the central hypersphere is $(-\frac{1}{10}, \frac{3}{10}, \frac{3}{10}, \frac{1}{10}, \frac{3}{10}, -\frac{1}{10})$. It is easily verified that this point is at a distance $\sqrt{\frac{5}{6}}$ from each of the vertices 23, 31, 12, 41, 24, 34, 15, 52, 53, 45.

MATHEMATICAL NOTES (8): ON FUNCTIONS SUBHARMONIC IN A CIRCLE (II).

By J. E. LITTLEWOOD.

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1. In Note 7 of this series† I proved that, if $v(r, \theta)$ is subharmonic in the unit circle $r < 1$ and satisfies

$$\int_{-\pi}^{\pi} |v(r, \theta)|^p \, d\theta = O(1) \quad (r < 1), \tag{1.1}$$

where $p > 1$, then there exists a function $V(\theta)$ such that, for any positive $q < p$,

$$\lim_{r \to 1} \int_{-\pi}^{\pi} |v(r, \theta) - V(\theta)|^q \, d\theta = 0;$$

v converges strongly to V with index q as $r \to 1$. The further question, whether $\lim_{r \to 1} v(r, \theta) = V(\theta)$ for almost all values of θ, I left unanswered. I am now able to answer it, and in the affirmative. We have, in fact, the following

THEOREM. *Let v be subharmonic in $r < 1$ and satisfy*

$$\int_{-\pi}^{\pi} |v(r, \theta)| \, d\theta = O(1) \quad (r < 1).$$

Then there exists a $V(\theta)$ such that

$$\lim_{r \to 1} v(r, \theta) = V(\theta)$$

for almost all values of θ.

† J. E. Littlewood, 2. (See the list of papers at the end of the note.)

In this theorem we suppose the definition of a subharmonic function to be the widest possible: v is to satisfy at every point z_0 the usual inequality

$$v(z_0) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} v(z_0+\rho e^{\phi i})\,d\phi$$

for all† sufficiently small ρ, and it is further to be upper-semi-continuous.

2. In what follows we use P, (x, y), (r, θ) indifferently to denote the current point; Q, (ξ, η), (ρ, ϕ) indifferently to denote a parametric point (representing variables of integration). Let

$$g(P, Q) = g(x, y\,;\ \xi, \eta) = g(r, \theta\,;\ \rho, \phi)$$

be the Green's function for the unit circle. For P interior to the circle, g is positive for interior Q, zero for Q on the circumference. It is harmonic in Q except for a logarithmic singularity at $Q = P$. The value of g is given by

$$g(P, Q) = \log\left(\frac{r\,.\,QP'}{QP}\right), \tag{2.1}$$

where P' is the point inverse to P in the circle.

We shall frequently use $s = 1-r$, $\sigma = 1-\rho$ in place of r, ρ. We write $\psi = \phi-\theta$. The angles θ, ϕ, ψ have a natural period 2π and it is more convenient sometimes to allow them the range $(-\pi, \pi)$, sometimes the range $(0, 2\pi)$; we shall in future use whichever range is convenient without further explanation.

We use throughout the symbol A to denote a positive absolute constant, not always the same from one occurrence to another.

The function $g(P, Q)$ satisfies (for interior P, Q) the following inequalities, in which the range of ψ is supposed to be $(-\pi, \pi)$.

$$0 < g(P, Q) \leqslant \frac{A\sigma s}{(s-\sigma)^2+\psi^2} \quad (s < \tfrac{1}{2})\ddagger, \tag{2.2}$$

$$0 < g(P, Q) \leqslant \log\left(1+\frac{A\sigma s}{(s-\sigma)^2+\psi^2}\right) \quad (s < \tfrac{1}{2}). \tag{2.3}$$

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† The definition obtained by writing "some arbitrarily" for "all sufficiently" is an equivalent one. See Note 6 of this series. (J. E. Littlewood, **1**.)

‡ Since $g(P, Q) = g(Q, P)$ the inequalities hold also subject to $\sigma < \frac{1}{2}$ (only): we do not use this fact.

Of these the first (needed for reference) follows from the second. For the second half of (2.3) we have

$$e^{2g} = \left(\frac{r,\ QP'}{QP}\right)^2 = \frac{r^2(r^{-2}-2r^{-1}\rho\cos\psi+\rho^2)}{r^2-2r\rho\cos\psi+\rho^2}$$

$$= 1+\frac{(1-r^2)(1-\rho^2)}{D} \leqslant 1+\frac{4\sigma s}{D},$$

where $D = r^2-2r\rho\cos\psi+\rho^2 = (s-\sigma)^2+4r(1-\sigma)\sin^2\tfrac{1}{2}\psi.$

Since $r > \frac{1}{2}$ (or $s < \frac{1}{2}$), we have either (i) $\sigma \leqslant \frac{3}{4}$ and then

$$D \geqslant (s-\sigma)^2+A\psi^2 \geqslant A\{(s-\sigma)^2+\psi^2\},$$

or else (ii) $\sigma > \frac{3}{4}$ and

$$D > (s-\sigma)^2 > (\tfrac{3}{4}-\tfrac{1}{2})^2 = A \geqslant A(1+\psi^2) \geqslant A\{(s-\sigma)^2+\psi^2\}.$$

In either case

$$e^{2g} \leqslant 1+\frac{A\sigma s}{(s-\sigma)^2+\psi^2},$$

and the desired result follows.

We require also the following property of g. Let

$$G(r,\ \rho) = \int_{-\pi}^{\pi} g(r,\ \theta;\ \rho,\ \phi)\,d\theta = \int_{-\pi}^{\pi} g(r,\ \theta;\ \rho,\ 0)\,d\theta$$

(evidently independent of ϕ). Then

$$G(r,\ \rho) \leqslant A\sigma \quad (r > \tfrac{1}{2}). \tag{2.4}$$

We must distinguish two cases:

$$\text{(i) } |\sigma-s| > \tfrac{1}{2}s, \quad \text{(ii) } |\sigma-s| \leqslant \tfrac{1}{2}s.$$

In (i) (2.2) gives

$$G \leqslant \int_{-\pi}^{\pi} \frac{A\sigma s}{s^2+\theta^2}\,d\theta \leqslant \int_{-\infty}^{\infty} = A\sigma.$$

In (ii) (2.3) gives

$$g \leqslant \log\left(1+\frac{A\sigma s}{\theta^2}\right) \leqslant A\frac{\sigma}{s}\log\left(1+\frac{As^2}{\theta^2}\right),$$

$$G \leqslant A\frac{\sigma}{s}\int_{-\infty}^{\infty} \log\left(1+\frac{As^2}{\theta^2}\right)d\theta = A\sigma.$$

3. We require also the following simple results on approximations.

LEMMA 1. *Suppose that $h(\phi)$ is non-negative and monotonic increasing [to a finite $h(\pi)$] in $0 \leqslant \phi \leqslant \pi$, and satisfies*

$$\overline{\lim_{\phi \to +0}} \frac{h(\phi)}{\phi} \leqslant c.$$

Then

$$\overline{\lim_{s \to +0}} \, s \int_{s-0}^{\pi} \frac{dh(\phi)}{\phi^2} \leqslant Ac, \tag{3.1}$$

$$\overline{\lim_{s \to +0}} \, s \int_{0}^{s} \frac{dh(\phi)}{s^2+\phi^2} \leqslant Ac, \tag{3.2}$$

$$\overline{\lim_{s \to +0}} \, \frac{1}{s} \int_{0}^{s} \log\left(\frac{As^2}{\phi^2}\right) dh(\phi) \leqslant Ac. \tag{3.3}$$

For (3.1) we have

$$s\int_{s-0}^{\pi} = s\left[\frac{h(\phi)}{\phi^2}\right]_{s-0}^{\pi} + 2s\int_{s}^{\pi} \frac{h(\phi)\,d\phi}{\phi^3} \leqslant s\frac{h(\pi)}{\pi^2} + 2s\int_{s}^{\pi} \frac{c\phi + o(\phi)}{\phi^3}\, d\phi$$

$$= o(1) + 2c + o(1) = 2c + o(1).$$

For (3.2) we have

$$s\int_{0}^{s} = s\left[\frac{h(\phi)}{s^2+\phi^2}\right]_{0}^{\pi} + 2s\int_{0}^{s} \frac{h(\phi)\phi\, d\phi}{(s^2+\phi^2)^2}$$

$$\leqslant s\,h(\pi) + 2s\int_{0}^{s} \frac{c\phi^2 + o(\phi^2)}{(s^2+\phi^2)^2}\, d\phi$$

$$= o(1) + 2\int_{0}^{s} \{c + o(1)\} \frac{s\phi^2\, d\phi}{(s^2+\phi^2)^2}$$

$$= o(1) + 2\int_{0}^{1} \{c + o(1)\} \frac{t^2\, dt}{(1+t^2)^2}$$

$$= Ac + o(1).$$

For (3.3) we have

$$\frac{1}{s}\int_{0}^{s} = \frac{1}{s}\left[\log\left(\frac{As^2}{\phi^2}\right) h(\phi)\right]_{0}^{s} + \frac{2}{s}\int_{0}^{s} \frac{h(\phi)\, d\phi}{\phi}$$

$$\leqslant A\frac{h(s)}{s} + \frac{2}{s}\int_{0}^{s} \{c + o(1)\}\, d\phi \leqslant Ac + o(1).$$

4. After these preliminaries we turn to our main problem, which, however, requires a few words of further explanation. For any function $f(r, \theta)$ with continuous second derivatives at all points of the closed unit

circle $r \leqslant 1$ we have, for $r < 1$, the familiar identity

$$(4.1)\quad f(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(1,\phi)\frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2}\,d\phi + \iint g(r,\theta;\rho,\phi)\Delta f(\rho,\phi)\,\rho\,d\rho\,d\phi,$$

where $\Delta f = \frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2}$, g is the Green's function for the circle, and the double integral is extended over the circle $\rho \leqslant 1$. (This formula reduces to the Poisson integral for a harmonic function when Δf vanishes identically, and it is what we obtain when we adapt the proof of Poisson's integral from the special to the general case†.)

Our original problem reduces fairly easily to proving that a non-negative superharmonic function tends radially to a limit for almost all θ. Suppose that in (4.1) f is a non-negative superharmonic function w‡, so that (see § 5)

$$\int_{-\pi}^{\pi} w(r,\theta)\,d\theta$$

is non-negative and decreasing in $r < 1$ (and in particular is bounded). Suppose now that (instead of being continuous in the closed circle) Δw is continuous at every interior point of the circle. It is easy to show that (4.1) still holds (with w for f) provided that we replace the first term on the right by

$$u^*(r,\theta) = \lim_{R\to 1}\frac{1}{2\pi}\int_{-\pi}^{\pi} w(R,\phi)\,P(r,\phi-\theta;R)\,d\phi,$$

where

$$(4.2)\qquad P(r,\psi;R) = \frac{R^2-r^2}{R^2-2Rr\cos\psi+r^2}.$$

This u^* is, in point of fact, a harmonic function for which $\int_{-\pi}^{\pi} u^*\,d\theta$ is bounded, and therefore, by a well known theorem, tends radially to a limit function, as $r\to 1$, for almost all θ. Proving the same thing for w reduces,

† The special case begins with the formula

$$\iint(f\Delta\phi-\phi\Delta f)\,d\xi\,d\eta + \int_C\left(f\frac{\partial\phi}{\partial n}-\phi\frac{\partial f}{\partial n}\right)ds = 0,$$

taking $\phi(\xi,\eta) = g(x,y;\xi,\eta)$, and C to be the circumference $\rho = 1$ plus a small circle round (x,y). Both $\Delta\phi$ and Δf are zero in the special case; (4.1) is what we obtain when Δf is not zero.

‡ We use u for harmonic, v for subharmonic, and w for superharmonic functions. A w is of type $-v$.

then, to proving that the second term on the right-hand side of (4 . 1) also tends radially to a limit for almost all θ. In solving this key problem we shall find that the limit just mentioned exists and is zero; thus, when $r \to 1$, we have

$$w^*(P) = \iint g(P, Q)\, \Delta w(Q)\, d\xi\, d\eta \to 0 \tag{4 . 3}$$

for almost all θ.

It will appear, further, that the continuity of Δw is irrelevant to the proof, and our argument applies whenever the integral in (4 . 3) exists (is finite) at a single point P (the integrand is non-negative); indeed whenever it exists at a P in a certain generalized sense. Now F. Riesz has recently completed his classical theory of subharmonic functions with the very beautiful theorem that *any* positive superharmonic function is the sum of a harmonic function (actually u^*) and a function w^* given by a generalized integral of type

$$\iint g\, dF(\rho, \phi),$$

F being a non-negative additive function of sets in the circle†. Since our argument is still available and establishes our theorem in a form free from any unnecessary hypothesis, it seems well worth while to deal with the general case, in spite of the fact that the details of the proof of Riesz's theorem are at the moment unpublished.

I proceed now to set out the proof in detail.

5. Let w be a superharmonic function satisfying

$$\int_{-\pi}^{\pi} |w(r, \theta)|\, d\theta = O(1) \quad (r < 1). \tag{5 . 1}$$

Let $0 < r < R < 1$ and

$$u(r, \theta;\ R) = \frac{1}{2\pi} \int_{-\pi}^{\pi} w(R, \psi) P(r, \psi - \theta;\ R)\, d\psi, \tag{5 . 2}$$

where P is given by (4 . 2); u is the Poisson integral of the boundary values of w for the circle C_R of radius R about the origin.

LEMMA 2‡. *For fixed* (r, θ) $u(r, \theta;\ R)$ [*given by* (5 . 2) *subject to* (5 . 1)] *is a decreasing function of* R, *and tends, as* $R \to 1$, *to a limit*

† F. Riesz, **1**, 100. The arguments are given only in outline.

‡ Cp. J. E. Littlewood, **2**, Lemma 1, where further references will be found. (The w there is subharmonic and what is there p is here unity.) I repeat the proof there given (with the necessary modifications) for the sake of completeness.

function $u^*(r, \theta)$, *harmonic in* $r < 1$, *and satisfying*

$$\int_{-\pi}^{\pi} |u^*(r, \theta)|\, d\theta = O(1) \quad (r < 1). \tag{5.3}$$

Also

$$w(r, \theta) \geqslant u^*(r, \theta). \tag{5.4}$$

A superharmonic function remains superharmonic under a conformal transformation†. Consider the linear transformation of the circle C_R into itself by which the point (r, θ) becomes the centre. The Poisson integral of $w(R, \psi)$ for the point (r, θ) becomes

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} w_1(R, \psi)\, d\psi,$$

where w_1 is the transform of w. Since w_1 is superharmonic this last expression is, by another theorem of Riesz‡, a decreasing function of R.

Next we have, for any point (r_1, θ_1) with $r_1 < 1$,

$$|u(r_1, \theta_1;\ R)| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |w(R, \psi)|\, \frac{R^2 - r_1^2}{(R - r_1)^2}\, d\psi < K\frac{R + r_1}{R - r_1} \tag{5.5}$$

by (5.1). Hence $\overline{\lim_{R \to 1}}\, |u(r_1, \theta_1| < \infty$. Since u is decreasing for increasing R it converges to a limit at (r_1, θ_1) as $R \to 1$; therefore, being harmonic in (r, θ) and decreasing in R, converges (by a well known theorem of Harnack) in $r < 1$, and uniformly in $r \leqslant r_0 < 1$, to a harmonic limit function $u^*(r, \theta)$. Next, by a theorem of Fatou and (5.2),

$$\int_{-\pi}^{\pi} |u^*(r, \theta)\ d\theta = \int_{-\pi}^{\pi} \lim_{R \to 1} |u(r, \theta;\ R)|\, d\theta \leqslant \underline{\lim} \int_{-\pi}^{\pi} |u(r, \theta;\ R)|\, d\theta$$

$$\leqslant \underline{\lim} \int_{-\pi}^{\pi} d\theta\, \frac{1}{2\pi}\int_{-\pi}^{\pi} |w(R, \psi)|\, P(r, \psi - \theta;\ R)\, d\psi,$$

$$\leqslant \underline{\lim} \int_{-\pi}^{\pi} |w(R, \psi)|\, d\psi,$$

by inverting the order of integration and recalling that $\frac{1}{2\pi}\int_{-\pi}^{\pi} P\, d\theta = 1$; and this limit is a finite constant.

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† F. Riesz, **2** (remark on p. 340). See also J. E. Littlewood, **1**, 190, f.n. ‡.

‡ F. Riesz, **2**, 338.

Finally $-u(r, \theta; R)$, harmonic in $r < R$ and continuous in $r \leqslant R$, is a majorant of $-w$ on the circumference $r = R$, so that, by Riesz's fundamental theorem†,

$$-w(r, \theta) \leqslant -u(r, \theta; R).$$

Making $R \to 1$ in this we have $-w \leqslant -u^*$, $w \geqslant u^*$.

6. LEMMA 3. *Let w be (finite almost everywhere and) superharmonic in $r < 1$, and satisfy*

$$\int_{-\pi}^{\pi} |w(r, \theta)| \, d\theta = O(1) \quad (r < 1). \tag{6.1}$$

Then

$$w = u^* + w^*, \tag{6.2}$$

where u^* is the function of Lemma 2, w^* is non-negative and superharmonic,

$$\lim_{r \to 1} \int_{-\pi}^{\pi} w^*(r, \theta) \, d\theta = 0, \tag{6.3}$$

and w^* is given by

$$w^*(P) = \iint g(P, Q) \, dF(Q)‡, \tag{6.4}$$

where

$$\iint (1-\rho) \, dF(Q) < \infty §. \tag{6.5}$$

The integrals are Stieltjes integrals extended over the circle $\rho < 1$; $F(Q)$ is a non-negative additive function of sets.

Let us denote $w - u^*$ for the moment by w_1. By Lemma 2 w_1 is a non-negative superharmonic function. By Riesz's theorem w_1 is the sum of a w^* given by (6.4) and a harmonic function u_1; $w_1 = w^* + u_1$. What remains to be proved is (6.3) and that u_1 is identically zero‖. But,

† F. Riesz, **2**, 331.

‡ g has an infinity at $Q = P$. The right-hand side of (6.4) means

$$\lim_{N \to \infty} \iint g_N \, dF,$$

where $g_N = \operatorname{Min}(g, N)$.

§ For fixed P $g \geqslant K(1-\rho)$ for all Q, where K depends on P. Hence (6.5) holds if (6.4) exists at any one P.

‖ That $u_1 = 0$ will doubtless appear incidentally in Riesz's final account of his theorem, but for the moment an *ad hoc* proof seems unavoidable.

integrating under the double integration sign in (6 . 4), we have, in the notation of §2,

$$\int_{-\pi}^{\pi} w^*(r, \theta)d\theta = \iint G(r, \rho)dF(Q) = \iint_{\rho \leqslant \rho_0} + \iint_{\rho > \rho_0}.$$

Now g, and so G, tends to zero as $r \to 1$, uniformly in $\rho \leqslant \rho_0 < 1$ (and all θ, ϕ); also $\iint dF(Q)$ taken over $\rho \leqslant \rho_0$ is finite. Hence

$$\overline{\lim_{r \to 1}} \int_{-\pi}^{\pi} w^*(r, \theta)d\theta = \overline{\lim} \iint_{\rho > \rho_0} G(r, \rho)dF(Q)$$

$$\leqslant A \iint_{\rho > \rho_0} \sigma dF(Q)$$

by (2 . 4). The last expression is of the form $\epsilon(\rho_0)$, arbitrarily small when ρ_0 is near enough to unity, in virtue of (6 . 5). It follows that the left side of the inequality is zero, and this gives (6 . 3).

To prove finally that $u_1 = 0$ we observe that

$$\int_{-\pi}^{\pi} w_1(R, \psi)d\psi = \int_{-\pi}^{\pi} w(R, \psi)d\psi - \int_{-\pi}^{\pi} u^*(R, \psi)\, d\psi$$

$$= u(0;\ R) - u^*(0)$$

(the two terms being values of u and u^* at the origin $r = 0$), which tends to zero as $R \to 1$; also that, by Lemma 2, w_1 is non-negative. In the equation

$$w(R, \psi) - u^*(R, \psi) = w^*(R, \psi) + u_1(R, \psi)$$

we multiply by $P(r, \psi - \theta;\ R)$ and integrate with respect to ψ. This gives (supposing $R > r$)

$$u(r, \theta;\ R) - u^*(r, \theta) = \int w^*(R, \psi)P(r, \psi - \theta;\ R)d\psi + u_1(r, \theta).$$

In this we fix (r, θ) and make $R \to 1$. The left-hand side tends to zero, and so, by (6 . 3), does the first term on the right [since $w^*(P)$ is non-negative and $P \leqslant (R+r)/(R-r)$]. Hence $u_1 = 0$, and the proof of the lemma is completed.

7. LEMMA 4. *The function w^* of Lemma 3 satisfies*

$$\lim_{r \to 1} w^*(r, \theta) = 0$$

for almost all values of θ.

This is the kernel of the proof. A theorem asserting the existence of a limit save for an exceptional set Z of zero measure in a parameter (not arising out of a Z in the *hypothesis*) reduces to an elementary problem of approximation once the set Z is given. Now sets Z are, in practice, almost always the sets where some function (not always the first to hand) fails to be the differential coefficient of its integral. Suppose for simplicity that our Stieltjes integral is an ordinary Lebesgue double integral $\iint \sigma D\, d\xi\, d\eta$. It is natural to try (not of course at random) for Z the set of points $\phi = \theta$ at which

$$I(\phi) = \int_0^1 \sigma(\rho, \phi)\, D(\rho, \phi)\, d\rho$$

fails to be the differential coefficient of its integral; *i.e.* to try to prove that w^* converges to zero as $\rho \to 1$ for values of θ for which $\dfrac{d}{d\theta}\int^\theta I(\phi)\, d\phi$ exists. This clue proves successful.

To come to details, let

$$\epsilon(\rho_0) = \iint_{\rho > \rho_0} \sigma dF,$$

so that $\lim\limits_{\rho_0 \to 1} \epsilon(\rho_0) = 0$. Let $\eta(\rho_0) = \sqrt{\epsilon(\rho_0)}$. We shall show that

$$\overline{\lim_{r \to 1}} \iint_{\rho > \rho_0} g dF \leqslant A\eta(\rho_0) \tag{7.1}$$

for all θ of a set $E = E(\rho_0)$ whose complementary CE has a measure less than $\eta(\rho_0)$. Since

$$\overline{\lim_{r \to 1}} \iint_{\rho \leqslant \rho_0} g dF = 0,$$

(uniformly) for all θ, the left side of (7.1) is unaltered when the symbol "$\rho > \rho_0$" is suppressed and the integration is taken over the whole of $\rho < 1$. But then the left side does not depend on ρ_0; hence, since $\eta(\rho_0)$ is arbitrarily small, (7.1), once proved, establishes the result of the lemma.

Let $\Phi(\omega)$ be $\iint \sigma dF$ taken over the area $\rho_0 \leqslant \rho < 1$, $0 \leqslant \phi \leqslant \omega$†. This is a non-negative monotonic increasing function of ω in $0 \leqslant \omega \leqslant 2\pi$, bounded by $2\epsilon(\rho_0)$‡. Hence $\Phi'(\omega)$ exists for almost all ω: moreover,

$$0 \leqslant \Phi'(\omega) \leqslant 2\eta(\rho_0)$$

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† Lines, or even single points, may contribute to the Stieltjes integral. Our arguments are arranged to err on the side of counting points and lines too often rather than not at all, and the difficulty is nowhere important.

‡ The contribution of the radius vector $\omega = 0$ is counted twice for $\omega = 2\pi$.

in a set $E = E(\rho_0)$ whose complementary set has a measure not exceeding $\eta(\rho_0)$; for

$$2\eta^2(\rho_0) = 2\epsilon(\rho_0) \geqslant \Phi(2\pi) \geqslant \int_0^{2\pi} \Phi'(\omega)\,d\omega \geqslant \int_{CE} \Phi'(\omega)\,d\omega \geqslant m(CE)\,.\,2\eta(\rho_0)$$

[since $\Phi' > 2\eta(\rho_0)$ almost everywhere in CE]. Let now $J(\theta, t)$ be $\iint \sigma dF$ taken over $\rho_0 \leqslant \rho < 1$ and the closed interval $(\theta, \theta+t)$ of ϕ. If now θ is a point of E we have, by the characteristic property of E,

$$\overline{\lim_{t\to 0}} \frac{J(\theta, t)}{|t|} \leqslant 2\eta(\rho_0). \tag{7.2}$$

I proceed to show that, for any θ for which (7.2) is true, (7.1) is also true; this establishes the lemma. In proving this we may suppose without loss of generality that $\theta = 0$, and may write $J(t)$ for $J(\theta, t)$, and

$$J^*(t) = J(t)+J(-t) \quad (t \geqslant 0).$$

Then

$$\overline{\lim_{t\to +0}} \frac{J^*(t)}{t} \leqslant 4\eta(\rho_0), \tag{7.3}$$

(and in particular J^* and $J^*(t)\log t$ are continuous at $t = 0$, with value 0).

Now

$$\Lambda(r) = \iint_{\rho > \rho_0} g dF \leqslant \Lambda_1+\Lambda_2+\Lambda_3,$$

where Λ_1, Λ_2, Λ_3 are the corresponding integrals taken over the areas $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$, for which respectively

(i) $\rho_0 \leqslant \rho < 1, \quad s \leqslant |\phi| \leqslant \pi;$

(ii) $\rho_0 \leqslant \rho < 1, \quad |\sigma-s| \geqslant \tfrac{1}{2}s, \quad |\phi| < s;$

(iii) $\rho_0 \leqslant \rho < 1, \quad |\sigma-s| \leqslant \tfrac{1}{2}s, \quad |\phi| < s.$

The number ψ in g is here $\phi-\theta=\phi$. In $\mathfrak{A}_1$ we have, by (2.2),

$$g \leqslant \frac{A\sigma s}{\phi^2}$$

$$\Lambda_1 \leqslant As \iint_{\mathfrak{A}_1} \phi^{-2}\sigma dF \leqslant As \int_{s-0}^{\pi} \phi^{-2} dJ^*(\phi),$$

$$\lim_{r\to 1} \Lambda_1 = \lim_{s\to +0} \Lambda_1 \leqslant A\eta(\rho_0),$$

by (7.3) and Lemma 1, (3.1).

In $\mathfrak{A}_2$ we have, by (2 . 2),

$$g \leqslant \frac{A\sigma s}{s^2+\phi^2}$$

$$\Lambda_2 \leqslant As \iint_{\mathfrak{A}_2} \frac{\sigma dF}{s^2+\phi^2} \leqslant As \int_0^s \frac{dJ^*(\phi)}{s^2+\phi^2},$$

$$\overline{\lim}\ \Lambda_2 \leqslant A\eta(\rho_0)$$

by (7 . 3) and Lemma 1, (3 . 2).

Finally in $\mathfrak{A}_3$ we have, by (2 . 3),

$$g \leqslant \log\left(\frac{A\sigma s}{\phi^2}\right)$$

$$\leqslant \frac{2\sigma}{s} \log\left(\frac{As^2}{\phi^2}\right).$$

Hence

$$\Lambda_3 \leqslant \frac{2}{s} \iint_{\mathfrak{A}_3} \log\left(\frac{As^2}{\phi^2}\right) \sigma dF \leqslant \frac{2}{s} \int_0^s \log\left(\frac{As^2}{\phi^2}\right) dJ^*(\phi),$$

$$\overline{\lim}\ \Lambda_3 \leqslant A\eta(\rho_0)$$

by (7 . 3) and Lemma 1, (3 . 3). Combining these results we have

$$\overline{\lim}\ \Lambda(r) \leqslant A\eta(\rho_0),$$

and this establishes the lemma.

8. It is now easy to deduce the main theorem. By Lemma 3, $w = -v = u^* + w^*$, where, by Lemma 4, $\lim_{r\to 1} w^* = 0$ for almost all θ. By Lemma 2, u^* satisfies (5 . 3). It follows, by a well known theorem, that $\lim_{r\to 1} u^*$ exists for almost all θ. Hence, finally, the same thing is true for v.

List of papers referred to.

J. E. Littlewood, **1.** "Mathematical notes (6)", *Journal London Math. Soc.*, 2 (1927), 189–192.

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F. Riesz, **1.** "Über subharmonische Funktionen u.s.w.", *Acta Univ. Franc. Jos. Szeged*, 2 (1925), 87–100.

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SIMULTANEOUS LINEAR EQUATIONS OVER A DIVISION ALGEBRA

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PREVIOUS work on the theory of equations and determinants in non-commutative algebras suffers from a lack of generality owing to an assumption of one-sidedness*. Moreover, information about such matters as the theory of equations and invariants, when this assumption of one-sidedness is adopted, is obtained only by means of a number of further artificial restrictions†, for the product of two one-sided expressions is not, in general, one-sided. In this paper I overcome these difficulties, and give a satisfactory definition of determinants which can be used almost as freely as ordinary determinants, which are, indeed, only particular examples.

My method consists in defining hypercomplex determinants, by means of which I give an explicit form to the solution of the general linear equation

$$\Sigma a_s x b_s = a,$$

in all cases including the singular. I then generalize the definition of determinant, and show how to obtain the solution of the most general system of simultaneous linear equations.

PART I. ONE-SIDED LINEAR EQUATIONS AND THEIR DETERMINANTS.

The following properties of the elements of a division algebra will be assumed‡ :

(*a*) There are no divisors of zero.

(*b*) There is a modulus which will be denoted by 1.

(*c*) Every element a, except zero, has a unique inverse a^{-1} such that

$$a a^{-1} = a^{-1} a = 1.$$

* E. Study, *Acta Math.*, 42 (1918), 1–61 (for the reference to this important paper, I am indebted to a referee); A. R. Richardson, *Messenger of Math.*, 55 (1926), 145–152.

† A. R. Richardson, *Messenger of Math.*, 55 (1926), 175–182.

‡ L. E. Dickson, *Algebras and their arithmetics* (1923), 61.

Definition of determinants.

The left-hand determinant

$$\Delta_{12}^{12} = \begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix}$$

is defined as follows:

$$\Delta_{12}^{12} = a_{22}a_{11} - a_{22}a_{12}a_{22}^{-1}a_{21} \quad (a_{22} \neq 0), \tag{1}$$

$$\Delta_{12}^{12} = -a_{12}a_{21}. \quad (a_{22} = 0). \tag{2}$$

The right-hand determinant

$$\nabla_{12}^{12} = \begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix}$$

is defined as follows:

$$\nabla_{12}^{12} = a_{11}a_{22} - a_{21}a_{22}^{-1}a_{12}a_{22} \quad (a_{22} \neq 0), \tag{3}$$

$$\nabla_{12}^{12} = -a_{21}a_{12} \quad (a_{22} = 0). \tag{4}$$

If the last column is zero, the determinant is defined to be zero.

In quaternions the conjugate of the left-hand determinant is seen to be the right-hand determinant of the conjugate elements.

The following results can be established by simple algebra.

Interchange of rows.

$$\left.\begin{aligned} \Delta_{12}^{12} &= -a_{22}a_{12}a_{22}^{-1}a_{12}^{-1}\Delta_{21}^{12} && (a_{22} \neq 0, \quad a_{12} \neq 0) \\ &= -\Delta_{21}^{12} && (a_{22} = 0, \quad a_{12} \neq 0) \\ &= -\Delta_{21}^{12} && (a_{22} \neq 0, \quad a_{12} = 0) \end{aligned}\right\}. \tag{5}$$

If two rows are identical, the determinant vanishes. Notice that the last two expressions can be obtained by striking out the zero elements in the first.

Interchange of columns.

$$\left.\begin{aligned} \Delta_{12}^{12} &= -a_{22}a_{21}^{-1}\Delta_{12}^{21}a_{22}^{-1}a_{21} && (a_{22} \neq 0, \quad a_{21} \neq 0) \\ &= -a_{21}^{-1}\Delta_{12}^{21}a_{21} && (a_{22} = 0, \quad a_{21} \neq 0) \\ &= -a_{22}\Delta_{12}^{21}a_{22}^{-1} && (a_{22} \neq 0, \quad a_{21} = 0) \end{aligned}\right\}. \tag{6}$$

If two columns are identical, the determinant vanishes. Notice that the last two expressions can be obtained by striking out the zero elements in the first.

Interchange of rows with columns.

$$\left.\begin{aligned} a_{22}^{-1}\begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix} &= \begin{vmatrix} a_{11}, & a_{21} \\ a_{12}, & a_{22} \end{vmatrix} a_{22}^{-1} \quad (a_{22} \neq 0) \\ \begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix} &= \begin{vmatrix} a_{11}, & a_{21} \\ a_{12}, & a_{22} \end{vmatrix} \quad (a_{22} = 0) \end{aligned}\right\}. \tag{7}$$

Multiplication of row or column.

$$\Delta^{12}_{(\lambda 1)2} = \left.\begin{vmatrix} \lambda a_{11}, & \lambda a_{12} \\ a_{21}, & a_{22} \end{vmatrix} \begin{aligned} &= a_{22}\lambda a_{22}^{-1}\Delta^{12}_{12} \quad (a_{22} \neq 0) \\ &= \lambda\Delta^{12}_{12} \quad (a_{22} = 0) \end{aligned}\right\}, \tag{8}$$

$$\Delta^{12}_{1(\lambda 2)} = \left.\begin{vmatrix} a_{11}, & a_{12} \\ \lambda a_{21}, & \lambda a_{22} \end{vmatrix} \begin{aligned} &= \lambda\Delta^{12}_{12} \quad (a_{22} \neq 0) \\ &= a_{12}\lambda a_{12}^{-1}\Delta^{12}_{12} \quad (a_{22} = 0) \end{aligned}\right\}, \tag{9}$$

$$\Delta^{(1\lambda)2}_{12} = \begin{vmatrix} a_{11}\lambda, & a_{12} \\ a_{21}\lambda, & a_{22} \end{vmatrix} = \Delta^{12}_{12}\lambda \quad \text{(in all cases)}, \tag{10}$$

$$\Delta^{1(2\lambda)}_{12} = \left.\begin{vmatrix} a_{11}, & a_{12}\lambda \\ a_{21}, & a_{22}\lambda \end{vmatrix} \begin{aligned} &= a_{22}\lambda a_{22}^{-1}\Delta^{12}_{12} \quad (a_{22} \neq 0) \\ &= a_{12}\lambda a_{12}^{-1}\Delta^{12}_{12} \quad (a_{22} = 0) \end{aligned}\right\}. \tag{11}$$

Definition. Expressions such as $a_{22}a_{12}a_{22}^{-1}a_{12}^{-1}$ and $a_{22}a_{21}^{-1}(\ )a_{22}^{-1}a_{21}$, in which each element is balanced by its inverse, will be termed *Permutators*. They are always of unit norm.

Addition of determinants.

$$\left.\begin{aligned} \begin{vmatrix} a_{11}+a'_{11}, & a_{12}+a'_{12} \\ a_{21}, & a_{22} \end{vmatrix} &= \begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix} + \begin{vmatrix} a'_{11}, & a'_{12} \\ a_{21}, & a_{22} \end{vmatrix} \\ \begin{vmatrix} a_{11}+a'_{11}, & a_{12} \\ a_{21}+a'_{21}, & a_{22} \end{vmatrix} &= \begin{vmatrix} a_{11}, & a_{12} \\ a_{12}, & a_{22} \end{vmatrix} + \begin{vmatrix} a'_{11}, & a_{12} \\ a'_{21}, & a_{22} \end{vmatrix} \end{aligned}\right\}. \tag{12}$$

Hence we may add a left (right) multiple of the elements of the second row (column) into the first row (column) without altering the determinant.

If we add a multiple of the first row (or column) into the second row (or column), the determinant is multiplied by a *permutator*; for example,

$$\begin{vmatrix} a_{11}, & a_{12} \\ a_{21}+a'_{21}, & a_{22}+a'_{22} \end{vmatrix} = -(a_{22}+a'_{22})\, a_{12}(a_{22}+a'_{22})^{-1}\, a_{12}^{-1}(\Delta_{21}^{12}+\Delta'^{12}_{21}), \quad (13)$$

provided that $a_{22}+a'_{22} \neq 0$ and $a_{12} \neq 0$.

Solution of two simultaneous equations in two unknowns.

If we take the equations to be

$$a_{11}x_1+a_{12}x_2 = a_1,$$

$$a_{21}x_1+a_{22}x_2 = a_2,$$

the necessary and sufficient condition for a unique solution is

$$\Delta_{12}^{12} \neq 0,$$

and this solution is given by

$$x_1 = (\Delta_{12}^{12})^{-1} \begin{vmatrix} a_1, & a_{12} \\ a_2, & a_{22} \end{vmatrix}, \qquad x_2 = (\Delta_{12}^{21})^{-1} \begin{vmatrix} a_1, & a_{11} \\ a_2, & a_{21} \end{vmatrix}.$$

If $\Delta_{12}^{12} = 0$, but both the determinants

$$\begin{vmatrix} a_1, & a_{12} \\ a_2, & a_{22} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1, & a_{11} \\ a_2, & a_{21} \end{vmatrix}$$

are not zero, the equations are inconsistent. If both of these determinants are zero, the equations are not independent and the most general solution for x_1 is

$$x_1 = a_{11}^{-1}a_1+(-a_{11}^{-1}a_{12})\, Y,$$

where Y is arbitrary*.

The determinant of order n.

The determinant

$$\Delta_{123\ldots n}^{123\ldots n} = \begin{vmatrix} a_{11}, & a_{12}, & \ldots, & a_{1n} \\ a_{21}, & a_{22}, & \ldots, & a_{2n} \\ \ldots & \ldots & & \ldots \\ a_{n1}, & a_{n2}, & \ldots, & a_{nn} \end{vmatrix}$$

* The Nabla function of Study is the norm of Δ_{12}^{12}.

is defined* to be equal to the following determinant of order $n-1$:

$$(-1)^{n-s}\begin{vmatrix} a_{sn}^{-1}\Delta_{1s}^{1n}, & \dots, & a_{sn}^{-1}\Delta_{1s}^{rn}, & \dots, & a_{sn}^{-1}\Delta_{1s}^{(n-1)n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{sn}^{-1}\Delta_{(s-1)s}^{1n}, & \dots, & a_{sn}^{-1}\Delta_{(s-1)s}^{rn}, & \dots, & a_{sn}^{-1}\Delta_{(s-1)s}^{(n-1)n} \\ a_{(s+1)1}, & \dots, & a_{(s+1)r}, & \dots, & a_{(s+1)(n-1)} \\ \dots & \dots & \dots & \dots & \dots \\ a_{(n-1)1}, & \dots, & a_{(n-1)r}, & \dots, & a_{(n-1)(n-1)} \\ a_{sn}a_{n1}, & \dots, & a_{sn}a_{nr}, & \dots, & a_{sn}a_{n(n-1)} \end{vmatrix} \tag{14}$$

where

$$\Delta_{rs}^{pq} = \begin{vmatrix} a_{rp}, & a_{rq} \\ a_{sp}, & a_{sq} \end{vmatrix}, \tag{15}$$

and s is defined by the conditions :

$$a_{nn} = a_{(n-1)n} = \dots = a_{(s+1)n} = 0, \quad a_{sn} \neq 0. \tag{16}$$

If all the elements of the last column are zero, the value of the determinant is defined to be zero.

Right-hand determinants are defined in a similar manner.

THEOREM 1. *In the successive reduction of the order of the determinant, the elements of a given row (say the t-th) remain in the t-th row, unless the diagonal minor of order $n-t$ in the lower right-hand corner contains a column of zeros in the last column at some stage in the reduction.*

For the elements in any column are never moved to a column on the right of the given one, and the elements of a given row are never moved from that row unless, at some stage in the process of reducing the order of the determinant, a zero appears as the right-hand element of every row below it. In symbols the elements of the t-th row remain in the t-th row unless

$$\Delta_{(t-1)\dots n}^{(t+1)\dots n} = 0.$$

The case in most frequent use is when $t = 1$. Hence

COROLLARY 1. *The elements of the first column always remain in the first column.*

* This modification of the definition given in the *Messenger of Math.* (*loc. cit.*, 146) is due to C. Slocombe, *University of Wales Thesis.*

The elements of the first row always remain in the first row, unless

$$\Delta_{23\ldots n}^{23\ldots n} = 0.$$

Hence, if $\Delta_{34\ldots n}^{34\ldots n} \neq 0$,

$$\Delta_{123\ldots n}^{123\ldots n} = \begin{vmatrix} (\Delta_{34\ldots n}^{34\ldots n})^{-1}\Delta_{134\ldots n}^{134\ldots n}, & (\Delta_{34\ldots n}^{34\ldots n})^{-1}\Delta_{134\ldots n}^{234\ldots n} \\ \Delta_{234\ldots n}^{134\ldots n}, & \Delta_{234\ldots n}^{234\ldots n} \end{vmatrix}, \tag{17}$$

and, if $\Delta_{r(r+1)\ldots n}^{r(r+1)\ldots n} \neq 0$,

$$\Delta_{123\ldots n}^{123\ldots n} = \begin{vmatrix} (\Delta_{r(r+1)\ldots n}^{r(r+1)\ldots n})^{-1}\Delta_{1r(r+1)\ldots n}^{1r(r+1)\ldots n}, & \ldots, & (\Delta_{r(r+1)\ldots n}^{r(r+1)\ldots n})^{-1}\Delta_{1r(r+1)\ldots n}^{(r-1)r(r+1))\ldots n} \\ \ldots \quad \ldots \quad \ldots \quad \ldots & \ldots & \ldots \quad \ldots \quad \ldots \quad \ldots \\ \Delta_{(r-1)r(r+1)\ldots n}^{1r(r+1)\ldots n}, & \ldots, & \Delta_{(r-1)r(r+1)\ldots n}^{(r-1)r(r+1)\ldots n} \end{vmatrix}. \tag{18}$$

THEOREM 2. *The coefficient of* a_{11} *is* $\Delta_{23\ldots n}^{23\ldots n}$.

For, from (16) and Cor. 1, a_{11} always remains in the leading position without multipliers, so that the determinant is certainly

$$\begin{vmatrix} a_{11}, & \ldots \\ \ldots, & \Delta_{23\ldots n}^{23\ldots n} \end{vmatrix}$$

if $\Delta_{23\ldots n}^{23\ldots n} \neq 0$. Hence the development begins with $\Delta_{23\ldots n}^{23\ldots n} a_{11} + \ldots$.

If $\Delta_{23\ldots n}^{23\ldots n} = 0$, the term a_{11} does not occur in the development.

Hence, in all cases,

$$\Delta_{123\ldots n}^{123\ldots n} = \Delta_{23\ldots n}^{23\ldots n} a_{11} + \ldots. \tag{19}$$

The following results may all be established by induction from the case $n = 2$.

THEOREM 3. *If all the elements of a row or column are zero, the determinant is equal to zero.*

THEOREM 4. *The determinant is equal to zero if two rows or two columns are identical.*

THEOREM 5. *The determinant is unaltered by adding to the elements of a row (or column) a left (right) multiple of the elements of any row (column) below (to the right of) it.*

THEOREM 6.

$$\left.\begin{aligned} \Delta_{12\ldots(r\lambda)\ldots n}^{12\ldots(r\lambda)\ldots n} &= P\lambda P^{-1}\Delta_{123\ldots n}^{123\ldots n} \\ \Delta_{12\ldots n}^{(1\lambda)2\ldots n} &= \Delta_{12\ldots n}^{12\ldots n}\lambda \end{aligned}\right\} \tag{20}$$

where $P(\ )P^{-1}$ *denotes a certain permutator, independent of* λ *and of the elements of the first column.*

In particular, if $\Delta^{r(r+1)\dots n}_{r(r+1)\dots n} \neq 0$,

$$P = \Delta^{r(r+1)\dots n}_{r(r+1)\dots n}. \tag{21}$$

THEOREM 7.

$$\left.\begin{aligned} \Delta^{12\dots n}_{12\dots(\lambda r)\dots n} &= P\lambda P^{-1}\Delta^{12\dots n}_{12\dots n} \\ \Delta^{12\dots n}_{(\lambda 1)2\dots n} &= \lambda\Delta^{12\dots n}_{12\dots n} \end{aligned}\right\}. \tag{22}$$

where $P(\ )P^{-1}$ *denotes a certain permutator, independent of* λ *and of the elements of the first row.*

In particular, if $\Delta^{r(r+1)\dots n}_{r(r+1)\dots n} \neq 0$,

$$P = \Delta^{r(r+1)\dots n}_{r(r+1)\dots n}. \tag{23}$$

THEOREM 8. *If* $\Delta^{23\dots n}_{23\dots n} \neq 0$, *then*

$$(\Delta^{23\dots n}_{23\dots(\lambda r)\dots n})^{-1}\Delta^{12\dots n}_{12\dots(\lambda r)\dots n} = (\Delta^{23\dots n}_{23\dots n})^{-1}\Delta^{123\dots n}_{123\dots n}. \tag{24}$$

THEOREM 9. *If* $ijk\dots q$ *is any permutation of* $123\dots n$, *and* κ *is the number of interchanges necessary to put the integers* $i, j, k, \dots, q$ *into their natural order, then*

$$\Delta^{123\dots n}_{123\dots n} = (-1)^{\kappa} P\Delta^{123\dots n}_{ijk\dots q}, \tag{25}$$

where P *is a certain permutator, independent of the elements of the first column.*

COROLLARY 2. *If* $\Delta^{123\dots n}_{123\dots n} = 0$, *then*

$$\Delta^{123\dots n}_{ijk\dots q} = 0. \tag{26}$$

COROLLARY 3. *If* $\Delta^{23\dots n}_{23\dots n} \neq 0$, *then*

$$\Delta^{123\dots n}_{123\dots n} = \Delta^{23\dots n}_{23\dots n}(\Delta^{23\dots n}_{jk\dots q})^{-1}\Delta^{123\dots n}_{1jk\dots q}. \tag{27}$$

COROLLARY 4. *If* $a_{nn} = a_{n-1,n} = \dots = a_{s+1,n} = 0$, $a_{sn} \neq 0$, *then*

$$\Delta^{12\dots n}_{12\dots s\dots t\dots n} = (-1)^{t-s}\Delta^{12\dots n}_{12\dots t\dots s\dots n}. \tag{28}$$

THEOREM 10. *Interchange of columns.*

$$\Delta^{123\dots n}_{123\dots n} = (-1)^{\kappa} P\Delta^{ijk\dots q}_{123\dots n} P', \tag{29}$$

where P *and* P' *are certain permutators independent of the elements of the first row.*

COROLLARY 5. *If* $\Delta_{123\ldots n}^{123\ldots n} = 0$, *then*

$$\Delta_{123\ldots n}^{ijk\ldots q} = 0. \tag{30}$$

COROLLARY 6. *If* $\Delta_{23\ldots n}^{23\ldots n} \neq 0$, *then*

$$\Delta_{123\ldots n}^{123\ldots n} = -\Delta_{23\ldots n}^{23\ldots n}(\Delta_{23\ldots n}^{jk\ldots q})^{-1}\,\Delta_{123\ldots n}^{ijk\ldots q}(\Delta_{23\ldots n}^{i2(i-1)(i+1)\ldots n})^{-1}\,\Delta_{22\ldots n}^{12\ldots(i-1)(i+1)\ldots n}. \tag{31}$$

COROLLARY 7. *If the first column is not involved in the interchange, then*

$$\Delta_{123\ldots n}^{123\ldots n} = \Delta_{23\ldots n}^{23\ldots n}(\Delta_{23\ldots n}^{jk\ldots q})^{-1}\,\Delta_{123\ldots n}^{1jk\ldots q}. \tag{32}$$

THEOREM 11. *Interchange of rows with columns. A left-hand determinant is the multiple by a permutator of the corresponding right-hand determinant in which rows have been interchanged with columns.*

If $\Delta_{23\ldots n}^{23\ldots n} \neq 0$, we have the important result that

$$\begin{vmatrix} a_{22}, \ldots, a_{2n} \\ \ldots \quad \ldots \\ a_{n2}, \ldots, a_{nn} \end{vmatrix}^{-1} \times \begin{vmatrix} a_{11}, a_{12}, \ldots, a_{1n} \\ a_{21}, a_{22}, \ldots, a_{2n} \\ \ldots \quad \ldots \quad \ldots \\ a_{n1}, a_{n2}, \ldots, a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}, a_{21}, \ldots, a_{n1} \\ a_{12}, a_{22}, \ldots, a_{n2} \\ \ldots \quad \ldots \quad \ldots \\ a_{1n}, a_{2n}, \ldots, a_{nn} \end{vmatrix} \times \begin{vmatrix} a_{22}, \ldots, a_{n2} \\ \ldots \quad \ldots \\ a_{2n}, \ldots, a_{nn} \end{vmatrix}^{-1}. \tag{33}$$

THEOREM 12. *The determinant, in which each element of a row (or column) is the sum of two numbers, is a permutator of the sum of the determinants in each of which only one of the two numbers appears in the corresponding row (or column).*

This follows from Theorems 6, 7, 9, and 10.

COROLLARY 8. *It follows from Theorems* 6, 7, 9, 10, *and* 12 *that the determinant resulting from the addition of a left (right) multiple of a row (column) to any other row (column) is a permutator of the original determinant.*

Theorem 5 should be compared with this result.

THEOREM 13. *Expansion in terms of the elements of the first row.*

It is evident from the work leading to (17) that, if $\Delta_{23\ldots n}^{23\ldots n} \neq 0$,

$$\Delta_{123\ldots n}^{123\ldots n} = \begin{vmatrix} (a_{11} - \ldots), & (a_{12} - \ldots) \\ \Delta_{234\ldots n}^{134\ldots n}, & \Delta_{234\ldots n}^{234\ldots n} \end{vmatrix} = \Delta_{23\ldots n}^{23\ldots n} a_{11} + \Delta_{23\ldots n}^{23\ldots n} a_{1s} B + \ldots, \tag{34}$$

for, since $\Delta_{23\ldots n}^{23\ldots n} \neq 0$, neither the first row nor the first column is moved. Similarly

$$\Delta_{123\ldots n}^{1s3\ldots 2\ldots n} = \begin{vmatrix} (a_{11}-\ldots), & (a_{1s}-\ldots) \\ \Delta_{234\ldots n}^{1s4\ldots 2\ldots n}, & \Delta_{234\ldots n}^{s34\ldots 2\ldots n} \end{vmatrix}$$

$$= \Delta_{23\ldots n}^{s3\ldots n} a_{11} - \Delta_{23\ldots n}^{s3\ldots 2\ldots n} a_{1s} (\Delta_{23\ldots n}^{s3\ldots 2\ldots n})^{-1} \Delta_{234\ldots n}^{1s4\ldots 2\ldots n} + \ldots . \quad (35)$$

The second expression is a permutator of the first, and consequently

$$B = (\Delta_{23\ldots n}^{s3\ldots 2\ldots n})^{-1} \Delta_{234\ldots n}^{1s4\ldots 2\ldots n}. \quad (36)$$

Hence, if $\Delta_{23\ldots n}^{23\ldots n} \neq 0$,

$$\Delta_{123\ldots n}^{123\ldots n} = \Delta_{23\ldots n}^{23\ldots n} \left[a_{11} - \sum_{s=2}^{n} a_{1s} (\Delta_{23\ldots n}^{s3\ldots (s-1)2(s+1)\ldots n})^{-1} \Delta_{234\ldots n}^{1s4\ldots (s-1)2(s+1)\ldots n} \right]. \quad (37)$$

COROLLARY 9. *If, in the development* (37), *we substitute for the* a_{1s} *the corresponding elements of any other row, the resulting expression vanishes.*

THEOREM 14. *Expansion in terms of the elements of the first column.*

We do not require the form of the coefficients in this expansion; Cor. 1 shows that the elements of the first column always remain in the first column, so that

$$\Delta_{123\ldots n}^{123\ldots n} = \sum_{s=1}^{r} A_{s1} \cdot a_{s1}. \quad (38)$$

When every $\Delta_{23\ldots (s-1)(s+1)\ldots n}^{34\ldots n} \neq 0$ and $\Delta_{23\ldots n}^{23\ldots n} \neq 0$, the actual expansion may be written out from considerations similar to those used in Theorem 13; the expansion is

$$\Delta_{123\ldots n}^{123\ldots n} = \Delta_{23\ldots n}^{23\ldots n} \left[a_{11} - \sum_{s=2}^{n} (\Delta_{23\ldots (s-1)(s+1)\ldots n}^{34\ldots n})^{-1} \Delta_{12\ldots (s-1)(s+1)\ldots n}^{23\ldots n} \right.$$

$$\left. \times (\Delta_{s2\ldots (s-1)(s+1)\ldots n}^{23\ldots n})^{-1} \Delta_{23\ldots (s-1)(s+1)\ldots n}^{34\ldots n} a_{s1} \right]. \quad (39)$$

In particular,

$$\Delta_{123}^{123} = \Delta_{23}^{23} \left[a_{11} - a_{33}^{-1} \Delta_{12}^{23} (\Delta_{23}^{23})^{-1} a_{33} a_{21} - a_{23}^{-1} \Delta_{12}^{23} (\Delta_{32}^{23})^{-1} a_{23} a_{31} \right], \quad (40)$$

except when $a_{33} = a_{23} = 0$; in other cases, where any of the symbols vanish they may be struck out.

When $a_{33} = a_{23} = 0$ and $a_{13} \neq 0$,

$$\Delta_{123}^{123} = a_{13} \Delta_{23}^{12}. \quad (41)$$

COROLLARY 10. *If, in the development* (38), *we substitute for the* a_{s1} *the corresponding elements of any other column, the resulting expression vanishes.*

One-sided linear equations.

Consider the system

$$\sum_{s=1}^{n} a_{rs}x_s = a_r \quad (r = 1, 2, 3, \ldots, n). \tag{42}$$

If $\Delta_{12\ldots n}^{123\ldots n} \neq 0$, there is one, and only one, solution, namely,

$$x_s = (\Delta_{123\ldots n}^{s12\ldots(s-1)(s+1)\ldots n})^{-1}\, \Delta_{123\ldots n}^{a12\ldots(s-1)(s+1)\ldots n}. \tag{43}$$

For consider

$$\begin{vmatrix} \beta_0, & \beta_1, & \ldots, & \beta_n \\ a_1, & a_{11}, & \ldots, & a_{1n} \\ a_2, & a_{21}, & \ldots, & a_{2n} \\ \ldots & \ldots & & \ldots \\ a_n, & a_{n1}, & \ldots, & a_{nn} \end{vmatrix}$$

Since $\Delta_{123\ldots n}^{123\ldots n} \neq 0$, it follows from Theorem 13 and Corollary 9 that the product of the elements of the r-th row by the algebraic complements of the first is zero.

Hence, from (37),

$$\Delta_{123\ldots n}^{123\ldots n}\left[a_r - \sum_{s=1}^{n} a_{rs}(\Delta_{123\ldots n}^{s12\ldots(s-1)(s+1)\ldots n}\, \Delta_{123\ldots n}^{a12\ldots(s-1)(s+1)\ldots n}\right] = 0,$$

so that $$a_r = \sum_{s=1}^{n} a_{rs}(\Delta_{123\ldots n}^{s12\ldots(s-1)(s+1)\ldots n})^{-1}\, \Delta_{123\ldots n}^{a12\ldots(s-1)(s+1)\ldots n}],$$

and consequently x_s, given by (43), is a solution of the equations (42).

Again, multiply the left of the r-th row of (42) by the algebraic complements of a_{rs}, and sum for $r = 1, 2, 3, \ldots, n$.

Hence $$\sum_{r-1}^{n} A_{rs} \sum_{s=1}^{n} a_{rs}x_s = \sum_{r-1}^{n} A_{rs}\, a_r,$$

that is to say

$$\Delta_{123\ldots n}^{s12\ldots(s-1)(s+1)\ldots n}\, x_s = \Delta_{123\ldots n}^{a12\ldots(s-1)(s+1)\ldots n},$$

so that the x_s of (43) must be the only solution. We thus obtain

THEOREM 15. *The system* (42) *has a unique solution, given by* (43), *if**

$$\Delta_{123\ldots n}^{123\ldots n} \neq 0.$$

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* In the case of quaternions, the Nabla function of Study (*loc. cit.*) is the norm of $\Delta_{123\ldots n}^{123\ldots n}$.

The case $\Delta_{123\ldots n}^{123\ldots n} = 0$.

Definition. The matrix

$$\begin{pmatrix} a_{11}, & a_{12}, & \ldots, & a_{1n} \\ a_{21}, & a_{22}, & \ldots, & a_{2n} \\ \ldots & \ldots & & \ldots \\ a_{p1}, & a_{p2}, & \ldots, & a_{pn} \end{pmatrix}$$

is said to be of *left (right) rank* r, if all $(r+1)$-rowed left (right) hand minor determinants of the matrix vanish, but at least one r-rowed left (right) hand minor determinant of the matrix does not vanish.

Evidently the rank is unaltered by an interchange of rows or columns, and by the addition of a left (right) multiple of the elements of any row (column) to the corresponding elements of any other row (column).

Definition. The p systems of numbers in the p rows of the matrix are said to be *left-linearly independent* if the n equations

$$\sum_{s=1}^{p} \lambda_s a_{sr} = 0 \quad (r = 1, 2, 3, \ldots, n)$$

are valid only for $\lambda_1 = \lambda_2 = \ldots = \lambda_p = 0$.

It is easily proved that a system of p left-hand linear expressions in n-variables are left-linearly independent if, and only if, the left rank of their matrix is p. Further, if the left rank is r $(\leqslant p)$, we may choose exactly r left-linearly independent expressions, the others being left-linear combinations of them.

If, therefore, the left rank of the system of homogeneous left-hand equations is r, we may write them as

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1r}x_r &= -(a_{1(r+1)}x_{r+1} + a_{1(r+2)}x_{r+2} + \ldots + a_{1n}x_n) \\ \ldots \quad \ldots \quad \ldots \quad

For convenience, write (46) in the form

$$x_s = \sum_{t=r+1}^{n} B_s^{-1} A_{ts}. \tag{47}$$

Hence every solution is a right-linear combination of

$$\left.\begin{array}{llllllllll} x_1, & x_2, & \ldots, & x_r; & x_{r+1}, & x_{r+2}, & \ldots, & x_n \\ B_1^{-1}A_{(r+1)1}, & B_2^{-1}A_{(r+1)2}, & \ldots, & B_r^{-1}A_{(r+1)r}; & 1, & 0, & \ldots, & 0 \\ B_1^{-1}A_{(r+2)1}, & B_2^{-1}A_{(r+2)2}, & \ldots, & B_r^{-1}A_{(r+2)r}; & 0, & 1, & \ldots, & 0 \\ \ldots \quad \ldots & \ldots & \ldots & \ldots \quad \ldots & \ldots & \ldots & \ldots & \ldots \\ B_1^{-1}A_{n1}, & B_2^{-1}A_{n2}, & \ldots, & B_r^{-1}A_{nr}; & 0, & 0, & \ldots, & 1 \end{array}\right\}, \tag{48}$$

Conversely every right-linear combination of these is a solution. All this is strictly analogous to the ordinary case over commutative systems.

The system $\sum_{s=1}^{n} a_{rs} x_r = a_r \quad (r = 1, 2, 3, \ldots, p),$

evidently has a solution only if the matrices

$$\begin{pmatrix} a_{11}, & a_{12}, & \ldots, & a_{1n} \\ a_{21}, & a_{22}, & \ldots, & a_{2n} \\ \ldots & \ldots & & \ldots \\ a_{p1}, & a_{p2}, & \ldots, & a_{pn} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1, & a_{11}, & a_{12}, & \ldots, & a_{1n} \\ a_2, & a_{21}, & a_{22}, & \ldots, & a_{2n} \\ \ldots & & \ldots & & \ldots \\ a_p, & a_{p1}, & a_{p2}, & \ldots, & a_{pn} \end{pmatrix}$$

have the same left rank. In such a case a solution is

$$x_t = x_t' = (\Delta_{12\ldots r}^{t1\ldots(t-1)(t+1)\ldots r})^{-1} \begin{vmatrix} a_1, & a_{11}, & \ldots, & a_{1(t-1)}, & a_{1(t+1)}, & \ldots, & a_{1r} \\ a_2, & a_{21}, & \ldots, & a_{2(t-1)}, & a_{2(t+1)}, & \ldots, & a_{2r} \\ \ldots & & \ldots & \ldots & \ldots & & \ldots \\ a_r, & a_{r1}, & \ldots, & a_{r(t-1)}, & a_{r(t+1)}, & \ldots, & a_{rr} \end{vmatrix} \tag{49}$$

by (43); and the most general solution is the sum of (49) and the general right-linear combinations of (48).

Multiplication of determinants. It is easily proved* that the left-hand determinant in which the element in the r-th row and s-th column is

$$\sum_{j=1}^{n} l_{sj} a_{rj}$$

is a permutator of the product of the determinants $|l_{rs}|$ and $|a_{rs}|$ in that order.

* *Messenger of Math.*, 55 (1926), 145–152 (151).

We require the following theorem :

THEOREM 16. *The expression B defined by the equation*

$$B = \begin{vmatrix} b_{11}, & b_{12}, & \ldots, & b_{1n} \\ b_{21}, & b_{22}, & \ldots, & b_{2n} \\ \ldots & \ldots & & \ldots \\ b_{n1}, & b_{n2}, & \ldots, & b_{nn} \end{vmatrix}^{-1} \times \begin{vmatrix} 1, & 0, & \ldots, & 0 \\ 0, & b_{22}, & \ldots, & b_{2n} \\ \ldots & \ldots & & \ldots \\ 0, & b_{n2}, & \ldots, & b_{nn} \end{vmatrix} \tag{50}$$

is unaltered if each of its constituent determinants is multiplied, row into row, by the non-vanishing scalar determinant

$$\delta = \begin{vmatrix} l_{11}, & l_{12}, & \ldots, & l_{1n} \\ 0, & l_{22}, & \ldots, & l_{2n} \\ \ldots & \ldots & & \ldots \\ 0, & l_{n2}, & \ldots, & l_{nn} \end{vmatrix}$$

For B is the value of x_1 which satisfies the system

$$\sum_{s=1}^{n} b_{rs} x_s = a_r \quad (r = 1, 2, 3, \ldots, n), \tag{51}$$

with $$a_1 = 1, \quad a_r = 0 \quad (r \neq 1).$$

Hence B is unaltered by all non-singular substitutions carried out on $x_2, x_3, \ldots, x_n$, and, in particular, by a linear substitution of the type specified by δ.

A direct proof is as follows :

The expression B is unaltered by any interchange of columns not involving the first columns, and it is also unaltered by the multiplication of the elements of a column by a scalar, the same operations being carried out on the two constituent determinants of B. Hence the following operations leave B unaltered, where, for brevity, the steps are specified for one of the two constituents of B only.

Consider

$$B' = \begin{vmatrix} \sum_{s=1}^{n} l_{1s} b_{1s}, & \sum_{s=2}^{n} l_{2s} b_{1s}, & \ldots, & \sum_{s=2}^{n} l_{ns} b_{1s} \\ \sum_{s=1}^{n} l_{1s} b_{2s}, & \sum_{s=2}^{n} l_{2s} b_{2s}, & \ldots, & \sum_{s=2}^{n} l_{ns} b_{2s} \\ \ldots & \ldots & \ldots & \ldots \\ \sum_{s=1}^{n} l_{1s} b_{ns}, & \sum_{s=2}^{n} l_{2s} b_{ns}, & \ldots, & \sum_{s=2}^{n} l_{ns} b_{ns} \end{vmatrix}$$

Multiply the r-th column ($r = 2, 3, \ldots, n$) by the algebraic complement L_{r1} of l_{r1} in δ, and add the results to the first column; B' becomes

$$\begin{vmatrix} \delta b_{11}, & \sum_{s=2}^{n} l_{2s} b_{1s}, & \ldots, & \sum_{s=2}^{n} l_{ns} b_{1s} \\ \delta b_{21}, & \sum_{s=2}^{n} l_{2s} b_{2s}, & \ldots, & \sum_{s=2}^{n} l_{ns} b_{2s} \\ \ldots & \ldots & \ldots & \ldots \\ \delta b_{n1}, & \sum_{s=2}^{n} l_{2s} b_{ns}, & \ldots, & \sum_{s=2}^{n} l_{ns} b_{ns} \end{vmatrix}.$$

Since $$L_{11} = \begin{vmatrix} l_{22}, & \ldots, & l_{2n} \\ \ldots & \ldots & \ldots \\ l_{n2}, & \ldots, & l_{nn} \end{vmatrix} \neq 0,$$

at least one of the minor determinants of the elements of the first row is non-zero; let it be L_{2r}. Multiply the second, third, ..., n-th columns respectively by the algebraic complements $L_{2r}, L_{3r}, \ldots, L_{nr}$, and add into the second column; B' becomes

$$\begin{vmatrix} \delta b_{11}, & L_{11} b_{1r}, & \sum_{s=2}^{n} l_{3s} b_{1s}, & \ldots, & \sum_{s=2}^{n} l_{ns} b_{1s} \\ \delta b_{21}, & L_{11} b_{2r}, & \sum_{s=2}^{n} l_{3s} b_{2s}, & \ldots, & \sum_{s=2}^{n} l_{ns} b_{2s} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \delta b_{n1}, & L_{11} b_{nr}, & \sum_{s=2}^{n} l_{3s} b_{ns}, & \ldots, & \sum_{s=2}^{n} l_{ns} b_{ns} \end{vmatrix},$$

where the elements b_{qr} have been removed from all columns after the second by interchanges and subtractions not involving the first column; B is unaltered by these operations. If we proceed in this way, it is evident that B' ultimately becomes

$$\begin{vmatrix} b_{11}, & b_{1r}, & \ldots, & b_{1t} \\ b_{21}, & b_{2r}, & \ldots, & b_{2t} \\ \ldots & \ldots & \ldots & \ldots \\ b_{n1}, & b_{nr}, & \ldots, & b_{nt} \end{vmatrix},$$

where $1, r, \ldots, t$ is some rearrangement of $1, 2, \ldots, n$; and scalar numbers δ, L_{11}, etc. have been removed.

Hence B is unaltered by the specified multiplication.

COROLLARY 11. *Similarly B is unaltered by multiplying rows of δ into columns of B, provided that the elements of B which are in a given column are kept in that column. The process is evidently equivalent to that of forming suitable linear combinations of the n equations* (51).

PART II. THE MOST GENERAL LINEAR SYSTEM.

We now proceed to generalize the preceding results by removing the restriction of one-sidedness from our equations and determinants.

Consider the single linear equation*

$$\sum_s b_s x b'_s = a.$$

It may be written

$$a_1 x + a_2 x e_2 + a_3 x e_3 + \ldots + a_n x e_n = a, \tag{52}$$

where $1, e_2, e_3, \ldots, e_n$ is a suitably chosen basis of the division algebra, that is to say the elements are linearly independent over the field F over which the algebra is defined†, and

$$e_r . e_s = \sum_{t=1}^{n} \gamma_{rst} e_t, \tag{53}$$

where the γ_{rst} are numbers of F, and consequently are commutative with all numbers of the algebra.

Let ϕ denote the operator $a_1(\ ) + a_2(\ )e_2 + a_3(\ )e_3 + \ldots + a_n(\ )e_n$, so that (52) may be written

$$\phi(x) = a. \tag{54}$$

To solve (52) multiply on the right in succession by $e_2, e_3, \ldots, e_n$, use (53), carry the scalar elements γ_{rst} through the x's (with which they are commutative) into the a's, and again write the expressions in the form (52).

Since $$e_r . 1 = 1 . e_r = e_r,$$

we have

$$\gamma_{r1t} = \gamma_{1rt} = 0 \quad (r \neq t), \qquad \gamma_{r1r} = \gamma_{1rr} = 1, \qquad \gamma_{111} = 1,$$

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* Sylvester, *Comptes rendus*, 99 (1884), 117-118, 473-476, 502-505; 409-412, 432-436, 527-529. [*Mathematical Papers*, 4 (1912), 181-187; 199-207.]

† Dickson, *Algebras and their arithmetics* (1923), 17.

and hence we get

$$\left.\begin{aligned}
&\sum_{t=1}^{n}\gamma_{t11}a_t.x+\sum_{t=1}^{n}\gamma_{t12}a_t.xe_2+\sum_{t=1}^{n}\gamma_{t13}a_t.xe_3+\ldots+\sum_{t=1}^{n}\gamma_{t1n}a_t.xe_n=\alpha\\
&\sum_{t=1}^{n}\gamma_{t21}a_t.x+\sum_{t=1}^{n}\gamma_{t22}a_t.xe_2+\sum_{t=1}^{n}\gamma_{t23}a_t.xe_3+\ldots+\sum_{t=1}^{n}\gamma_{t2n}a_t.xe_n=\alpha e_2\\
&\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\
&\sum_{t=1}^{n}\gamma_{ts1}a_t.x+\sum_{t=1}^{n}\gamma_{ts2}a_t.xe_2+\sum_{t=1}^{n}\gamma_{ts3}a_t.xe_3+\ldots+\sum_{t=1}^{n}\gamma_{tsn}a_t.xe_n=\alpha e_s\\
&\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\
&\sum_{t=1}^{n}\gamma_{tn1}a_t.x+\sum_{t=1}^{n}\gamma_{tn2}a_t.xe_2+\sum_{t=1}^{n}\gamma_{tn3}a_t.xe_3+\ldots+\sum_{t=1}^{n}\gamma_{tnn}a_t.xe_n=\alpha e_n
\end{aligned}\right\}. \tag{55}$$

Multiply, in succession, on the left by the algebraic complements A of the elements of the first column in

$$\Delta=\begin{vmatrix}
\sum_{t=1}^{n}\gamma_{t11}a_t, & \sum_{t=1}^{n}\gamma_{t12}a_t, & \sum_{t=1}^{n}\gamma_{t13}a_t, & \ldots, & \sum_{t=1}^{n}\gamma_{t1n}a_t\\
\sum_{t=1}^{n}\gamma_{t21}a_t, & \sum_{t=1}^{n}\gamma_{t22}a_t, & \sum_{t=1}^{n}\gamma_{t23}a_t, & \ldots, & \sum_{t=1}^{n}\gamma_{t2n}a_t\\
\ldots\quad\ldots & \ldots & \ldots & \ldots & \ldots\quad\ldots\\
\sum_{t=1}^{n}\gamma_{ts1}a_t, & \sum_{t=1}^{n}\gamma_{ts2}a_t, & \sum_{t=1}^{n}\gamma_{ts3}a_t, & \ldots, & \sum_{t=1}^{n}\gamma_{tsn}a_t\\
\ldots\quad\ldots & \ldots & \ldots & \ldots & \ldots\quad\ldots\\
\sum_{t=1}^{n}\gamma_{tn1}a_t, & \sum_{t=1}^{n}\gamma_{tn2}a_t, & \sum_{t=1}^{n}\gamma_{tn3}a_t, & \ldots, & \sum_{t=1}^{n}\gamma_{tnn}a_t
\end{vmatrix}, \tag{56}$$

and add : we get

$$\left.\begin{aligned}
\Delta x&=\sum_{s=1}^{n}A_s\alpha e_s,\\
\Delta&=\sum_{s=1}^{n}A_s.\sum_{t=1}^{n}\gamma_{ts1}a_t,\\
\text{and}\qquad 0&=\sum_{s=1}^{n}A_s\sum_{t=1}^{n}\gamma_{tsr}a_t\quad(r\neq 1)
\end{aligned}\right\}. \tag{57}$$

We write $$\phi^{-1}=\sum_{s=1}^{n}\Delta^{-1}A_s(\ \ )e_s,$$

when $\Delta\neq 0$, or, more conveniently,

$$\phi^{-1}=\sum_{s=1}^{n}H_s(\ \ )e_s,\quad H_s=\Delta^{-1}A_s.$$

We shall now prove that

$$x = \phi^{-1}(a)$$

satisfies the equation (52).

Substituting on the left of (52) we get

$$a_1 \sum_{s=1}^{n} H_s a e_s + a_2 \sum_{s=1}^{n} H_s a e_s \cdot e_2 + a_3 \sum_{s=1}^{n} H_s a e_s \cdot e_3 + \ldots + a_n \sum_{s=1}^{n} H_s a e_s \cdot e_n.$$

If we use the multiplication table (53) and collect corresponding terms, this becomes

$$\left.\begin{aligned}
&\Big[\sum_{t=1}^{n} \gamma_{1t1} a_t \cdot H_1 + \sum_{t=1}^{n} \gamma_{2t1} a_t \cdot H_2 + \sum_{t=1}^{n} \gamma_{3t1} a_t \cdot H_3 + \ldots + \sum_{t=1}^{n} \gamma_{nt1} a_t \cdot H_n\Big] a \\
+&\Big[\sum_{t=1}^{n} \gamma_{1t2} a_t \cdot H_1 + \sum_{t=1}^{n} \gamma_{2t2} a_t \cdot H_2 + \sum_{t=1}^{n} \gamma_{3t2} a_t \cdot H_3 + \ldots + \sum_{t=1}^{n} \gamma_{nt2} a_t \cdot H_n\Big] a e_2 \\
+&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
+&\Big[\sum_{t=1}^{n} \gamma_{1tr} a_t \cdot H_1 + \sum_{t=1}^{n} \gamma_{2tr} a_t \cdot H_2 + \sum_{t=1}^{n} \gamma_{3tr} a_t \cdot H_3 + \ldots + \sum_{t=1}^{n} \gamma_{ntr} a_t \cdot H_n\Big] a e_r \\
+&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
+&\Big[\sum_{t=1}^{n} \gamma_{1tn} a_t \cdot H_1 + \sum_{t=1}^{n} \gamma_{2tn} a_t \cdot H_2 + \sum_{t=1}^{n} \gamma_{3tn} a_t \cdot H_3 + \ldots + \sum_{t=1}^{n} \gamma_{ntn} a_t \cdot H_n\Big] a e_n
\end{aligned}\right\}, \tag{58}$$

We shall prove that this expression is equal to a by showing that the equations in $H_1, H_2, \ldots, H_n$ obtained from (58) by direct comparison of (58) with a have the same values for $H_1, H_2, \ldots, H_n$ as equations (57).

It is therefore a question of identifying

$$H_s = \begin{vmatrix}
\Sigma\gamma_{st1}a_t, & \Sigma\gamma_{2t1}a_t, & \ldots, & \Sigma\gamma_{(s-1)t1}a_t, & \Sigma\gamma_{1t1}a_t, & \Sigma\gamma_{(s+1)t1}a_t, & \ldots, & \Sigma\gamma_{nt1}a_t \\
\Sigma\gamma_{st2}a_t, & \Sigma\gamma_{2t2}a_t, & \ldots, & \Sigma\gamma_{(s-1)t2}a_t, & \Sigma\gamma_{1t2}a_t, & \Sigma\gamma_{(s+1)t2}a_t, & \ldots, & \Sigma\gamma_{nt2}a_t \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\Sigma\gamma_{str}a_t, & \Sigma\gamma_{2tr}a_t, & \ldots, & \Sigma\gamma_{(s-1)tr}a_t, & \Sigma\gamma_{1tr}a_t, & \Sigma\gamma_{(s+1)tr}a_t, & \ldots, & \Sigma\gamma_{ntr}a_t \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\Sigma\gamma_{stn}a_t, & \Sigma\gamma_{2tn}a_t, & \ldots, & \Sigma\gamma_{(s-1)tn}a_t, & \Sigma\gamma_{1tn}a_t, & \Sigma\gamma_{(s+1)tn}a_t, & \ldots, & \Sigma\gamma_{ntn}a_t
\end{vmatrix}^{-1}$$

$\times$ (the minor outlined by dots). (59)

with

$H'_s =$ (the minor outlined by dots in the following determinant)

$$\times \left| \begin{array}{c:c:c:c:c:c:c:c:c} \Sigma\gamma_{ts1}a_t, & \Sigma\gamma_{t21}a_t, \ldots, & \Sigma\gamma_{t(s-1)1}a_t, & \Sigma\gamma_{t11}a_t, & \Sigma\gamma_{t(s+1)1}a_t, & \ldots, & \Sigma\gamma_{tn1}a_t \\ \Sigma\gamma_{ts2}a_t, & \Sigma\gamma_{t22}a_t, \ldots, & \Sigma\gamma_{t(s-1)2}a_t, & \Sigma\gamma_{t12}a_t, & \Sigma\gamma_{t(s+1)2}a_t, & \ldots, & \Sigma\gamma_{tn2}a_t \\ \cdots & \cdots \quad \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Sigma\gamma_{tsr}a_t, & \Sigma\gamma_{tsr}a_t, \ldots, & \Sigma\gamma_{t(s-1)r}a_t, & \Sigma\gamma_{t1r}a_t, & \Sigma\gamma_{t(s+1)r}a_t, & \ldots, & \Sigma\gamma_{tnr}a_t \\ \cdots & \cdots \quad \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Sigma\gamma_{tsn}a_t, & \Sigma\gamma_{t2n}a_t, \ldots, & \Sigma\gamma_{t(s-1)n}a_t, & \Sigma\gamma_{t1n}a_t, & \Sigma\gamma_{t(s+1)n}a_t, & \ldots, & \Sigma\gamma_{tnn}a_t \end{array} \right|^{-1}$$

If $H'_s \neq 0$, we may interchange rows with columns in this expression, by Theorem 11, and get

$$H'_s = \left| \begin{array}{c:c:c:c:c:c:c:c:c} \Sigma\gamma_{ts1}a_t, & \Sigma\boldsymbol{\gamma}_{ts2}a_t, & \ldots, & \Sigma\gamma_{tsr}a_t, & \ldots, & \Sigma\gamma_{tsn}a_t \\ \Sigma\gamma_{t21}a_t, & \Sigma\gamma_{t22}a_t, & \ldots, & \Sigma\gamma_{t2r}a_t, & \ldots, & \Sigma\gamma_{t2n}a_t \\ \cdots \quad \cdots & \cdots \quad \cdots & \cdots & \cdots & \cdots & \cdots \quad \cdots \\ \Sigma\gamma_{t(s-1)1}a_t, & \Sigma\gamma_{t(s-1)2}a_t, & \ldots, & \Sigma\gamma_{t(s-1)r}a_t, & \ldots, & \Sigma\gamma_{t(s-1)n}a_t \\ \Sigma\gamma_{t11}a_t, & \Sigma\gamma_{t12}a_t & \ldots, & \Sigma\gamma_{t1r}a_t, & \ldots, & \Sigma\gamma_{t1n}a_t \\ \Sigma\gamma_{t(s+1)1}a_t, & \Sigma\gamma_{t(s+1)2}a_t, & \ldots, & \Sigma\gamma_{t(s+1)r}a_t, & \ldots, & \Sigma\gamma_{t(s+1)n}a_t \\ \cdots \quad \cdots & \cdots \quad \cdots & \cdots & \cdots & \cdots & \cdots \quad \cdots \\ \Sigma\gamma_{tn1}a_t, & \Sigma\gamma_{tn2}a_t, & \ldots, & \Sigma\gamma_{tnr}a_t, & \ldots, & \Sigma\gamma_{tnn}a_t \end{array} \right|^{-1}$$

$\times$(the minor outlined by dots). (60)

Now, since the algebra is a division algebra, the equation $\beta x = e_s$ regarded as an equation in β has one and only one solution for every x which is not zero.

Consequently the equation

$$\sum_{r=1}^{n} \beta_r e_r \,.\, \sum_{t=1}^{n} x_t e_t = e_s,$$

that is to say

$$\sum_{r=1}^{n}\sum_{t=1}^{n} \beta_r x_t \,.\, e_r e_t = \sum_{r=1}^{n}\sum_{t=1}^{n} \beta_r x_t \sum_{k=1}^{n} \gamma_{tr,k} e_k = e_s$$

has a solution for all x_t which are not all zero.

Hence the determinant of which the element in the r-th row and t-th column is γ_{trs} is non-zero for any of the values 1, 2, ...,, n of s. By

Theorem (16) multiplication of the constituent determinants of H_s and H'_s by non-vanishing determinants which contain only one non-zero element in the first row and first column leaves them unaltered.

Hence H_s is unaltered if we multiply its constituent determinants (rows into columns) by

$$\begin{vmatrix} \gamma_{1ss}, & \gamma_{2ss}, & \cdots, & \gamma_{jss}, & \cdots, & \gamma_{nss} \\ \gamma_{12s}, & \gamma_{22s}, & \cdots, & \gamma_{j2s}, & \cdots, & \gamma_{n2s} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{1(s-1)s}, & \gamma_{2(s-1)s}, & \cdots, & \gamma_{j(s-1)s}, & \cdots, & \gamma_{n(s-1)s} \\ \gamma_{11s}, & \gamma_{21s}, & \cdots, & \gamma_{j1s}, & \cdots, & \gamma_{n1s} \\ \gamma_{1(s-1)s}, & \gamma_{2(s-1)s}, & \cdots, & \gamma_{j(s+1)s}, & \cdots, & \gamma_{n(s+1)s} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{1ns}, & \gamma_{2ns}, & \cdots, & \gamma_{jns}, & \cdots, & \gamma_{nns} \end{vmatrix},$$

a determinant which does not vanish and whose first column (except for the leading element) consists of zeros.

There results

$$H_s = \begin{vmatrix} \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{jss}\gamma_{stj}, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{jss}\gamma_{2tj}, & \cdots, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{jss}\gamma_{rtj}, & \cdots, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{jss}\gamma_{ntj} \\ \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{j2s}\gamma_{stj}, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{j2s}\gamma_{2tj}, & \cdots, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{j2s}\gamma_{rtj}, & \cdots, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{j2s}\gamma_{ntj} \\ \cdots \quad \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots & & \cdots \quad \cdots \quad \cdots & & \cdots \quad \cdots \quad \cdots \\ \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{j1s}\gamma_{stj}, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{j1s}\gamma_{2tj}, & \cdots, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{j1s}\gamma_{rtj}, & \cdots, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{j1s}\gamma_{ntj} \\ \cdots \quad \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots & & \cdots \quad \cdots \quad \cdots & & \cdots \quad \cdots \quad \cdots \\ \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{jns}\gamma_{stj}, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{jns}\gamma_{2tj}, & \cdots, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{jns}\gamma_{rtj}, & \cdots, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{jns}\gamma_{ntj} \end{vmatrix}^{-1}$$

$\times$ (the minor outlined by dots).

But, since the algebra is associative*,

$$\sum_{j=1}^{n} \gamma_{srj} \cdot \gamma_{ijk} = \sum_{j=1}^{n} \gamma_{isj}\gamma_{jrk} \quad (i, s, r, k = 1, 2, 3, \ldots, n),$$

* Dickson, *Algebras and their arithmetics* (1923), 92.

and hence

$$H_s = \begin{vmatrix} \sum_{t,j} \gamma_{tsj}\cdot\gamma_{sjs}a_t, & \sum_{t,j} \gamma_{tsj}\cdot\gamma_{2js}a_t, & \cdots, & \sum_{t,j} \gamma_{tsj}\cdot\gamma_{rjs}a_t, & \cdots, & \sum_{t,j} \gamma_{tsj}\cdot\gamma_{njs}a_t \\ \sum_{t,j} \gamma_{t2j}\cdot\gamma_{sjs}a_t, & \sum_{t,j} \gamma_{t2j}\cdot\gamma_{2js}a_t, & \cdots, & \sum_{t,j} \gamma_{t2j}\cdot\gamma_{rjs}a_t, & \cdots, & \sum_{t,j} \gamma_{t2j}\cdot\gamma_{njs}a_t \\ \cdots \quad \cdots & \cdots \quad \cdots & \cdots & \cdots \quad \cdots & \cdots & \cdots \quad \cdots \\ \sum_{t,j} \gamma_{t1j}\cdot\gamma_{sjs}a_t, & \sum_{t,j} \gamma_{t1j}\cdot\gamma_{2js}a_t, & \cdots, & \sum_{t,j} \gamma_{t1j}\cdot\gamma_{rjs}a_t, & \cdots, & \sum_{t,j} \gamma_{t1j}\cdot\gamma_{njs}a_t \\ \cdots \quad \cdots & \cdots \quad \cdots & \cdots & \cdots \quad \cdots & \cdots & \cdots \quad \cdots \\ \sum_{t,j} \gamma_{tnj}\cdot\gamma_{sjs}a_t, & \sum_{t,j} \gamma_{tnj}\cdot\gamma_{2js}a_t, & \cdots, & \sum_{t,j} \gamma_{tnj}\cdot\gamma_{rjs}a_t, & \cdots, & \sum_{t,j} \gamma_{tnj}\cdot\gamma_{njs}a_t \end{vmatrix}^{-1}$$

$\times$ (the minor outlined by dots). (61)

But this is precisely the result of multiplying the rows in the constituent determinants of H'_s by the rows of

$$\begin{vmatrix} \gamma_{s1s}, & \gamma_{s2s}, & \cdots, & \gamma_{sjs}, & \cdots, & \gamma_{sns} \\ \gamma_{21s}, & \gamma_{22s}, & \cdots, & \gamma_{2js}, & \cdots, & \gamma_{2ns} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{(s-1)1s}, & \gamma_{(s-1)2s}, & \cdots, & \gamma_{(s-1)js}, & \cdots, & \gamma_{(s-1)ns} \\ \gamma_{11s}, & \gamma_{12s}, & \cdots, & \gamma_{1js}, & \cdots, & \gamma_{1ns} \\ \gamma_{(s+1)1s}, & \gamma_{(s+1)2s}, & \cdots, & \gamma_{(s+1)js}, & \cdots, & \gamma_{(s+1)ns} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{n1s}, & \gamma_{n2s}, & \cdots, & \gamma_{njs}, & \cdots, & \gamma_{nns} \end{vmatrix},$$

an operation which leaves H'_s unchanged.

Hence $$H_s = H'_s,$$

provided that neither is zero.

If H_s is zero, the minor which is outlined in (59) vanishes, and hence, by the theorem on multiplication of determinants, the determinant shown in (61) also vanishes; and therefore H'_s also vanishes.

Hence, in all cases, $H_s = H'_s$ and the expression (58) is equal to a. Consequently

$$\phi\{\phi^{-1}(x)\} = x,$$

provided that $\Delta \neq 0$.

Therefore the necessary and sufficient condition for the equation (52) *to have one and only one solution is that* $\Delta \neq 0$, *and the solution is then given by* $x = \phi^{-1}(a)$.

We shall now prove that

$$\phi^{-1}\{\phi(z)\} = z.$$

For

$$\phi^{-1}\{\phi(z)\} = \Delta^{-1}\begin{vmatrix} \phi(z), & \sum_{t=1}^{n}\gamma_{t12}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{t1j}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{t1n}a_t \\ \phi(z)\,e_2, & \sum_{t=1}^{n}\gamma_{t22}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{t2j}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{t2n}a_t \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \phi(z)\,e_s, & \sum_{t=1}^{n}\gamma_{ts2}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{tsj}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{tsn}a_t \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \phi(z)\,e_n, & \sum_{t=1}^{n}\gamma_{tn2}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{tnj}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{tnn}a_t \end{vmatrix}$$

$$= \Delta^{-1}\begin{vmatrix} \sum_{j,t}\gamma_{t1j}a_t ze_j, & \sum_{t=1}^{n}\gamma_{t12}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{t1j}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{t1n}a_t \\ \sum_{j,t}\gamma_{t2j}a_t ze_j, & \sum_{t=1}^{n}\gamma_{t22}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{t2j}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{t2n}a_t \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{j,t}\gamma_{tsj}a_t ze_j, & \sum_{t=1}^{n}\gamma_{ts2}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{tsj}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{tsn}a_t \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{j,t}\gamma_{tnj}a_t ze_j, & \sum_{t=1}^{n}\gamma_{tn2}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{tnj}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{tnn}a_t \end{vmatrix}$$

If we multiply the j-th column on the right by ze_j $(j = 2, 3, \ldots, n)$, and subtract the results from the first column, this becomes

$$\Delta^{-1}\begin{vmatrix} \sum_{t=1}^{n}\gamma_{t11}a_t\cdot z, & \sum_{t=1}^{n}\gamma_{t12}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{t1j}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{t1n}a_t \\ \sum_{t=1}^{n}\gamma_{t21}a_t\cdot z, & \sum_{t=1}^{n}\gamma_{t22}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{t2j}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{t2n}a_t \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{t=1}^{n}\gamma_{ts1}a_t\cdot z, & \sum_{t=1}^{n}\gamma_{ts2}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{tsj}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{tsn}a_t \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{t=1}^{n}\gamma_{tn1}a_t\cdot z, & \sum_{t=1}^{n}\gamma_{tn2}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{tnj}a_t, & \cdots, & \sum_{t=1}^{n}\gamma_{tnn}a_t \end{vmatrix}$$

$$= \Delta^{-1}\cdot\Delta\cdot z = z.$$

Hence $$\phi^{-1}\{\phi(z)\} = z,$$

so that ϕ is commutative with ϕ^{-1}.

Similarly, $$(\phi^{-1})^{-1}(z) = \phi(z),$$

so that $$(\phi^{-1})^{-1} = \phi.$$

We see, therefore, that ϕ may be treated as if it were a number; and provided that we always interpret $\phi_1\phi_2(z)$ to mean $\phi_1\{\phi_2(z)\}$, no difficulties arise. In fact, the ϕ's constitute an algebra with n^2 units, namely $\phi_{rs} = e_r(\ )e_s$, but this algebra is not, in general, a division algebra.

Singular linear operators. The case $\Delta = 0$ remains for consideration. Let equations (55) be of left rank r, so that at least one left-hand determinant of order r in Δ is non-zero. It follows from the investigation in Part I following on the definition of left-linear independence that the corresponding matrices which involve a must also have left rank r, or the equations are inconsistent.

This is the case with the equation

$$xa - ax = 1,$$

which arises in quantum algebras; this equation has no solution in any *finite* associative algebra, nor, indeed, in a very extensive class of infinite algebras.

If there is more than one solution, the difference of two solutions is a solution of (55) with the right-hand sides replaced by zeros; and when we multiply by the algebraic complements of the elements of a column in Δ and add, we get

$$\Delta(x' - x) = 0.$$

Since there are no divisors of zero in a division algebra we must have $\Delta = 0$.

A method of solution in the case of quaternions makes use of a formula given by Sylvester*, where the operator ϕ is shown to satisfy an equation of the fourth degree with real coefficients†

$$\phi^4 - A\phi^3 + B\phi^2 + C\phi - D = 0.$$

* *Comptes rendus*, 99 (1884), 504.

† He does not calculate the coefficients.

In our case, since ϕ is singular, $D=0$; and, if we suppose that $C\neq 0$, we have

$$\phi(\phi^3-A\phi^2+B\phi+C)=0.$$

Hence the general solution of $\phi(x)=a$ is

$$x=X+(\phi^3-A\phi^2+B\phi+C)Y,$$

where Y is an arbitrary quaternion and X is a particular solution of the equation.

A more direct method, based on the theory of determinants, is the following, which has the advantage of exhibiting the solution in an explicit form. We confine our attention to quaternions, since the work for the general algebra is much more involved and tedious.

Let our equations be first of rank three, and, for clearness, suppose that the non-vanishing minor of order three is in the top left-hand corner of Δ.

Then, if we multiply the first three equations of (55) by the algebraic complements of the first column in

$$\delta=\begin{vmatrix} a_1, & a_2, & a_3 \\ -a_2, & a_1, & a_4 \\ -a_3, & -a_4, & a_1 \end{vmatrix}$$

and add, we see that x must be a solution of

$$\delta\,.\,x=\begin{vmatrix} a-a_1xk, & a_2, & a_3 \\ ai+a_3xk, & a_1, & a_4 \\ aj-a_2xk, & -a_4, & a_1 \end{vmatrix}=X'(a)+H'\,.\,xk,$$

X' being a linear operator on a, and H' being a left multiple of xk.

That is to say
$$x=X(a)+H\,.\,xk, \tag{62}$$

where
$$H=\delta^{-1}H', \quad X=\delta^{-1}X',$$

since $\delta\neq 0$; and consequently

$$(1+H^2)x=X(a)+H\,.\,X(a)\,k.$$

If* $1+H^2\neq 0$, the solution is obvious. In general, however, $1+H^2=0$, so that

$$X(a)+HX(a)\,k=0.$$

* This means that there is a unique solution in spite of Δ vanishing, and, from other considerations, this seems to be ruled out.

A particular solution of (62) is therefore $x = \frac{1}{2}X(a)$, and the most general solution is

$$x = \tfrac{1}{2}X(a)+(Y+HYk),$$

where Y is an arbitrary quaternion.

Δ *of rank two.* Again, for convenience, let the non-vanishing minor of order two be

$$d = \begin{vmatrix} a_1, & a_2 \\ -a_2, & a_1 \end{vmatrix},$$

so that, as above, x must be a solution of the equation

$$dx = \begin{vmatrix} a, & a_2 \\ ai, & a_1 \end{vmatrix} - \begin{vmatrix} a_3, & a_2 \\ a_4, & a_1 \end{vmatrix} xj + \begin{vmatrix} -a_4, & a_2 \\ a_3, & a_1 \end{vmatrix} xk,$$

that is, of

$$x = X(a)+Jxj+Kxk, \tag{63}$$

where $X(a)$ is a linear operator on a, and J and K are left-hand multiples of xj and xk respectively.

Hence

$$(1+J^2+K^2)\,x = X(a)+JX(a)\,j+KX(a)\,k+(KJ-JK)\,xi. \tag{64}$$

If both $\qquad 1+J^2+K^2 = 0 \quad$ and $\quad KJ-JK = 0,$

then

$$X(a)+JX(a)\,j+KX(a)\,k = 0. \tag{65}$$

A particular solution is

$$x = \tfrac{1}{2}X(a).$$

The general solution is

$$x = \tfrac{1}{2}X(a)+(Y+JYj+KYk),$$

where Y is an arbitrary quaternion.

If one of $1+J^2+K^2$ and $KJ-JK$ is zero, and the other is not zero, there is only one solution, and it is obtained by direct division from (64).

There remains the case in which neither is zero; (64) may then be rewritten as

$$A'x+B'xi = \Pi'(a),$$

which is reducible to the form

$$x+Bxi = \Pi(a);$$

the general solution of this, as in (62), is

$$x = \tfrac{1}{2}\Pi(a)+(Y-BYi),$$

where Y is an arbitrary quaternion and

$$\Pi(a) = (1+J^2+K^2)^{-1}[X(a)+JX(a)j+KX(a)k],$$

$$B = (1+J^2+K^2)^{-1}(JK-KJ).$$

Hence, in all cases, we can write down explicit expressions for the solution of the equation $\phi(x) = a$ in terms of our determinants.

In the case of quaternions the method leading to (64) simplifies the actual calculations, for, from (64), the equation $\phi(x) = a$ is now

$$x+Bxi = \Pi(a),$$

that is
$$(1+B^2)x = \Pi(a)-B\Pi(a)i,$$

so that
$$x = (1+B^2)^{-1}[\Pi-B\Pi(a)i].$$

Solution of the general system of simultaneous linear equations in m unknowns.

Let ϕ_{rp} denote the linear operator

$$a_{r1}^{(1)}(\quad)+a_{r1}^{(2)}(\quad)e_2+\ldots+a_{r1}^{(n)}(\quad)e_n.$$

The most general set of linear equations in m unknowns may be written

$$\left.\begin{aligned}
\phi_{11}(x_1)+\phi_{12}(x_2)+\ldots+\phi_{1m}(x_m) &= \alpha_1\\
\phi_{21}(x_1)+\phi_{22}(x_2)+\ldots+\phi_{2m}(x_m) &= \alpha_2\\
\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad &\\
\phi_{p1}(x_1)+\phi_{p2}(x_2)+\ldots+\phi_{pm}(x_m) &= \alpha_p
\end{aligned}\right\}. \tag{66}$$

Since the solution cannot depend on the order in which the equations and variables are written, if not all the linear operators are singular, we may suppose ϕ_{rp} to be non-singular, and use the theory of determinants set out in this paper, avoiding always divisions by singular or zero operators.

In particular, if the symbolic determinant

$$\Delta = \begin{vmatrix}
\phi_{11}, & \phi_{12}, & \ldots, & \phi_{1m}\\
\phi_{21}, & \phi_{22}, & \ldots, & \phi_{2m}\\
\ldots & \ldots & & \ldots\\
\phi_{m1}, & \phi_{m2}, & \ldots, & \phi_{mm}
\end{vmatrix} \tag{67}$$

is non-singular, there is one and only one solution, namely

$$x_s = \begin{vmatrix} \phi_{1s}, & \phi_{12}, & \dots, & \phi_{11}, & \dots, & \phi_{1m} \\ \phi_{2s}, & \phi_{22}, & \dots, & \phi_{21}, & \dots, & \phi_{2m} \\ \dots & \dots & & \dots & & \dots \\ \phi_{ms}, & \phi_{m2}, & \dots, & \phi_{m1}, & \dots, & \phi_{mm} \end{vmatrix}^{-1} \begin{vmatrix} \alpha_1, & \phi_{12}, & \dots, & \phi_{11}, & \dots, & \phi_{1m} \\ \alpha_2, & \phi_{22}, & \dots, & \phi_{21}, & \dots, & \phi_{2m} \\ \dots & \dots & & \dots & & \dots \\ \alpha_m, & \phi_{m2}, & \dots, & \phi_{m1}, & \dots, & \phi_{mm} \end{vmatrix}$$

$(s = 1, 2, 3, \dots, m).$

If all the ϕ_{sm} are singular, it may still be possible to eliminate the other variables and get

$$\begin{vmatrix} \phi_{1m}, & \phi_{12}, & \dots, & \phi_{11} \\ \phi_{2m}, & \phi_{22}, & \dots, & \phi_{21} \\ \dots & \dots & & \dots \\ \phi_{mm}, & \phi_{m2}, & \dots, & \phi_{m1} \end{vmatrix} x_m = \begin{vmatrix} \alpha_1, & \phi_{12}, & \dots, & \phi_{11} \\ \alpha_2, & \phi_{22}, & \dots, & \phi_{21} \\ \dots & \dots & & \dots \\ \alpha_m, & \phi_{m2}, & \dots, & \phi_{m1} \end{vmatrix},$$

the general solution of which may be obtained by the method just given. The substitution of these values in (66) now leaves a set of equations which can be solved by the above process.

The outstanding case is that in which all the ϕ_{rp} are singular. As an example, we take

$$\left.\begin{aligned} \phi_{11}(x_1) + \phi_{12}(x_2) &= \alpha_1 \\ \phi_{21}(x_1) + \phi_{22}(x_2) &= \alpha_2 \end{aligned}\right\}. \tag{68}$$

As on p. 410, operators Φ_{12} and Φ_{22} exist such that $\Phi_{12}\phi_{12}$ and $\Phi_{22}\phi_{22}$ are zero. Hence x_1 satisfies the two equations

$$\Phi_{12}\{\phi_{11}(x_1)\} = \Phi_{12}(\alpha_1), \qquad \Phi_{22}\{\phi_{21}(x_1)\} = \Phi_{22}(\alpha_2).$$

If these have a common solution, x_2 may be obtained from one of the set (68) by the method for solving equations in one unknown.

Generalization of determinants. Symbolic determinants such as (67) include both the right-hand and the left-hand determinants of Part I as special cases, and furnish their natural generalization. By using them we overcome the grave restrictions of one-sidedness in such matters as the theories of equations and of invariants over division algebras*.

* *Messenger of Math.*, 55 (1926), 175.

BINARY AND TERNARY FORMS WITH PRESCRIBED POLAR SYSTEMS

By J. H. GRACE.

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1. There are two familiar results in invariant theory which may be stated as follows :

(i) Two binary cubics are, in general, first polars of a unique binary quartic.

(ii) Three ternary quadratics are, in general, first polars of a unique ternary cubic.

My purpose is to discuss the exact conditions under which these theorems are true and to find all possible extensions. It may save trouble to state the main results. Necessary and sufficient conditions are, for (i), that no linear combination of the cubics should be a perfect cube and, for (ii), that no linear combination of the quadratics should be a perfect square. There is an infinite number of analogues of (i) and none of (ii).

2. I shall begin with binary forms, but it may be well to state here that by a k-th polar of a form I mean any linear combination of its differential coefficients of order k with respect to the variables.

For convenience I recall some elementary theorems used in the sequel

2.1. Two binary forms f, ϕ of orders n, m $(m \leqslant n)$ are said to be *conjugate* or *apolar* if

$$\phi\left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right) f(x, y) = 0,$$

x, y being the variables.

2.2. If ϕ is conjugate to f, it is conjugate to any polar of f of order not exceeding $n-m$; and, conversely, if ϕ is conjugate to all polar forms of order

$k \leqslant n-m$, it is conjugate to f. These are simple exercises in differential operators.

2.3. If f is conjugate to ϕ, then it is conjugate to the product of ϕ and any form of order not exceeding $n-m$; and conversely.

2.4. To a set of h linearly independent forms of order n there is a conjugate set of $n+1-h$ linearly independent forms of order n.

This is an immediate consequence of the conjugacy condition being, for forms of equal order, a single equation linear in the coefficients of each.

2.5. If ϕ is the m-th power of a linear form p, then the conjugacy condition is equivalent to f containing the factor p^{n-m+1}; and conversely.

3. Pass now to the theorem of § 1 (i).

Suppose that f_1, f_2 are two binary cubics and suppose that f is a quartic of which they are first polars.

Let ϕ_1, ϕ_2, be two binary cubics conjugate to f_1, f_2; f must be conjugate to ϕ_1, ϕ_2 (2.2) and hence to

$$\phi_1 x, \ \phi_1 y, \ \phi_2 x, \ \phi_2 y. \qquad (2.2) \text{ and } (2.3)$$

If these four quartics are linearly independent, f will be uniquely determined by the conjugacy conditions; and its first polars, being conjugate to ϕ_1, ϕ_2, will belong to the conjugate system*

$$(f_1, \ f_2).$$

Consequently, if f has two independent first polars, the linear system of its first polars must be identical with the linear system

$$(f_1, \ f_2).$$

Subject to the provisions mentioned, the theorem is proved; it is true "in general".

Now in the first place, if the four forms (2.3) are not linearly independent, there must be an identity of the type

$$\phi_1 p_1 + \phi_2 p_2,$$

* I use $(f_1, f_2, f_3, \ldots)$ to denote the whole linear system of forms

$$l_1 f_1 + l_2 f_2 + l_3 f_3 + \ldots,$$

where l_1, l_2, l_3, ..., are constants.

where p_1, p_2 are linear forms ; ϕ_1 and ϕ_2 must in such case have a common linear factor q and the conjugate system (f_1, f_2) must contain q^3 as a member. Secondly, if f has not two linearly independent first polars, it must be a fourth power, say l^4 ; and l^3 must be contained in the system

$$(f_1, f_2),$$

for, being a first polar of f, it is conjugate to ϕ_1, ϕ_2. Thus both provisos in the proof lead to the fact that the linear system (f_1, f_2) must contain a perfect cubic ; it follows that a sufficient condition for the truth of § 1 (i) is the absence of such a cube from the prescribed polar system.

4. Let us now examine this exceptional case. Since f, if it exists, has a cube as first polar, it has a conjugate quadratic ; for a cube will be annihilated by a proper linear operator and it was in turn produced from f by a linear operator.

Without loss of generality (since it is an affair of invariants) we may suppose the factors of the quadratic to be

$$x, y \text{ or } x, x,$$

and the general form of f is

$$ax^4+by^4 \text{ or } ax^4+bx^3y,$$

the polar systems being

$$(x^3, y^3) \text{ or } (x^3, x^2y).$$

It follows that if the system (f_1, f_2) contains a cube, say x^3, there is no solution unless it contains a second cube or another form with x^2 as factor.

In each of the latter cases there is an infinite number of solutions, namely

$$ax^4+by^4 \text{ or } ax^4+bx^3y,$$

corresponding to

$$(x^3, y^3) \text{ or } (x^3, x^2y) ;$$

incidentally it follows that the absence of a perfect cube from the system (f_1, f_2) is not only a sufficient but a necessary condition for the existence of a unique solution.

5. Coming now to the general problem, observe that the k-th polars of a form of order n form a linear system*

$$(f_1, f_2, \ldots, f_{k+1})$$

* If $k > \frac{1}{2}n$, the polars cannot be linearly independent ; if $k \leqslant \frac{1}{2}n$, they usually are independent, but in special cases they are not.

of order $n-k$ and the question arises when such a linear system is given: When is it the k-th polar system of a unique form of order n ?

Suppose that f, then, is a form of order n of which a given linear system

$$(f_1, f_2, \ldots, f_{k+1})$$

of order $n-k$ is the system of k-th polars.

Just as in § 3, f must be conjugate to the conjugate system

$$(\phi_1, \phi_2, \ldots, \phi_{n-2k})$$

of forms of order $n-k$; and thence to each of these base forms multiplied by

$$x^k, x^{k-1}y, \ldots, xy^{k-1}, y^k.$$

We thus get $(n-2k)(k+1)$

forms of order n conjugate to f. It is natural*, if a unique solution is desired, to equate this to n, which gives

$$\theta = (n-2k)(k+1)-n = 0$$

or

$$n = 2(k+1)$$

and leads one to expect that a linear system

$$(f_1, f_2, \ldots, f_{k+1})$$

is in general the k-th polar system of a unique form of order $2k+2$. The order of the forms is $k+2$, and we revert to the problem of § 3 on writing $k = 1$.

To examine this question note that the conjugate system contains only two terms ϕ_1, ϕ_2; and f must be conjugate to each multiplied by

$$x^k, x^{k-1}y, \ldots, y^k.$$

The form f is thus determined uniquely unless (cf. § 3) there is an identity of the type

$$\phi_1\psi_1+\phi_2\psi_2 = 0,$$

where ψ_1, ψ_2 are of order k.

If there is such an identity, ϕ_1 and ϕ_2 must have at least one common linear factor l, and then l^{k+2} must occur in the conjugate system

$$(f_1, f_2, \ldots, f_{k+1}).$$

* Cf. *infra*, § 7.

An f so determined will be such that all its k-th polars are conjugate to ϕ_1, ϕ_2; and thus, if the k-th polars form a genuine system with $(k+1)$ terms, they must be identical with the given "f_r" system.

If the k-th polars are not so independent, then clearly f has a conjugate form of order k, from which it easily follows that among its k-th polars is at least one perfect $(k+2)$-th power; since this must be included in the "f_r" system, we arrive at the same conclusion as in § 3, namely that the solution is unique provided that the prescribed polar system does not contain the power of a linear form.

6. The discussion of the exceptional case follows exactly the lines of § 4. It will suffice to state the complete results for $k = 2$.

Three binary quartics are, in general, second polars of a unique binary sextic.

It is a necessary and sufficient condition that no linear combination of the three should be a perfect fourth power.

If the condition is not satisfied, there is no solution unless the system of quartics is reducible to one of the three forms

$$(x^4, y^4, z^4); \quad (x^4, x^3y, z^4); \quad (x^4, x^3y, x^2y^2);$$

then there are ∞^2 solutions, namely

$$ax^6+by^6+cz^6; \quad ax^6+bx^5y+cz^6; \quad ax^6+bx^5y+cx^4y^2.$$

7. There remains the question as to assuming $\theta = 0$ in § 5.

If θ is negative, there will be a linear series of values of f satisfying the conditions of conjugacy: though none of them may have the assigned polar forms, it is clear that, if one of them has, then by continuity an infinite number of others have. A unique solution is out of the question.

If θ is positive, there are apparently too many forms to be conjugate to one form f.

The condition that f should exist may be expressed as the vanishing of a certain number of determinants and it must be made clear that they do not vanish identically. It will suffice to find a single case in which all forms included in

$$\phi_1\psi_1+\phi_2\psi_2+\ldots+\phi_{n-2k}\psi_{n-2k},$$

the ψ's, being arbitrary and of order k, are not conjugate to a single form f.

For this purpose take

$$\phi_1 = l_1^{n-k}, \quad \phi_2 = l_2^{n-k}, \quad \ldots,$$

where l_1, l_2, ... are linear forms.

The form f must then be conjugate to

$$l_1^{n-k},\ l_2^{n-k},\ \ldots,$$

and so it must have the factors

$$l_1^{k+1},\ l_2^{k+1},\ \ldots.$$

Since $$(n-2k)(k+1) > n,$$

this will be impossible if the l's are all different. It follows, as was assumed, that the problem has a unique solution in a general case only when $\theta = 0$.

Ternary forms.

8. A form of order n $\quad f(x, y, z)$

and one of class m $(m \leqslant n)$ $\quad \phi(u, v, w)$

are said to be conjugate if

$$\phi\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right) f = 0.$$

8.1. If ϕ is conjugate to f, it is conjugate to all polars of f of order not exceeding $n-m$; conversely, if it is conjugate to all such polars, it is conjugate to f.

8.2. In the same case f is conjugate to the product of ϕ and any form of class not exceeding $n-m$; and conversely.

8.3. For $m = n$, the condition of conjugacy is a single lineo-linear relation between the two sets of coefficients; thus a system of k linearly independent forms of order n determines a system of $\{\frac{1}{2}(n+1)(n+2)-k\}$ conjugate forms of class n.

8.4. If ϕ is the m-th power of a linear form p, then the point given by $p = 0$ is of multiplicity $n-m+1$ on the curve defined by $f = 0$.

9. Let f_1, f_2, f_3 be three linearly independent ternary quadratics, and let f be a cubic of which they are first polars. The conjugate system of quadratics (or conics) being

$$\phi_1,\ \phi_2,\ \phi_3,$$

f must be conjugate to each and consequently to the nine class forms obtained by multiplying each by u, v, w (§ 7). Unless there is a linear relation of the type

$$\phi_1 p_1 + \phi_2 p_2 + \phi_3 p_3 = 0,$$

f will thus be uniquely determined; and, so determined, all its first polars will be conjugate to ϕ_1, ϕ_2, ϕ_3 and will therefore be included in the linear system

$$(f_1,\ f_2,\ f_3).$$

If f has three linearly independent first polars, its polar system will be identical with the given (f) system and the theorem of § 1 (ii) is proved in general, *i.e.* subject to the two provisos.

Now the relation $$\phi_1 p_1+\phi_2 p_2+\phi_3 p_3=0$$

leads readily to the conclusion that the three curves represented by

$$\phi_1=0,\ \ \phi_2=0,\ \ \phi_3=0$$

have at least one common tangent. Denoting its equation by $Z=0$, the form Z^2, being conjugate to each ϕ, must be included in the system

$$(f_1,\ f_2,\ f_3).$$

Again, if f has not three linearly independent first polars, it must have a vanishing one, and, taking this (as we may) to be $\partial f/\partial z$, f will be a cubic in the variables x, y only, and so it will be reducible to one of the forms

$$x^3+y^3,\ \ x^2y,\ \ x^3.$$

In any case x^2 will be a first polar, and as such contained in the set

$$(f_1,\ f_2,\ f_3).$$

It follows that the existence of the unique cubic is certain *unless the prescribed polar system contains a perfect square.*

10. The exceptional case can be treated directly.

Without loss of generality we may suppose the square to be x^2, and this to be (i) $\partial f/\partial x$ or (ii) $\partial f/\partial y$ according as it is the polar of a point not on $x=0$ or on $x=0$.

(i) f must be of the type

$$x^3+py^3+qy^2z+ryz^2+sz^3,$$

which can be reduced to one of the forms

$$x^3+y^3+z^3,$$

$$x^3+y^2z,$$

$$x^3+y^3,$$

and the polar systems of these are

$$(x^2, y^2, z^2) ; \quad (x^2, y^2, yz) ; \quad (x^2, y^2),$$

the last case being worthless because it is only two-fold.

(ii) f must be of the type

$$x^2y+px^3+qy^2z+ryz^2+sz^3,$$

If $s \neq 0$, this can be reduced to the form

$$x^2y+px^3+qy^2z+sz^3,$$

and finally to $$x^2y+x^3+z^3.$$

If $s = 0$, it can be reduced to

$$x^2y+x^3+xz^2.$$

The polar systems of these two types are

$$(x^2, xy, z^2) \text{ and } (x^2, xz, z^2+2xy).$$

Thus when the linear system (f_1, f_2, f_3) contains a square, there is no solution unless it can be reduced to one of the forms

$$(x^2, y^2, z^2); \quad (x^2, xy, z^2); \quad (x^2, xz, z^2+2xy),$$

for (x^2, y^2, yz) is, of course, the same as the second.

In each of these cases there is a doubly infinite number of solutions, namely

$$ax^3+by^3+cz^3, \quad ax^3+bx^2y+cz^3, \quad ax^3+b(z^2x+x^2y)+czx^2.$$

It may illustrate the general method to find the third form, which corresponds to

$$(f_1, f_2, f_3) = (x^2, xz, z^2+2xy).$$

The conjugate system is

$$(\phi_1, \phi_2, \phi_3) = (v^2, vw, w^2-uv),$$

and f is conjugate to each of these multiples by u, v, w; repeating redundant forms, this system becomes

$$uv^2, v^3, v^2w, uvw, vw^2, w^3, uw^2-u^2v,$$

and the general form of f is

$$ax^3+b(z^2x+x^2y)+czx^2$$

as stated.

11. Pass now to the general problem.

The k-th polars of a form of order n are a linear series of $\frac{1}{2}(k+1)(k+2)$ terms of order $n-k$, and the question is under what conditions a general series of this nature is identical with the k-th polar system of a form of order n.

The conjugate system of forms is of class $n-k$ and contains

$$\tfrac{1}{2}(n-k+1)(n-k+2)-\tfrac{1}{2}(k+1)(k+2) = \tfrac{1}{2}(n-2k)(n+3)$$

terms.

Denoting these class forms by

$$\phi_1, \ \phi_2, \ \ldots, \ \phi_r,$$

$$r = \tfrac{1}{2}(n-2k)(n+3),$$

the required form f must be conjugate to each ϕ and so to each ϕ multiplied by an arbitrary form of order k in u, v, w (§ 8).

In all, then, f must be conjugate to

$$\tfrac{1}{2}(n-2k)(n+3)\tfrac{1}{2}(k+1)(k+2)$$

forms of class n. It is natural to equate this to $\frac{1}{2}n(n+3)$, the number of constants* in f, if a unique solution is desired, and we are led to

$$\theta = \tfrac{1}{2}(n-2k)(n+3)\tfrac{1}{2}(k+1)(k+2)-\tfrac{1}{2}n(n+3) = 0,$$

and hence to $$n = \frac{2(k+1)(k+2)}{k+3};$$

the only integer value of k consistent with n being an integer is $k=1$, giving $n=3$. The problem solved is thus the only one of its class.

12. There remains only the discussion of the assumption $\theta = 0$.

If $\theta < 0$, an argument like that of § 7 leads readily to the fact that the problem cannot have a unique solution.

If $\theta > 0$, there is, in general no solution. To see this (cf. § 7) we have to construct a system of ϕ's in number

$$r = \tfrac{1}{2}(n-2k)(n+3)$$

of class $n-k$ such that they have not a conjugate form f. Taking each ϕ to be a power of a linear form, this reduces (8.4) to finding r points such that they are not each of multiplicity $k+1$ on any curve f of order n. The fact that θ is positive implies, of course, that there are too many conditions

* See below, § 2.

for f, but this does not in turn imply the non-existence of f. For example, five triple points give 30 conditions, three too many for a sextic curve; but there is always a sextic with five given triple points, namely, the conic through the five taken thrice.

The problem is similar to such as arise in applications of Wakeford's principle in Canonical Forms*; it will suffice to give an outline of the argument.

The condition $\theta > 0$ is equivalent to

$$n > \frac{2(k+1)(k+2)}{k+3},$$

which is always satisfied for $n \geqslant 2k+2$ and for $n = 2k+1$ if $k > 1$.

Consider $n = 2k+1$.

We have $k+2$ points of multiplicity $k+1$ for a C_{2k+1}. Taking the points to be general, *i.e.* no three on a line, the join of any two is part of the C_{2k+1}, and so the C_{2k+1} is impossible if

$$\tfrac{1}{2}(k+2)(k+1) > 2k+1$$

or

$$k > 1$$

as required.

$n = 2k+2$.

We have $2k+5$ points of multiplicity $k+1$ for a C_{2k+2}. If these are placed on a line, this is part of the C_{2k+2}, and we are left with $2k+2$ points for a C_{2k+1}; this, as above, is impossible in a general case.

I shall deal with only one other case, viz. $k = 1$, which leads to

$$r = \tfrac{1}{2}(n-2)(n+3)$$

double points for a C_n, and, since $\theta > 0$, $n > 3$.

Since

$$r = \tfrac{1}{2}n(n-1)+(n-3),$$

if we place $\frac{1}{2}n(n-1)$ of the points at the corners of a complete n line, the C_n must consist of the n-line only, and the $n-3$ extra double points are in general clearly impossible.

Arguments of this type suffice throughout.

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* *Proc. London Math. Soc.* (2), 18 (1919), 403-410. Cf. *Journal London Math. Soc.*, 2 (1927), 185.

ON A THEOREM, DUE TO F. S. MACAULAY, CONCERNING THE ENUMERATION OF POWER-PRODUCTS

By F. J. W. WHIPPLE.

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1. *Introduction.*

The theorem discussed in the present note was enunciated and proved by Dr. F. S. Macaulay in a recent paper* on the theory of modular systems. By a slight modification of his procedure I have succeeded in obtaining a more direct proof of the theorem. I am much indebted to Dr. Macaulay for helpful criticism and advice with regard to the form of this note.

The theorem is concerned with the number of power-products which are obtained when a given set of power-products are all multiplied by the variables in succession. We define the "natural" order† for writing down the power-products of degree l in n variables $x_1, x_2, \ldots, x_n$ as that in which $x_1^{p_1}x_2^{p_2}x_3^{p_3}\ldots x_n^{p_n}$ comes before $x_1^{q_1}x_2^{q_2}\ldots x_n^{q_n}$ if the first of the indices $p_1, p_2, \ldots, p_n$, which differs from the corresponding index in $q_1, q_2, \ldots, q_n$, exceeds it. Thus with products of degree 5 in four variables u, v, w, x, the p.p.'s in the natural order are

$$u^5,\ u^4v,\ u^4w,\ u^4x,\ u^3v^2,\ u^3vw,\ u^3vx,\ u^3w^2,\ u^3wx,\ u^3x^2,\ u^2v^3,\ u^2v^2w,\ \ldots.$$

A set of products in natural order, starting with the first possible term and without any omissions, I propose to call a "natural sequence."

It is an immediate deduction from the definition that, if the terms of a natural sequence are rearranged in powers of any of the variables, say x_r, then the coefficient of any power, say x_r^p, will be a natural

* *Proc. London Math. Soc.* (2), 26 (1927), 531–555.

† Dr. Macaulay's term is "ascending order". We follow him in writing p.p. for power product.

sequence in the remaining variables x_1, $x_2 \dots x_{r-1}$, $x_{r+1} \dots x_n$. For, if a term $x_1^{p_1} x_2^{p_2} \dots x_{r-1}^{p_{r-1}} x_{r+1}^{p_{r+1}} \dots x_n^{p_n}$ had been missing, whilst a junior term $x_1^{q_1} x_2^{q_2} \dots x_{r-1}^{q_{r-1}} x_{r+1}^{q_{r+1}} \dots x_n^{q_n}$ (such that the first p to differ from a q exceeded it) was present, then in the original series the term $x_1^{p_1} \dots x_n^{p_n}$ would have been absent whilst $x_1^{q_1} \dots x_{r-1}^{q_{r-1}} x_r^{p_r} x_{r+1}^{q_{r+1}} \dots x_n^{p_n}$ was present. But by hypothesis this was not so and the deduction is therefore justified.

Macaulay demonstrates* that, if each of the terms of a natural sequence is multiplied in turn by each of the variables and duplicates are rejected, the aggregate is a natural sequence. Thus, in our illustration, the derived system would be the natural sequence from u^6 to $u^2 v^2 wx$. If N be the number of terms in the original sequence, then $Q(N)$ is defined as the number of terms in the derived sequence.

If N p.p.'s which do not form a natural sequence are multiplied in turn by the variables, the number of different derived terms will generally be other than $Q(N)$. The proposition to be investigated is that *this number is not less than* $Q(N)$.

The proof is by mathematical induction, but it has an unusual feature in that the extension from two to three variables has to be treated specially. The further extensions from three to four variables, etc. are all on the same lines and the argument is simpler than with three variables.

2. *The guiding principle of the proof.*

The first part of the proof is taken from Macaulay's paper. In his notation S_p is written for a collection of N_p p.p.'s of degree p in the n variables $x_1, x_2, \dots, x_n$ and OS_p denotes the collection of p.p.'s derived from S_p by multiplying the terms by $x_1, x_2, \dots, x_n$. It is assumed that our theorem is true for n variables so that, N_p being given, the number of terms in OS_p is a minimum when S_p is a natural sequence.

With $n+1$ variables $x_0, x_1, \dots, x_n$, we write

$$S \equiv x_0^{l-a} S_a, \quad x_0^{l-a-1} S_{a+1}, \quad \dots, \quad x_0^{l-b} S_b.$$

The derived series consists of all the different terms in the array

$$x_0^{l-a+1} S_a, \quad x_0^{l-a} S_{a+1}, \quad \dots, \quad x_0^{l-b+1} S_b,$$

$$x_0^{l-a} OS_a, \quad \dots, \quad x_0^{l-b+1} OS_{b-1}, \quad x_0^{l-b} OS_b.$$

The total number of derived p.p.'s will be reduced if we choose the p.p.'s in each S_p so that the number of p.p.'s in OS_p may be a minimum and at

* *Loc. cit.*, 534.

the same time these p.p.'s coincide as far as possible with those in S_{p+1}. Both conditions are fulfilled by choosing the p.p.'s S_p to be the first N_p p.p.'s in the natural sequence of degree p, for the number of p.p.'s in OS_p is then a minimum and the longer of the sequences OS_p and S_{p+1} includes all the terms of the shorter.

This principle will be referred to as the "guiding principle".

3. *The cases of two and three variables.*

With two variables the number of derived p.p.'s is clearly a minimum when the p.p.'s are in sequence: for the addition of an extra term to the original sequence only adds one term to the derived sequence. In this case $Q(N) = N+1$ for all values greater than 0, whilst $Q(0) = 0$.

Proceeding to the case of three variables, we find it convenient to take the variables as u, v, w rather than x_1, x_2, x_3. We also introduce the symbol $|\not<|$ to denote "has not fewer derived p.p.'s than". Before dealing with the general case, we take a particular example to illustrate the method. We start with an arbitrary group of ten homogeneous products of the 8-th degree and make transformations in accordance with the "guiding principle".

$$S \equiv u^7w\,;\ u^6vw\,;\ u^5(v^3,\ v^2w,\ vw^2,\ w^3)\,;\ u^3(v^5,\ v^2w^3,\ vw^4)\,;\ u^2v^3w^3$$

$$|\not<|\ u^7v\,;\ u^6v^2\,;\ u^5(v^3,\ v^2w,\ vw^2,\ w^3)\,;\ u^3(v^5,\ v^4w,\ v^3w^2)\,;\ u^2v^6$$

$$\equiv w^3u^5\,;\ w^2(u^5v,\ u^3v^3)\,;\ w(u^5v^2,\ u^3v^4)\,;\ (u^7v,\ u^6v^2,\ u^5v^3,\ u^3v^5,\ u^2v^6)$$

$$|\not<|\ w^3u^5\,;\ w^2(u^6,\ u^5v)\,;\ w(u^7,\ u^6v)\,;\ (u^8,\ u^7v,\ u^6v^2,\ u^5v^3,\ u^4v^4)$$

$$\equiv u^8\,;\ u^7(v,\ w)\,;\ u^6(v^2,\ vw,\ w^2)\,;\ u^5(v^3,\ vw^2,\ w^3)\,;\ u^4v^4$$

$$|\not<|\ u^8\,;\ u^7(v,\ w)\,;\ u^6(v^2,\ vw,\ w^2)\,;\ u^5(v^3,\ v^2w,\ vw^2)\,;\ u^4v^4.$$

The terms in the last line do not form a natural sequence. They would, if u^5w^3 could be substituted for u^4v^4. We see at once that the substitution would reduce by 1 the number of derived p.p.'s, u^4v^5 and u^4v^4w being dropped and u^5w^4 introduced. Thus in this case the number of p.p.'s derived from S is less than the number derived from the natural sequence of ten terms. Actually the number derived from S would be 22 whilst the number derived from 10 terms of the natural sequence would be 15.

In dealing with the general group of p.p.'s of degree l in these variables the same procedure is to be followed. Writing S for any set of p.p.'s and P for a natural sequence, we set out the transformations

symbolically in the following forms:

$$S \equiv \Sigma u^{l-r} S_r(v, w) \tag{1}$$

$$|\lessdot| \Sigma u^{l-r} P_r(v, w) \tag{2}$$

$$\equiv \sum_{s=0} w^s S'_{l-s}(u, v) \tag{3}$$

$$|\lessdot| \sum_{s=0} w^s P'_{l-s}(u, v) \tag{4}$$

$$\equiv \sum_{t=0} u^{l-t} S''_t(v, w) \tag{5}$$

$$|\lessdot| \sum_{t=0} u^{l-t} P''_t(v, w). \tag{6}$$

Consideration of the form (2) shows that the number of terms containing w^s is not less than the number containing w^{s+1}. Accordingly (3) begins with $s = 0$ and the numbers of terms in successive S''s are in descending order. The same rule holds for the P''s in (4).

Hence, if P'_{l-s} in (4) does not proceed beyond v'^{l-1}, no succeeding P' proceeds beyond v'^{l-1}; *i.e.*, if $w^s v'^t$ is absent from $w^s P'_{l-s}$, there is no term in (4) divisible by $w^s v'^t$, no previous P' having a term divisible by w^s and no succeeding P' having a term divisible by v'^t. This property also holds for (5) and therefore, if S''_t has any terms absent S''_{t+1} has all derived terms absent. Accordingly, if S''_t is incomplete, S''_{t+1} can not have more terms than S''_t. The same must be true for P''_t, P''_{t+1} in (6). On the other hand if, P''_t is complete, then P''_{t+1} may be complete also, and in that case it will have one more term than P''_t. Dropping the dashes, we write

$$S_l(u, v, w) |\lessdot| u^l;\ u^{l-1} P_1(v, w);\ u^{l-2} P_2(v, w);\ \ldots;\ u^{l-\kappa} P_\kappa(v, w).$$

If $N_1, N_2, \ldots, N_\kappa$ are the numbers of terms in $P_1, P_2, \ldots, P_\kappa$, the conditions just obtained are equivalent to $N_{t+1} \not> N_t + 1$, and it follows that the product terms derived by multiplying $u^{l-t-1} P_{t+1}$ by u are all included amongst the terms derived by multiplying $u^{l-t} P_t$ by v and by w.

Accordingly, if $N_1 N_2 \ldots N_\kappa$ are the numbers of terms in $P_1 P_2 \ldots P_\kappa$ so that $1 + N_1 + \ldots N_\kappa = N$, then the number of derived terms is

$$1 + 2 + (N_1 + 1) + (N_2 + 1) + \ldots (N_\kappa + 1)$$

or

$$N + \kappa + 2.$$

To make κ as small as possible the successive N's must be as large as possible. This will happen if the whole series of terms forms a natural sequence.

It follows that in the case of three variables the proposition is true.

4. *The cases of four or more variables.*

With four variables u, v, w, x we proceed by arranging the terms alternately by powers of u and by powers of w, and applying the guiding principle at each stage.

The process is illustrated by the following example:

$$u^5(v, x);\ u^4(w^2);\ u^3(vx^2);\ u^2(v^3x^2, vx^3);\ u(v^2wx^2, vx^4);\ w^4x^2, w^3x^3, x^6$$

$$|\prec|\ u^5(v, w);\ u^4v^2;\ u^3v^3;\ u^2(v^4, v^3w);\ u(v^5, v^4w);\ v^6, v^5w, v^5x$$

$$\equiv\ w(u^5, u^2v^3, uv^4, v^5);\ (u^5v, u^4v^2, u^3v^3, u^2v^4, uv^5, v^6, v^5x)$$

$$|\prec|\ w(u^5, u^4v, u^4x, u^3v^2);\ (u^6, u^5v, u^5x, u^4v^2, u^4vx, u^4x^2, u^3v^3)$$

$$\equiv\ u^6;\ u^5(v, w, x);\ u^4(v^2, vw, vx, wx, x^2);\ u^3(v^3, v^2w)$$

$$|\prec|\ u^6;\ u^5(v, w, x);\ u^4(v^2, vw, vx, w^2, wx);\ u^3(v^3, v^2w)$$

$$\equiv\ w^2u^4;\ w(u^5, u^4v, u^4x, u^3v^2);\ u^6;\ u^5(v, x);\ u^4(v^2, vx);\ u^3v^3$$

$$|\prec|\ w^2u^4;\ w(u^5, u^4v, u^4x, u^3v^2);\ u^6;\ u^5(v, x);\ u^4(v^2, vx, x^2)$$

$$\equiv\ u^6;\ u^5(v, w, x);\ u^4(v^2, vw, vx, w^2, wx, x^2);\ u^3v^2w$$

$$|\prec|\ u^6;\ u^5(v, w, x);\ u^4(v^2, vw, vx, w^2, wx, x^2);\ u^3v^3.$$

It will be seen that, in each application of the guiding principle, the validity of our theorem for three variables, established in § 3, is assumed. The terms in the last line form a natural sequence, and the validity of the theorem for four variables is established in this particular instance.

In generalizing, we suppose that at any stage we have the relation

$$S_l(u, v, w, x)\,|\prec|\,\Sigma u^{l-r}S_r(v, w, x). \tag{1}$$

We proceed by applying the guiding principle and rearranging the terms. Thus

$$S_l(u, v, w, x)\,|\prec|\,\Sigma u^{l-r}P_r(v, w, x) \tag{2}$$

$$\equiv \sum_{s=0} w^s S'_{l-s}(u, v, x) \tag{3}$$

$$|\prec|\ \sum_{s=0} w^s P'_{l-s}(u, v, x) \tag{4}$$

$$\equiv \sum_{t=0} u^{l-t} S''_t(v, w, x) \tag{5}$$

$$|\prec|\ \sum_{t=0} u^{l-t} P''_t(v, w, x). \tag{6}$$

It will be seen that in (3) the powers of w are consecutive in ascending order beginning with w^0, so the same is true of (4). In (5) the powers

of u which appear are consecutive in descending order beginning with u^l. The same is true of (6). At this stage the leading terms are already part of a natural sequence. Dropping the dashes we write

$$S_l \,|\, \lessdot \,|\, \psi_1(u, v, w, x);\ \ \psi_2(u, v, w, x),$$

where

$$\psi_1(u, v, w, x) \equiv u^l;\ \ u^{l-1}(v, w, x);\ \ \ldots;\ \ u^{l-p+1}(v, w, x)^{p-1};\ \ u^{l-p} P_p(v, w, x)$$

and

$$\psi_2(u, v, w, x) \equiv u^{l-p-1} P_{p+1}(v, w, x);\ \ \ldots;\ \ u^{l-\kappa} P_\kappa(v, w, x).$$

Here $(v, w, x)^r$ is written for the aggregate of all the p.p.'s of degree r of the three variables, *i.e.* for a complete natural sequence of that degree. The natural sequence P_p is the first which is incomplete. When, as in step (3), the terms are rearranged by powers of w the group $w^s S'_{l-s}(u, v, x)$ will, in general, be made of terms from ψ_1 and terms from ψ_2. Those belonging to ψ_1 form a natural sequence; accordingly, at step (4) these terms are unaltered and at step (5) ψ_1 will be reconstituted. Thus at the end of the four steps 3, 4, 5, 6 the terms of ψ_1 are unchanged.

In $\psi_1(u, v, w, x)$ the natural sequence P_p is incomplete. This implies that there is a vacancy for $u^{l-p} x^p$ in the array* and there may be vacancies for higher terms. Let us suppose that the first term independent of w for which there is a vacancy is $u^{l-p} v^q x^{p-q}$, where $p > q \geqslant 0$. Further, we know that the next group $u^{l-p-1} P_{p+1}(v, w, x)$ must contain as its leading term $u^{l-p-1} v^{p+1}$.

At step (3) the group independent of w, *i.e.* $S'_l(u, v, x)$, will contain this term $u^{l-p-1} v^{p+1}$, which ranks below the missing term $u^{l-p} v^q x^{p-q}$. Accordingly at stage (4), when the guiding principle is applied, the term $u^{l-p} v^q x^{p-q}$ appears in P'_l. It follows that the number of terms in u^{l-p} in stages (5) and (6) is increased by at least one as compared with the number in ψ_1.

By repetition of the steps (3) to (6) as many times as may be needed, the coefficient of u^{l-p} will either become the complete $(v, w, x)^p$ or else will absorb all the terms of ψ_2.

As long as there are any terms which do not belong to the natural sequence the process can be repeated, until finally we have

$$S_l(u, v, w, x) \,|\, \lessdot \,|\, u^l;\ \ u^{l-1}(v, w, x);\ \ u^{l-2}(v, w, x)^2$$

$$\ldots\ u^{l-m+1}(v, w, x)^{m-1};\ \ u^{l-m} P_m(v, w, x),$$

* At this step the proof fails for three variables.

which signifies that the number of terms derived from the N terms of $S_l(u, v, w, x)$ by multiplication by u, v, w or x does not exceed $Q(N)$, *i.e.* that our proposition is established for four variables.

Exactly the same argument may be used to established the proposition successively for five, six, and more variables. It is therefore true for any number of variables.

It is of interest to notice that the argument by which we pass from n variables to $n+1$ involves the use of two of the variables u and w, as bases of the powers dominating the alternate arrangements of the terms, and of at least two other variables, one ranking above and the other below w. When there are three variables the four rôles can not all be filled. That is why the demonstration for three variables has to be treated specially.

SOME SOLUTIONS OF THE ELECTROMAGNETIC EQUATIONS, AND OF THE ELASTIC EQUATIONS, WITH APPLICATIONS TO THE PROBLEM OF SECONDARY WAVES

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1. *Introduction.*

The solutions of the electromagnetic wave-equations, which are given in § 2 below, were constructed for use in other problems, in the first place; but on examining them in connexion with an advancing plane wave, the formulae (7) and (7′) of § 3 were obtained. The form of (7′) at once suggested a comparison with Stokes's formula for resolving a plane wave into secondary waves, and a closer examination showed that there was a slight difference between the formulae. At first sight it seemed possible that the discrepancy might arise from Stokes's use of the elastic-solid theory, while (7′) was derived from the electromagnetic equations; this led to the investigation of (16) in § 4, assuming sources of discontinuity in the displacements and rotations. But actually (16) is precisely the same as (7), when the velocity of dilatational waves is assumed very large (as in Stokes's formula); thus the cause of the discrepancy must lie in Stokes's method of using the initial displacements and velocities.

Actually the part of Stokes's solution which depends upon the velocities of the primary wave agrees precisely with the first part of the solutions (7) in § 3 or (16) in § 4, and the discrepancy comes from the terms containing the initial *displacements* in Stokes's solution. There are, however, two points in which criticisms may be directed against Stokes's method; the first is that his so-called *initial disturbance* is, in reality, treated as a disturbance which is constantly being renewed, and so varies with the time. In this respect the idea of sources of discontinuity, seems simpler, since the strength of the source will naturally

vary with the time. The second objection arises from the fact that sharp discontinuities have to be introduced into the initial disturbances in order to isolate the contribution from a particular point of the wave-front; but if we use sources of discontinuity, we can arrange a *gradual* transition from space free from discontinuities to the region where discontinuities exist.

The solutions given in (7) and (7′) have the same advantages as Stokes's over most other formulae; that is, each *separate** secondary wave satisfies the fundamental equations, and also the waves give zero resultant behind the wave front. But (in contrast to the solution given by Stokes) the resultant vector in (7) and (7′) is *not* in the same plane as the vector in the primary waves; the deviation is not very great, but it is zero only for special positions.

The symbolical methods adopted in these solutions are essentially the same as in Heaviside's treatment of wave-problems. The fundamental principle of the method is to write

$$p = \frac{\partial}{\partial t}, \quad q = \frac{1}{c}\frac{\partial}{\partial t},$$

and to treat p, q as mere symbols obeying the laws of algebra. The formulae so obtained are finally interpreted by formulae such as†

$$e^{-t_0 p} f(t) = f(t-t_0), \quad e^{-rq} F(ct) = F(ct-r),$$

which are suggested by Taylor's theorem (written in the symbolic form) The effect of applying p, p^2, q, q^2 to these results is immediately evident, and similarly $1/p$, $1/q$ are interpreted as integrals.

When the system of equations to be solved is derived from a dynamical system (which is initially undisturbed) the functions of t, on which the symbols operate, will be the *applied forces*; and the simplest types are zero for negative values of t, and are constant for positive values of t. Thus the interpretation of a symbolic operator $\Phi(p)$ is obtained by regarding the operator as an abbreviation for

$$\Phi(p)\, H(t),$$

* So that, not merely the integrated forces, but also the integrands themselves, represent true waves.

† Since this paper was originally worked out, Dr. Harold Jeffreys has published a tract, *Operational methods in mathematical physics* (Cambridge, 1927), which makes the principles of the process more easily available. A discussion of these special formulae is given in § 1.8 and in Ch. II of his tract; it is perhaps well to warn the reader that the notation used in this tract differs from that of earlier writers, σ being used (in place of p) for $\partial/\partial t$ and p for $\partial/\partial x$.

where* $H(t) = 0$ if $t < 0$; $H(t) = 1$ if $t > 0$.

Now we can express $H(t)$ in an anaytical form, due to Cauchy, by the complex integral

$$H(t) = \int_{a-i\infty}^{a+i\infty} \frac{e^{\lambda t} d\lambda}{2\pi i \lambda} \quad (a > 0),$$

the path of integration being parallel to the imaginary axis (and to the right of the origin) in the λ-plane. Thus the interpretation of the operator $\Phi(p)$ is given by†

$$\Phi(p) = \int_{a-i\infty}^{a+i\infty} \Phi(\lambda) \frac{e^{\lambda t} d\lambda}{2\pi i \lambda} \quad (a > 0),$$

the path of integration being unchanged.

Similarly we can write

$$\Psi(q) = \int_{a-i\infty}^{a+i\infty} \Psi(\mu) \frac{e^{\mu t} d\mu}{2\pi i \mu}.$$

It follows that the algebraic properties of p or q are the same as those of λ or μ respectively, and so the symbols p, q may be treated as having their real parts positive in any preliminary calculations (keeping the paths of integration on the positive side of $\lambda = 0$ or $\mu = 0$). But the final formulae may often be most easily evaluated by a change of path: this can be carried out according to the ordinary rules of complex integrals, residues being evaluated as may be necessary (at the poles of the integrand).

In the simple problems considered below, the interpretation by complex integrals is not necessary (except to ensure the logical development of the theory), and the operators can be evaluated from first principles. Thus, when t is positive, we have

$$\frac{1}{p} = \frac{1}{p} H(t) = t, \quad \frac{1}{p^2} = \frac{1}{p} t = \tfrac{1}{2} t^2, \quad \frac{1}{p^3} = \frac{1}{p} (\tfrac{1}{2} t^2) = \tfrac{1}{6} t^3$$

and

$$\frac{1}{p^n} = \frac{t^n}{n!},$$

where n is any positive integer.

* The notation $H(t)$ is suggested by Jeffreys (*loc. cit.* 17); the letter H serves to recall the fact that the use of the function in this connexion is due to Heaviside.

† *Proc. London Math. Soc.* (2), 15 (1916), 401; see, in particular, § 2, § 4, and § 5. The above interpretation assumes zero initial disturbances; the modifications necessary to take account of such initial values will be found in §§ 1, 8. See, also §§ 5, 6 below and Jeffreys (*loc. cit.*, Chs. I and II).

Similarly, if $t > t_0$, we have

$$\frac{1}{p}e^{-t_0 p} = t - t_0, \quad \frac{1}{p^2}e^{-t_0 p} = \tfrac{1}{2}(t-t_0)^2, \quad \frac{1}{p^3}e^{-t_0 p} = \tfrac{1}{6}(t-t_0)^3$$

and so on; but these functions are all zero when $t < t_0$.

Occasionally it becomes necessary to make a distinction between e^{-qr} and e^{+qr}, on account of the fact that only one of these can be used in a region where r may extend to infinity. In effect, all that is needed is to treat p or q as *positive* (or, more exactly, as *complex with a positive real* part); and then we must reject e^{+qr} and retain e^{-qr}.

When the applied forces vary with t, it is usually most convenient to convert the given functions of t into operational form; in the present group of problems we shall use linear or quadratic functions of t only, which can be replaced by symbolic formulae at once, by using the results already mentioned, such as

$$t = 1/p, \quad t^2 = 2/p^2.$$

In the same way, we can write

$$e^{\beta t} = p/(p-\beta),$$

by expanding in powers of t and $1/p$; this amounts to replacing the λ in the denominator of the complex integrals by $\lambda - \beta$. More general functions are considered in the paper just quoted*.

In the above account of the origin of the complex integrals reference has been made to *applied forces*, which was the natural phraseology to use when the original problem was stated in a *dynamical form*. But in the electromagnetic equations, as used in § 2 below, the dynamical form is lost, and the vectors (X_0, Y_0, Z_0), $(\alpha_0, \beta_0, \gamma_0)$ appear merely as representing *discontinuities* in the magnetic and electric fields: thus (X_0, Y_0, Z_0) is proportional to the conduction-current if the field is due to such currents. And, speaking generally, in any system of equations, those terms, which do not arise from the internal mechanism of the vibrating system, are to be regarded as belonging to the *applied forces*, irrespective of their actual physical meaning. That is to say, they are to be expressed by complex integrals of the standard type, when that

* The general case may be dealt with by the Fourier double integral theorem, in the form (*loc. cit.*, 412)

$$f(t) = \int_{\alpha - i\infty}^{\alpha + i\infty} g(\lambda) e^{\lambda t} \frac{d\lambda}{2\pi i},$$

where

$$g(\lambda) = \int_0^\infty e^{-\lambda\theta} f(\theta)\, d\theta;$$

see, also, Jeffreys (*loc. cit.* § 2.2). In Heaviside's notation $f(t) = pg(p)$: see, for instance, *Electromagnetic theory*, 3, 235–239.

method of interpretation is found to be the easiest way of evaluating the symbolic results.

In order to complete the proof that the symbolic methods adopted here are adequate to replace the older methods, a direct comparison of some results is carried out in § 5; from this it appears that Poisson's integral can be deduced from the symbolic solutions. In the case of an initially *constant* disturbance, the results can be derived rather more easily from Poisson's integral than from the symbolic solution; but if the disturbance varies from point to point, the symbolic method appears to give results, such as (31) in § 5, with greater ease and certainty.

2. *Propagation of electromagnetic waves from a source of limited extent.*

Let us suppose that sources producing discontinuities, both in electric and in magnetic forces, are maintained within a region of limited extent, which we denote by T. Thus we write the electromagnetic equations in the forms (taking the dielectric constant K and the magnetic constant μ each to be unity),

$$(1)\qquad \left.\begin{aligned} qX &= X_0+\frac{\partial}{\partial y}(c\gamma)-\frac{\partial}{\partial z}(c\beta), & -qc\alpha &= -c\alpha_0+\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\\ qY &= Y_0+\frac{\partial}{\partial z}(c\alpha)-\frac{\partial}{\partial x}(c\gamma), & -qc\beta &= -c\beta_0+\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x}\\ qZ &= Z_0+\frac{\partial}{\partial x}(c\beta)-\frac{\partial}{\partial y}(c\alpha), & -qc\gamma &= -c\gamma_0+\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y} \end{aligned}\right\},$$

where $$q=\frac{1}{c}\frac{\partial}{\partial t},$$

c being the velocity of radiation; here (X_0, Y_0, Z_0) and $(\alpha_0, \beta_0, \gamma_0)$ represent the distribution of discontinuities*, and, in general, they will vary both with t and with x, y, z, but their values will be zero outside the limited space T. To avoid the discussion of surface integrals over the boundary of T, it will be convenient to suppose that (X_0, Y_0, Z_0), $(\alpha_0, \beta_0, \gamma_0)$, as well as their differential coefficients with respect to x, y, z, are zero over the boundary of T.

To give a concrete illustration, let us consider a similar problem in potential-theory; our present hypothesis corresponds to the assumption that the volume-density is zero, and also its differential coefficients at the surface of the volume T which contains the matter considered. This restriction, however, is needed only in the preliminary analysis; and it is easy to pass to limiting cases in which a surface-density is to be considered. For instance, suppose that

* Actually (X_0, Y_0, Z_0) arises from discontinuities in the magnetic force, $(\alpha_0, \beta_0, \gamma_0)$ from those in the electric force; thus $c^{-1}(X_0, Y_0, Z_0)$ corresponds to what Heaviside denoted by $-\mathrm{curl}\,\mathbf{h}$, and $c(\alpha_0, \beta_0, \gamma_0)$ to what he called $\mathrm{curl}\,\mathbf{e}$.

a surface-distribution of constant amount σ, per unit area of the xy plane, is required; we can then begin with a volume-density of the type $\rho = k(h^2-z^2)^2$ between the two planes $z = \pm h$, where $\frac{16}{15}kh^5 = \sigma$, so that

$$\rho = \tfrac{15}{16}\frac{\sigma}{h}\left(1-\frac{z^2}{h^2}\right)^2.$$

We can make h tend to zero afterwards, when dealing with points at a distance from the distribution of matter considered.

On eliminating (α, β, γ) between the two sets of equations in (1), we find three equations for X, Y, Z, the first of which is

$$q^2X = q\left(X_0-\frac{1}{q^2}\frac{\partial\delta_0}{\partial x}\right)+\frac{\partial}{\partial y}(c\gamma_0)-\frac{\partial}{\partial z}(c\beta_0)+\Delta_2 X, \tag{2}$$

where

$$\delta_0 = \frac{\partial X_0}{\partial x}+\frac{\partial Y_0}{\partial y}+\frac{\partial Z_0}{\partial z},$$

and

$$\Delta_2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}$$

is Laplace's operator.

It follows at once from potential-theory that the solution of (2) can be written in the form

$$4\pi qX = \left(q^2-\frac{\partial^2}{\partial x^2}\right)P_1-\frac{\partial^2 P_2}{\partial x\partial y}-\frac{\partial^2 P_3}{\partial x\partial z}+q\left(\frac{\partial M_3}{\partial y}-\frac{\partial M_2}{\partial z}\right),$$

where

$$\left\{\begin{aligned} &P_1 = \int\frac{e^{-qr}}{r}X_0'\,d\tau_1, \quad P_2 = \int\frac{e^{-qr}}{r}Y_0'\,d\tau_1, \quad P_3 = \int\frac{e^{-qr}}{r}Z_0'\,d\tau_1,\\ &M_1 = \int\frac{e^{-qr}}{r}c\alpha_0'\,d\tau_1, \quad M_2 = \int\frac{e^{-qr}}{r}c\beta_0'\,d\tau_1, \quad M_3 = \int\frac{e^{-qr}}{r}c\gamma_0'\,d\tau_1,\\ &r^2 = (x-x_1)^2+(y-y_1)^2+(z-z_1)^2, \end{aligned}\right. \tag{3}$$

and dashes indicate that the values of the functions are to be taken at (x_1, y_1, z_1), the point where $d\tau_1$ is situated, while the integrals extend throughout the volume T. The significance of these symbolic formulae (in most cases of interest) can usually be found easily by direct reasoning; but if there is any uncertainty, the interpretation by means of complex integrals can be used in the way explained in § 1 above.

For the formal proof of (3) it will be enough to take one term in (2); suppose that we examine the term arising from Z_0; omitting all other terms, (2) becomes

$$(\Delta_2-q^2)X = +\frac{1}{q}\frac{\partial^2 Z_0}{\partial x\,\partial z}, \tag{2'}$$

the solution of which is given by

$$4\pi qX = -\int\left(\frac{e^{-qr}}{r}\right)\frac{\partial^2 Z_0'}{\partial x_1\,\partial z_1}\,d\tau_1. \tag{3'}$$

Now integrate by parts, using Green's theorem, and remembering that $\partial Z_0'/\partial x_1$ is zero at the surface of T; then we see that

$$4\pi q X = + \int \frac{\partial}{\partial z_1}\left(\frac{e^{-qr}}{r}\right)\frac{\partial Z_0'}{\partial x_1}\,d\tau_1.$$

But

$$\frac{\partial}{\partial z_1}\left(\frac{e^{-qr}}{r}\right) = -\frac{\partial}{\partial z}\left(\frac{e^{-qr}}{r}\right),$$

and we can now remove the differentiation with respect to z from under the sign of integration. Thus

$$4\pi q X = -\frac{\partial}{\partial z}\int \frac{e^{-qr}}{r}\frac{\partial Z_0'}{\partial x_1}\partial\tau_1.$$

Similarly we prove that*

$$\int \frac{e^{-qr}}{r}\frac{\partial Z_0'}{\partial x_1}\,d\tau_1 = \frac{\partial}{\partial x}\int \frac{e^{-qr}}{r}Z_0'\,d\tau_1 = \frac{\partial P_3}{\partial x},$$

and so the final contribution of P_3 to X is given by

$$4\pi q X = -\frac{\partial^3 P_3}{\partial x\,\partial z}.$$

Having obtained this formula for X, it is evident that the equation (2′) is certainly satisfied provided that

$$(\Delta_2 - q^2)P_3 = -4\pi Z_0;$$

and this holds good in virtue of the definition of P_3 contained in equations (3).

Having found that the general solution of (2) is expressible in the form (3), we can see that the restrictions† imposed on the discontinuities (X_0, Y_0, Z_0) and $(\alpha_0, \beta_0, \gamma_0)$ are, in reality, superfluous; for, in effect, these functions play the same parts in the integrals (3) as the volume-density in elementary potential-theory. Thus these restrictions have simply enabled us to avoid writing out the necessary relations between the discontinuities in the *second* differential coefficients of the potentials, and the values of $X_0, Y_0, Z_0, \alpha_0, \beta_0, \gamma_0$ at the boundary.

Further, so long as we are dealing with *points outside the region* T, we can obtain the values of X, Y, Z by differentiating under the signs of integration, when convenient; the difficulties in this matter (referred to in the first footnote below) arise only at points within T. As a rule, we need the formulae for points at a considerable distance from T, so that these questions of convergence need not trouble us‡.

In the simple case when the space T is reduced to a point, the electromagnetic field (3) can be regarded as due to a Hertzian oscillator, having

* To avoid difficulties of convergence, we should not use the integral of $\frac{\partial^2}{\partial x\,\partial z}\left(\frac{e^{-qr}}{r}\right)$ through T; in the proof as given above it is not necessary to do so.

† That the discontinuities (as well as their first differential coefficients) should vanish at the boundary of T.

‡ Of course, in the investigation given above, we must consider *internal*, as well as *external*, points.

moments proportional to (P_1, P_2, P_3), and to a magnetic oscillator having moments proportional to (M_1, M_2, M_3).

The formulae (3) are natural extensions of those due to Prof. H. M. Macdonald (*Electric waves*, Chapter II) for the case when the discontinuities are due to conduction-currents* of wave-length $2\pi/\kappa$ (so that we can simply write $q = i\kappa$, when the time-factor is taken to be $e^{i\kappa ct}$, assuming that the disturbance has lasted long enough for the initial transient waves to be radiated away).

3. *Application to obtain the representation of a plane wave by means of secondary waves.*

We can treat the plane wave as made up of a sequence of thin pulses; and let us consider the circumstances arising when one such pulse arrives at the wave-front, which we suppose to be the plane $z = 0$. We take the incident wave as given by the electric and magnetic forces

$$X = f(ct-z), \quad c\beta = f(ct-z),$$

the other components of force being zero. The region T of § 2 is then a thin layer just behind $z = 0$; we denote its thickness by θ, and if we integrate through the layer the equations (1) of § 2, we see that

$$\int X_0 d\theta - c(\beta_2 - \beta_1) = 0, \quad -\int c\beta_0 d\theta + (X_2 - X_1) = 0,$$

the other discontinuities being zero. Thus we are to take

$$\left.\begin{aligned} &\int X_0 d\theta = f(ct), \quad \int Y_0 d\theta = 0, \qquad \int Z_0 d\theta = 0 \\ &\int c\alpha_0 d\theta = 0, \qquad \int c\beta_0 d\theta = f(ct), \quad \int c\gamma_0 d\theta = 0 \end{aligned}\right\},$$

for the field (X_1, β_1) just *behind* the pulse is *zero*.

It follows from (3), that

$$P_1 = \int \frac{e^{-qr}}{r} f(ct)\, dS = M_2,$$

where the integral is taken over the wave-front; and the other terms P_2, P_3, M_1, M_3 are zero.

* It is only necessary to write $(X_0, Y_0, Z_0) = -4\pi c\,(u, v, w)$ to transform the terms in (P_1, P_2, P_3) into those found, *loc. cit.*, p. 13; more complete formulae are given on p. 17. Compare also H. M. Macdonald, *Proc. Royal Soc.* (A), 113 (1926), 237; in particular, formulae (9) and (10).

It is easy to confirm these results by integrating out; we can put $dS = 2\pi r dr$; and then the lower limit in the integral will be $r = \pm z$, according to the sign of z (r being essentially positive). Thus we find that

$$P_1 = M_2 = \frac{2\pi}{q} e^{-qz} f(ct) \quad (z > 0), \quad \text{or} \quad \frac{2\pi}{q} e^{+qz} f(ct) \quad (z < 0), \tag{4}$$

for, since the real part of q is positive in our complex integrals, the value of e^{-qr} tends to zero as $r \to \infty$ at the outer boundary of the wave.

We can also confirm these formulae by observing that

$$\frac{\partial^2 P_1}{\partial z^2} - q^2 P_1 = 0;$$

that $P_1 \to 0$, as $z \to \pm\infty$; and that, at $z = 0$, the discontinuity in $\partial P_1/\partial z$ is equal to $4\pi f(ct)$. These are the properties to be expected for the generalized potential due to a surface-density $f(ct)$ on the plane $z = 0$.

If we substitute from (4) in the equations (3) of § 2, we see that

$$4\pi q X = q^2 P_1 - q \frac{\partial M_2}{\partial z}, \quad Y = 0, \quad Z = 0.$$

Thus

$$X = \frac{q}{4\pi} P_1 - \frac{1}{4\pi} \frac{\partial M_2}{\partial z},$$

that is,

$$X = \tfrac{1}{2} e^{-qz} f(ct) - \frac{1}{2q} \frac{\partial}{\partial z} \{ e^{-qz} f(ct) \} \qquad (z > 0),$$

or

$$= \tfrac{1}{2} e^{+qz} f(ct) - \frac{1}{2q} \frac{\partial}{\partial z} \{ e^{+qz} f(ct) \} \qquad (z < 0).$$

On evaluating the differential coefficients, we see that

$$X = (\tfrac{1}{2} + \tfrac{1}{2}) e^{-qz} f(ct) = f(ct - z) \qquad (z > 0), \tag{5}$$

or

$$= (\tfrac{1}{2} - \tfrac{1}{2}) e^{+qz} f(ct) = 0 \qquad (z < 0),$$

which is in agreement with our original assumption that *the pulse is moving forwards leaving a zero field in the rear*.

The corresponding formula for the magnetic force is

$$c\mathcal{B} = \frac{q}{4\pi} M_2 - \frac{1}{4\pi} \frac{\partial P_1}{\partial z}. \tag{5'}$$

And, since $M_2 = P_1$, this formula will give

$$c\mathcal{B} = X,$$

in accordance with the form originally assumed for the plane-wave.

It is now easy to see how to express the disturbance in the secondary waves, so as to be equivalent to the original plane-wave. In fact, from (3) of § 2 we see that the contributions due to dS will be

$$X_1 dS, \; Y_1 dS, \; Z_1 dS,$$

where

$$\left\{ \begin{aligned} 4\pi q X_1 &= \left(q^2 - \frac{\partial^2}{\partial x^2} \right) \frac{e^{-qr}}{r} f(ct) - q \frac{\partial}{\partial z} \left(\frac{e^{-qr}}{r} \right) f(ct), \\ 4\pi q Y_1 &= -\frac{\partial^2}{\partial x \partial y} \frac{e^{-qr}}{r} f(ct), \\ 4\pi q Z_1 &= -\frac{\partial^2}{\partial x \partial z} \frac{e^{-qr}}{r} f(ct) + q \frac{\partial}{\partial x} \left(\frac{e^{-qr}}{r} \right) f(ct). \end{aligned} \right. \tag{6}$$

and the corresponding formulae for the magnetic forces are found by interchanging* x, y, and replacing X_1 by $c\beta_1$, Y_1 by $c\alpha_1$, Z_1 by $c\gamma_1$.

The results in (6) can be evaluated by differentiation in the forms

$$(6') \quad \begin{cases} 4\pi q X_1 = \frac{1}{r^3}\{(1+qr+q^2r^2) - l^2(3+3qr+q^2r^2) + nqr(1+qr)\} f(ct-r), \\ 4\pi q Y_1 = \frac{1}{r^3}\{ \quad -lm(3+3qr+q^2r^2) \quad \} f(ct-r), \\ 4\pi q Z_1 = \frac{1}{r^3}\{ \quad -ln(3+3qr+q^2r^2) - lqr(1+qr)\} f(ct-r), \end{cases}$$

where l, m, n are the direction-cosines of the radius r, so that

$$l = (x-x_1)/r, \quad m = (y-y_1)/r, \quad n = (z-z_1)/r.$$

Similar formulae hold for $\alpha_1, \beta_1, \gamma_1$, which can be written down by the rule just given. For the important special case when the waves are simple harmonic, so that $f(ct-r)$ is replaced by $\sin\kappa(ct-r)$ or $\cos\kappa(ct-r)$, results equivalent to (6) have been found by Prof. A. E. H. Love†, using an entirely different method.

The vector on the right of equations (6') can be expressed conveniently as the resultant of three others,

(lL, mL, nL) along the radius (l, m, n),

and

$$\left.\begin{matrix} (nM, & 0, & -lM) \\ \{(m^2+n^2)N, & -lmN, & -lnN\} \end{matrix}\right\} \text{ perpendicular to } (l, m, n),$$

where

$$\left.\begin{aligned} L &= -2l(1+qr) f(ct-r)/r^3 \\ M &= qr(1+qr) f(ct-r)/r^3 \\ N &= (1+qr+q^2r^2) f(ct-r)/r^3 \end{aligned}\right\}.$$

From these results we can readily transform to spherical polars by observing that the second and third vectors are the vector-products of $(0, M, 0)$ by (l, m, n) and of $(0, nN, -mM)$ by (l, m, n), respectively. Using this property, the second and third vectors are found to give $(N\cos\theta + M)\cos\phi$, $-(N + M\cos\theta)\sin\phi$, in the directions of θ, ϕ increasing (see the figure on p. 449 below).

We can now readily express (6') in terms of spherical polar coordinates; thus

$$(7) \quad \begin{cases} 4\pi q R_1 = -\dfrac{2\sin\theta\cos\phi}{r^3}(1+qr) f(ct-r), \\ 4\pi q \Theta_1 = \dfrac{\cos\theta\cos\phi}{r^3}(1+qr+q^2r^2) f(ct-r) + \dfrac{q\cos\phi}{r^2}(1+qr) f(ct-r), \\ 4\pi q \Phi_1 = -\dfrac{\sin\phi}{r^3}(1+qr+q^2r^2) f(ct-r) - \dfrac{q\cos\theta\sin\phi}{r^2}(1+qr) f(ct-r), \end{cases}$$

where θ is measured from the axis of z and ϕ from the plane of zx.

* This appears at first sight to introduce left-handed axes, and so to alter our standard equations; actually the necessary changes of sign are found to neutralize each other in the final formulae.

† *Phil. Trans. Royal Soc.* (A), 197 (1901), 18.

The general formulae (7) can be confirmed by writing

$$U = -\sin\theta\cos\phi\,\frac{1}{r}(1+qr)\,e^{-qr}f(ct),$$

$$V = -\sin\theta\sin\phi\,\frac{1}{r}(1+qr)\,e^{-qr}f(ct),$$

in the general spherical polar solutions*

$$4\pi q R_1 = \frac{\partial^2 U}{\partial r^2} - q^2 U,$$

$$4\pi q \Theta_1 = \frac{1}{r}\frac{\partial^2 U}{\partial r\,\partial\theta} - \frac{q}{r\sin\theta}\frac{\partial V}{\partial\phi},$$

$$4\pi q \Phi_1 = \frac{1}{r\sin\theta}\frac{\partial^2 U}{\partial r\,\partial\phi} + \frac{q}{r}\frac{\partial V}{\partial\theta}.$$

An easy independent verification of the transformation from (6) to (7) is found by comparing only the terms in $1/r$; these arise in the original formulae from differentiating the indices in the exponentials (and nothing else).

When we are concerned with distances which are large compared with the wave-lengths of the problem, it will suffice to retain only the terms involving $1/r$ (that is, we shall reject $1/r^2$ and $1/r^3$). The result (7) is then found to reduce to

$$(7')\quad \begin{cases} R_1 = 0, \\ \Theta_1 = \dfrac{(1+\cos\theta)\cos\phi}{4\pi r}\,qe^{-qr}f(ct) = \dfrac{(1+\cos\theta)\cos\phi}{4\pi r}\,f'(ct-r), \\ \Phi_1 = -\dfrac{(1+\cos\theta)\sin\phi}{4\pi r}\,qe^{-qr}f(ct) = -\dfrac{(1+\cos\theta)\sin\phi}{4\pi r}\,f'(ct-r). \end{cases}$$

This result at first sight appears to be almost identical with that found by Stokes on the elastic-solid theory (*Math. Papers*, 2, 284–286); but it is to be remembered that the angle denoted by ϕ in Stokes's formula is the angle between r and the direction, which is taken as Ox in the present analysis. It is also to be noticed that Stokes's vector is in the plane containing r and Ox; but our resultant force is *not* in that plane†. Also the resultant force in (7) is equal to

$$\frac{1+\cos\theta}{4\pi r}\,f'(ct-r),$$

without the factor called $\sin\phi$ by Stokes.

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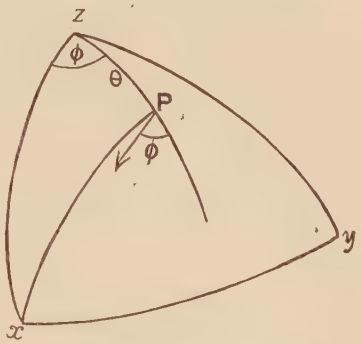
* *Phil. Mag.* (6), 38 (July, 1919), 143; see, in particular, p. 150; a direct transformation of a similar (but rather simpler) field of force is carried out on p. 154.

† The part arising from the first column in (7) is in the plane rOx, but not that which comes from the second column. This is readily confirmed from (6).

It is perhaps easiest to compare Stokes's formula with (7′) by resolving into similar components. Then we find that

$$\Theta_s = \Theta_1 \cos\theta, \quad \Phi_s = \Phi_1.$$

The rough sketch indicates the arrangement; x, y, z are points on a sphere where the axes cut the surface, P being a point on the radius r. Stokes's vector is then along the great circle Px: but our resultant force makes an angle ϕ with the arc zP, as sketched. When θ is small, as happens in many applications, the arc Px is near to $\frac{1}{2}\pi$; and so Stokes's extra factor $\sin Px$ is practically unity. Also the angle zPx is nearly $\pi-\phi$, and so the directions of the two vectors are nearly coincident.



Although the formulae (7) and (7′) are the easiest to appreciate geometrically, yet, for practical calculation, the most convenient forms are given by (6′), *retaining only terms of order* $1/r$. When $1/r^2$ and $1/r^3$ are rejected, it will be found that we may write

$$L = 0, \quad M = N = f'(ct-r)/r,$$

so that

(7″) $$4\pi X_1 = (m^2+n^2+n)N, \quad 4\pi Y_1 = -lmN, \quad 4\pi Z_1 = -lnN,$$

where l, m, n are the direction-cosines of the radius*.

It has been remarked that the case of chief interest is that in which the radius r is nearly normal to the wave-front, so that l, m are small and n^2 is nearly unity. Then (7″) reduces to the simple form

(7‴) $$X_1 = \frac{1}{2\pi r} f'(ct-r), \quad Y_1 = 0, \quad Z_1 = -lX_1,$$

neglecting terms of the second order in l, m. And, if the radius is produced backwards *behind the wave-front*, $X_1 = 0$, $Y_1 = 0$, $Z_1 = 0$.

It is hardly necessary to check the facts that, on integration,

$$\int X_1 dS = f(ct-z), \quad \int Y_1 dS = 0, \quad \int Z_1 dS = 0,$$

since we have already verified them in a more complete way in (5) above;

* Stokes's formulae, which correspond to these, are given on p. 287, and in our notation are proportional to $(1+n)(1-l^2)$, $-lm(1+n)$, $-ln(1+n)$. And the corresponding differences of the two sets of formulae are proportional to

$$l^2 n, \quad lmn, \quad l(n^2-1),$$

which are of the second order in θ, when θ is small or near to π (n being $\cos\theta$).

and, further, the actual integrations arising from (7''') are in no essential respect different from those carried out by Stokes in verifying his solution.

The advantage of the formulae (6) and (7) over others that can be obtained is that (in these forms) each particular area dS can be seen to contribute *a set of waves satisfying the electromagnetic equations*; in other forms, although the final *integrals* reduce to the original plane wave, and so satisfy the electromagnetic equations, yet the separate parts do not.

For example, if we apply Green's theorem to the space for which z is positive, we find at once that

$$\int \left\{ \frac{\partial}{\partial z_1}\left(\frac{e^{-qr}}{r}\right) X - \frac{e^{-qr}}{r}\frac{\partial X}{\partial z_1} \right\} dS = 4\pi X_P,$$

where P is the point from which r is measured, the integral being taken over the plane $z = 0$.

Similarly take P', the image of P in the plane $z = 0$, and measure r' from P', then the corresponding integral (with r' in place of r) is zero, because P' is outside the space of integration.

But $r' = r$ at the surface, and $\partial r'/\partial z_1 = -\partial r/\partial z_1$.

Hence

$$\int \left\{ \frac{\partial}{\partial z_1}\left(\frac{e^{-qr}}{r}\right) X + \frac{e^{-qr}}{r}\frac{\partial X}{\partial z_1} \right\} dS = 0,$$

and so, if we eliminate $\partial X/\partial z$, we have the result

$$X_P = \frac{1}{2\pi}\int \frac{\partial}{\partial z_1}\left(\frac{e^{-qr}}{r}\right) X\, dS;$$

thus X is given as the resultant of $\int X_2 dS$, where

$$X_2 = \frac{\cos\theta}{2\pi r^2}(1+qr)\, e^{-qr} f(ct) = \frac{\cos\theta}{2\pi r^2}\left\{ f(ct-r) + rf'(ct-r) \right\}, \tag{8}$$

while the corresponding terms in Y_2, Z_2 are zero.

However, these forms *cannot* give a solution of the electromagnetic wave-equations: for if Y, Z are both zero, the fundamental equations give $\alpha = 0$, and then it will be found that X, β, γ are *independent of* x—a condition which is *not* satisfied by X_2.

We can obtain an alternative solution by eliminating X from the surface-integrals, and then

$$X_P = -\frac{1}{2\pi}\int \frac{e^{-qr}}{r}\frac{\partial X}{\partial z_1} dS = \frac{q}{2\pi}\int \frac{e^{-qr}}{r}(c\beta)\, dS.$$

This leads at once to the formula $\int X_3 dS$, with

$$X_3 = \frac{q}{2\pi r}\, e^{-qr} f(ct) = \frac{1}{2\pi r} f'(ct-r), \tag{8'}$$

which is, intrinsically, the simplest of all the secondary wave formulae.

The formulae (8) and (8') are essentially due to Lord Rayleigh (*Theory of sound*, § 278).

We could also use $X_4 = \frac{1}{2}(X_2 + X_3)$, which has the advantage that it represents waves travelling *forwards* leaving zero field behind the wave-front.

It will be seen that, when the angle θ is small, so that the radii are nearly normal to the wave-front, the formulae (8), (8') reduce to (7'''), and so these simpler methods give formulae which are quite adequate to obtain all that is wanted for ordinary optical calculations.

Before leaving these results it may be useful to note that, with a slight modification of the formulae, the waves need not be restricted to be *plane*.

In fact, it follows from Heaviside's investigations* that (in any pulse) every discontinuity in the electric force E is accompanied by a corresponding discontinuity in the magnetic force H; these discontinuities are at right angles to each other in the wave-front and are such that $E-cH$ is continuous in numerical value. Consider, then, any point Q in the pulse (supposed to be bounded by simple curved surfaces), and take for the moment the axis of x parallel to the discontinuity in E; thus the axis of y may be parallel to the discontinuity in H (the two discontinuities being perpendicular). Then the axis of z is normal to the wave-front at Q, and its positive direction is that of the advancing wave.

Consequently the formulae found at the beginning of the present section (§ 3) will apply without change, so that (at the point Q)

$$\int X_0 d\theta = c(\beta_2-\beta_1), \quad \int c\beta_0 d\theta = X_2-X_1,$$

and there are no other discontinuities.

To obtain corresponding formulae when the axes are not chosen in special directions is easy; in fact, we shall have

$$\int X_0 d\theta = c[\nu\beta-\mu\gamma], \quad \int Y_0 d\theta = c[\lambda\gamma-\nu\alpha], \quad \int Z_0 d\theta = c[\mu\alpha-\lambda\beta],$$

and

$$-\int c\alpha_0 d\theta = [\nu Z-\mu Y], \quad -\int c\beta_0 d\theta = [\lambda Z-\nu X], \quad -\int c\alpha_0 d\theta = [\mu X-\lambda Y],$$

where λ, μ, ν are the direction-cosines of the normal, and brackets [ ] are used to denote the discontinuities in the enclosed forces.

This is almost obvious; actually the two vectors on the right-hand sides are in the form of vector-products, and will therefore be of the same form, whatever axes are chosen; for the special set of axes used above, we have $\lambda = 0$, $\mu = 0$, $\nu = 1$, and the equations reduce to those already found.

Thus the six potential-integrals are of the types given by

$$P_1 = c\int \frac{e^{-qr}}{r}[\nu\beta'-\mu\gamma']\,dS, \qquad M_1 = -\int \frac{e^{-qr}}{r}[\nu Y'-\mu Z']\,dS.$$

After these values are substituted in the formulae for X given in § 2, equation (3), and after the differentiations are carried out, the

* See, for example, *Electrical papers*, 2, 405; or *Electromagnetic theory*, 3, *passim*. Compare also Love, *Proc. London Math. Soc.* (2), 1 (1903), 56; Larmor, *ibid.*, 9–11.

results are found to agree with those given by Prof. Love, who used an entirely different process*; it is not, therefore, necessary to write these results out at length. But it may be useful to write down the approximation (neglecting $1/r^2$, $1/r^3$) which corresponds to (7'''). This is

$$(7^{\text{iv}}) \quad 4\pi X_1 = \frac{qe^{-qr}}{r}\left[c\{(m^2+n^2)(\nu\beta-\mu\gamma)-lm(\nu\gamma-\mu\alpha)-ln(\mu\alpha-\lambda\beta)\}\right.$$
$$\left.+\{m(\mu X-\lambda Y)-n(\lambda Z-\nu X)\}\right],$$

where l, m, n are the direction-cosines of the radius r.

We can write (7iv) rather more compactly, by using the property that $E-cH$ is continuous, so that six relations hold, two of which are

$$[X] = c[\nu\beta-\mu\gamma], \quad c[\alpha] = -[\nu Y-\mu Z].$$

Then the formula takes the simpler shape

$$(7^{\text{v}}) \qquad 4\pi X_1 = \frac{qe^{-qr}}{r}\{[X]-l[R]+cn[\beta]-cm[\gamma]\},$$

where
$$[R] = [lX+mY+nZ].$$

4. *The corresponding problem for an elastic solid.*

The fundamental equations of motion are three of the type

$$(9) \qquad p^2\xi = (a^2-b^2)\frac{\partial\delta}{\partial x}+b^2\Delta_2\xi,$$

where
$$\delta = \frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}+\frac{\partial\zeta}{\partial z}, \quad p = \frac{\partial}{\partial t}.$$

To introduce sufficient surface discontinuities, it will be easier to subdivide (9) into two groups of three, by introducing $(\omega_1, \omega_2, \omega_3)$, which will be proportional to the time-integrals of the ordinary rotations

$$\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}, \quad \frac{\partial\xi}{\partial z}-\frac{\partial\zeta}{\partial x}, \quad \frac{\partial\eta}{\partial x}-\frac{\partial\xi}{\partial y}.$$

Thus we replace (9) by the two groups of equations

$$(10) \qquad \begin{cases} p\xi = X_0+\dfrac{a^2}{p}\dfrac{\partial\delta}{dx}+b\left(\dfrac{\partial\omega_3}{\partial y}-\dfrac{\partial\omega_2}{\partial z}\right), \\ p\eta = \quad \dfrac{a^2}{p}\dfrac{\partial\delta}{\partial y}+b\left(\dfrac{\partial\omega_1}{\partial z}-\dfrac{\partial\omega_3}{\partial x}\right), \\ p\zeta = \quad \dfrac{a^2}{p}\dfrac{\partial\delta}{\partial z}+b\left(\dfrac{\partial\omega_2}{\partial x}-\dfrac{\partial\omega_1}{\partial y}\right), \end{cases}$$

* A. E. H. Love, *Phil. Trans. Royal Soc.* (A), 197 (1901), 18.

and

$$(11)\qquad \begin{cases} -p\omega_1 = & b\left(\dfrac{\partial\xi}{\partial y}-\dfrac{\partial\eta}{\partial z}\right), \\ -p\omega_2 = -\Omega_0+ & b\left(\dfrac{\partial\xi}{\partial z}-\dfrac{\partial\zeta}{\partial x}\right), \\ -p\omega_3 = & b\left(\dfrac{\partial\eta}{\partial x}-\dfrac{\partial\xi}{\partial y}\right), \end{cases}$$

where, for brevity, we have retained only two discontinuities, X_0, Ω_0.

By differentiating (10) with respect to x, y, z, and adding, we obtain

$$p^2\delta = p\frac{\partial X_0}{\partial x}+a^2\Delta_2\delta.$$

Thus, proceeding as in § 2, we find that

$$(12)\qquad \delta = \frac{p}{4\pi a^2}\frac{\partial P}{\partial x}, \quad \text{where} \quad P=\int\frac{e^{-pr/a}}{r}X_0'\,d\tau_1.$$

It is natural to expect that the solution will contain the terms

$$(13)\qquad \xi_1 = \frac{a^2}{p^2}\frac{\partial\delta}{\partial x} = \frac{1}{4\pi p}\frac{\partial^2 P}{\partial x^2}, \quad \eta_1 = \frac{a^2}{p^2}\frac{\partial\delta}{\partial y} = \frac{1}{4\pi p}\frac{\partial^2 P}{\partial x\partial y},$$

$$\zeta_1 = \frac{a^2}{p^2}\frac{\partial\delta}{\partial z} = \frac{1}{4\pi p}\frac{\partial^2 P}{\partial x\partial z}.$$

Now

$$(14)\qquad \begin{cases} \Delta_2\xi_1 = \dfrac{1}{p^2}\dfrac{\partial}{\partial x}(a^2\Delta_2\delta) = \dfrac{\partial\delta}{\partial x}-\dfrac{1}{p}\dfrac{\partial^2 X_0}{\partial x^2}, \\ \Delta_2\eta_1 = \dfrac{\partial\delta}{\partial y}-\dfrac{1}{p}\dfrac{\partial^2 X_0}{\partial x\partial y}, \\ \Delta_2\zeta_1 = \dfrac{\partial\delta}{\partial z}-\dfrac{1}{p}\dfrac{\partial^2 X_0}{\partial x\partial z}. \end{cases}$$

But, eliminating ω_1, ω_2, ω_3 from (10) by means of (11), we see that

$$p^2\xi = pX_0-b\frac{\partial\Omega_0}{\partial z}+a^2\frac{\partial\delta}{\partial x}-b^2\frac{\partial\delta}{\partial x}+b^2\Delta_2\xi.$$

Also, from (13) and (14), we have

$$a^2\frac{\partial\delta}{\partial x} = p^2\xi_1, \quad b^2\frac{\partial\delta}{\partial x} = b^2\frac{\partial^2 X_0}{\partial x^2}+b^2\Delta_2\xi_1;$$

thus, eliminating $\partial\delta/\partial x$, we obtain the result

$$(15)\qquad \begin{cases} p^2(\xi-\xi_1) = \dfrac{b^2}{p}\left(\dfrac{p^2}{b^2}X_0 - \dfrac{\partial^2 X_0}{\partial x^2}\right) - b\dfrac{\partial \Omega_0}{\partial z} + b^2\Delta_2(\xi-\xi_1), \\ p^2(\eta-\eta_1) = -\dfrac{b^2}{p}\dfrac{\partial^2 X_0}{\partial x\partial y} + b^2\Delta_2(\eta-\eta_1), \\ p^2(\zeta-\zeta_1) = -\dfrac{b^2}{p}\dfrac{\partial^2 X_0}{\partial x\partial z} + b\dfrac{\partial \Omega_0}{\partial x} + b^2\Delta_2(\zeta-\zeta_1). \end{cases}$$

The solution of (15) is given (as in §§ 2, 3) by

$$(16)\qquad \begin{cases} \xi-\xi_1 = \dfrac{1}{4\pi p}\left(\dfrac{p^2}{b^2}Q - \dfrac{\partial^2 Q}{\partial x^2}\right) - \dfrac{1}{4\pi b}\dfrac{\partial R}{\partial z}, \\ \eta-\eta_1 = -\dfrac{1}{4\pi p}\dfrac{\partial^2 Q}{\partial x\partial y}, \\ \zeta-\zeta_1 = -\dfrac{1}{4\pi p}\dfrac{\partial^2 Q}{\partial x\partial z} + \dfrac{1}{4\pi b}\dfrac{\partial R}{\partial x}, \end{cases}$$

where $$Q = \int \frac{e^{-pr/b}}{r} X_0' d\tau_1, \quad R = \int \frac{e^{-pr/b}}{r}\Omega_0' d\tau_1.$$

From (16) it is easy to obtain the solution of the problem proposed. If we assume that the dilatational velocity a is very large, $e^{-pr/a} \to 1$, so that, from (12), we see that the function P becomes a Newtonian potential; hence the displacements (ξ_1, η_1, ζ_1) are seen to be of order $1/r^3$, which are to be neglected in accordance with our work in § 3.

The remaining terms in (16) are exactly the same as those obtained in (6) of § 3: the only point outstanding is the expression of Ω_0, X_0 in terms of the prescribed displacement in the original wave. We proceed exactly as in § 3, assuming the original wave to be given by

$$\xi = f(bt-z), \quad \eta = 0, \quad \zeta = 0,$$

and then $$\omega_1 = 0, \quad \omega_2 = f(bt-z), \quad \omega_3 = 0.$$

If we integrate (10) and (11) through the thin pulse just behind $z = 0$, we obtain

$$\int X_0 d\theta - b[\omega_2] = 0, \quad -\int \Omega_0 d\theta + b[\xi] = 0,$$

where the brackets [ ] represent discontinuities, as before. Thus

$$(17)\qquad \int X_0 d\theta = bf(bt) = \int \Omega_0 d\theta.$$

Hence the formulae

$$(18)\qquad Q = b\int \frac{e^{-pr/b}}{r} f(bt)\, dS = R$$

will give the values of the potentials in (16). It will be seen that (16) then becomes identical in form with (6) (remembering that q corresponds to p/b), and it is therefore superfluous to repeat the development of the formulae into their final shapes.

The point to be noticed is that the same discrepancies persist in regard to Stokes's formulae, so that their explanation does not lie in the fact that his equations are derived from the elastic-solid theory and not from the electromagnetic equations. The discrepancy* would therefore seem to be due to a fundamental difference in the mode of taking account of the effect of a pulse.

As a further comparison with Stokes's work, let us determine the displacements due to a force applied to a limited region; this can be obtained quickly from our previous analysis. Suppose that the force is parallel to z, and to be of amount F per unit mass, then, if ρ is the density, the third equation of (9) takes the modified form

$$p^2\zeta = \frac{F}{\rho} + (a^2-b^2)\frac{\partial\delta}{\partial z} + b^2\Delta_2\zeta. \tag{9'}$$

Then, as in (12), we obtain (after omitting X_0, Ω_0, since now we want to trace only the effect of F)

$$\delta = \frac{1}{4\pi a^2}\frac{\partial S_1}{\partial z}, \quad \text{where} \quad S_1 = \int \frac{e^{-pr/a}}{r}\frac{F'}{\rho}\,d\tau_1. \tag{12'}$$

Thus we take as the particular integrals

$$\begin{cases} \xi_1 = \dfrac{a^2}{p^2}\dfrac{\partial\delta}{\partial x} = \dfrac{1}{4\pi p^2}\dfrac{\partial^2 S_1}{\partial x\,\partial z}, \\ \eta_1 = \dfrac{a^2}{p^2}\dfrac{\partial\delta}{\partial y} = \dfrac{1}{4\pi p^2}\dfrac{\partial^2 S_1}{\partial y\,\partial z}, \\ \zeta_1 = \dfrac{a^2}{p^2}\dfrac{\partial\delta}{\partial z} = \dfrac{1}{4\pi p^2}\dfrac{\partial^2 S_1}{\partial z^2}, \end{cases} \tag{13'}$$

and, by the same method as in (15), we obtain

$$\begin{cases} p^2(\xi-\xi_1) = -\dfrac{b^2}{\rho p^2}\dfrac{\partial^2 F}{\partial x\,\partial z} + b^2\Delta_2(\xi-\xi_1), \\ p^2(\eta-\eta_1) = -\dfrac{b^2}{\rho p^2}\dfrac{\partial^2 F}{\partial y\,\partial z} + b^2\Delta_2(\eta-\eta_1), \\ p^2(\zeta-\zeta_1) = -\dfrac{b^2}{\rho p^2}\dfrac{\partial^2 F}{\partial z^2} + b^2\Delta_2(\zeta-\zeta_1). \end{cases} \tag{15'}$$

* In effect, Stokes's formulae would contain terms in the second column of (16), which are of the same type as the first column after another differentiation with respect to z.

Hence the complete solution is given by

$$(19)\qquad \begin{cases} \xi-\xi_1 = & -\dfrac{1}{4\pi p^2}\dfrac{\partial^2 S_2}{\partial x\,\partial z}, \\ \eta-\eta_1 = & -\dfrac{1}{4\pi p^2}\dfrac{\partial^2 S_2}{\partial y\,\partial z}, \\ \zeta-\zeta_1 = \dfrac{S_2}{4\pi b^2} & -\dfrac{1}{4\pi p^2}\dfrac{\partial^2 S_2}{\partial z^2}, \end{cases}$$

where

$$S_2 = \int \frac{e^{-pr/b}}{r}\,\frac{F'}{\rho}\,d\tau_1.$$

These formulae can be more conveniently written as

$$(19')\qquad \begin{cases} \xi = \dfrac{1}{4\pi p^2}\dfrac{\partial^2}{\partial x\,\partial z}(S_1-S_2), \\ \eta = \dfrac{1}{4\pi p^2}\dfrac{\partial^2}{\partial y\,\partial z}(S_1-S_2), \\ \zeta = \dfrac{1}{4\pi p^2}\dfrac{\partial^2}{\partial z^2}(S_1-S_2)+\dfrac{S_2}{4\pi b^2}, \end{cases}$$

and (19′) is effectively the same as the solution due to Heaviside [*Electromagnetic theory*, 3, 389]. It is easy now to identify (19′) with Stokes's solution {*l.c.* 279; equations (39) and (40)}.

We begin by reducing the region T to a point, with

$$(20)\qquad \int F'\,d\tau_1 = f(t),$$

so that

$$S_1 = \frac{e^{-pr/a}}{r\rho}f(t) = \frac{1}{r\rho}f(t-r/a), \qquad S_2 = \frac{e^{-pr/b}}{r\rho}f(t) = \frac{1}{r\rho}f(t-r/b).$$

Let us write now

$$(21)\qquad \phi = \frac{1}{4\pi p^2}\frac{\partial}{\partial z}(S_1-S_2) = \frac{1}{4\pi\rho p^2}\frac{\partial}{\partial z}\left\{\frac{e^{-pr/a}}{r}-\frac{e^{-pr/b}}{r}\right\}f(t)$$

or

$$\phi = -\frac{\cos\theta}{4\pi\rho p\,r^2}\left\{\left(1+\frac{pr}{a}\right)e^{-pr/a}-\left(1+\frac{pr}{b}\right)e^{-pr/b}\right\}f(t).$$

Then the formulae (19′) become, in spherical polar coordinates,

$$(22)\qquad \begin{cases} \xi_r = & \dfrac{\partial\phi}{\partial r}+\dfrac{S_2\cos\theta}{4\pi b^2}, \\ \xi_\theta = \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} & -\dfrac{S_2\sin\theta}{4\pi b^2}. \end{cases}$$

Stokes's formulae are expressed in terms of the integral

$$\chi = \int_{r/a}^{r/b} t'f(t-t')\,dt',$$

which can be transformed into a symbolic form without difficulty. For we have

$$f(t-t') = e^{-t'\partial/\partial t}f(t),$$

and so

$$(23)\qquad \chi = \int_{r/a}^{r/b} t'e^{-pt'}\,dt'\,f(t) = \frac{1}{p^2}\left\{\left(1+\frac{pr}{a}\right)e^{-pr/a}-\left(1+\frac{pr}{b}\right)e^{-pr/b}\right\}f(t).$$

By comparison of (23) and (21), we see that

$$\phi = -\frac{\cos\theta}{4\pi\rho r^2}\chi.$$

In calculating ξ_r we shall need the value of $\partial\chi/\partial r$, which is found to be

$$\frac{\partial\chi}{\partial r} = -r\left(\frac{1}{a^2}e^{-pr/a} - \frac{1}{b^2}e^{-pr/b}\right)f(t) = -\frac{r}{a^2}f\left(t-\frac{r}{a}\right) + \frac{r}{b^2}f\left(t-\frac{r}{b}\right);$$

a result which can be verified by the direct differentiation of Stokes's integral for χ.

Thus (22) gives now

$$\xi_r = \frac{2\cos\theta}{4\pi\rho r^3}\chi - \frac{\cos\theta}{4\pi\rho r^2}\frac{\partial\chi}{\partial r} + \frac{\cos\theta}{4\pi\rho b^2 r}f\left(t-\frac{r}{b}\right),$$

$$\xi_\theta = \frac{\sin\theta}{4\pi\rho r^3}\chi \qquad\qquad - \frac{\sin\theta}{4\pi\rho b^2 r}f\left(t-\frac{r}{b}\right).$$

That is, after substituting for $\partial\chi/\partial r$, we have

$$(24)\qquad \begin{cases} \xi_r = \dfrac{\cos\theta}{4\pi\rho a^2 r}f\left(t-\dfrac{r}{a}\right) + \dfrac{2\cos\theta}{4\pi\rho r^3}\chi, \\ \xi_\theta = -\dfrac{\sin\theta}{4\pi\rho b^2 r}f\left(t-\dfrac{r}{b}\right) + \dfrac{\sin\theta}{4\pi\rho r^3}\chi. \end{cases}$$

The formulae (24) agree with those given by Stokes [*l.c.* 278 (40)].

To show the actual working out of these formulae, we may consider an example given by Prof. Love*, in which the force rises uniformly to a maximum, and is afterwards maintained at a steady value.

Suppose the maximum to be Z, and let it be reached at the end of an interval τ, so that†

$$f(t) = \frac{Zt}{\tau} \quad \text{or } Z, \quad \text{according as } t < \tau \text{ or } t > \tau.$$

Thus, here the symbolic formula is $f(t) = Z(1-e^{-p\tau})/p\tau$.

Then, to evaluate ϕ, the first term only is needed in (23) until the time $t = r/b$; and so we have at first

$$\frac{1}{p^2}\left(1+\frac{p\tau}{a}\right)f(t) = \frac{Z}{\tau}\left(\tfrac{1}{6}t^3 + \tfrac{1}{2}t^2\frac{r}{a}\right) = F(t) \quad (0 < t < \tau),$$

or

$$= F(t) - F(t-\tau), \qquad \text{if } t > \tau.$$

Applying the operation $e^{-pr/a}$ simply converts t into $t-r/a$, and thus we see that

$$\phi = -\frac{Z\cos\theta}{4\pi\rho\tau r^2}\left\{\tfrac{1}{6}\left(t-\frac{r}{a}\right)^2\left(t+\frac{2r}{a}\right)\right\}, \qquad \text{if } r < at < r + a\tau,$$

$$= -\frac{Z\cos\theta}{4\pi\rho\tau r^2}\left(\tfrac{1}{6}t^3 - \frac{t}{2}\frac{r^2}{a^2} + \tfrac{1}{3}\frac{r^3}{a^3}\right).$$

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* *Proc. London Math. Soc.* (2), 1 (1903), 300: if the reader is interested in following the course of such waves he may be advised to examine the graphs given on pp. 303, 304.

† In more complicated examples, when it is best to convert $f(t)$ to a symbolic form, we may use the formula $\int pe^{-p\theta}f(\theta)\,d\theta$, found in *Proc. London Math. Soc.* (2), 15 (1916), 412; see also the footnote on p. 441 above.

The corresponding displacement is irrotational and the dilatation is given by

$$\Delta_2\phi = \frac{p^2}{a^2}\phi = -\frac{Z\cos\theta}{4\pi\rho a^2 r^2}\frac{t}{\tau}.$$

These two results correspond to the formulae (20) and (21) given by Love.*

At the next stage, by evaluating $F(t-r/a)-F(t-r/a-\tau)$, we have the result

$$\phi = -\frac{Z\cos\theta}{4\pi\rho r^2}\left\{\tfrac{1}{2}(t^2-t\tau-r^2/a^2)+\tfrac{1}{6}\tau^2\right\}, \quad \text{if } at > r+a\tau,$$

and

$$\Delta_2\phi = -\frac{Z\cos\theta}{4\pi\rho a^2 r^2},$$

which agree with (23) and (24) of Love's work.

When t reaches the value r/b we must add to ϕ similar terms containing $(t-r/b)$, and we obtain, after reduction,

$$\phi = \frac{Z\cos\theta}{4\pi\rho\tau r^2}\left\{\tfrac{1}{6}(t-\tau)^3+\frac{\tau}{2}\frac{r^2}{a^2}-\frac{t}{2}\frac{r^2}{b^2}+\frac{1}{3}\frac{r^3}{b^3}\right\}, \quad \text{if } r < bt < r+b\tau,$$

which agrees with Love's formula (27), except in the last term, where the numerical factor is given as $-\frac{2}{3}$ (allowing for the change of sign just mentioned).

We need now the function

$$S_2 = \frac{1}{r\rho}\,e^{-pr/b}f(t) = \frac{Z}{r\rho\tau}(t-r/b).$$

In this way we complete Love's formula (26), the term $-Z/b\rho\tau$ in S_2 being just sufficient to balance the change in ϕ mentioned above; and the final formula agrees completely with Love's.

The dilatation is now

$$\Delta_2\phi+\frac{1}{4\pi b^2}\frac{\partial S_2}{\partial z} = \frac{1}{4\pi p^2}\frac{\partial}{\partial z}\left(\frac{p^2}{a^2}S_1-\frac{p^2}{b^2}S_2\right)+\frac{1}{4\pi b^2}\frac{\partial S_2}{\partial z} = \frac{1}{4\pi a^2}\frac{\partial S_1}{\partial z},$$

which is the same as in the previous interval; and the doubled rotations are given by

$$\frac{1}{4\pi b^2}\frac{\partial S_2}{\partial y}, \quad -\frac{1}{4\pi b^2}\frac{\partial S_2}{\partial x}, \quad 0,$$

and the resultant of these is found to be proportional to $Zt\sin\theta/(4\pi\rho b^2\tau r^2)$.

Finally, when $bt > r+b\tau$, the steady state is attained; the formulae are worked out most easily† by expanding in powers of p, which gives

$$\begin{aligned}\phi &= -\frac{\cos\theta}{4\pi\rho p^2 r^2}\left\{\left(1-\tfrac{1}{2}\frac{p^2r^2}{a^2}\right)-\left(1-\tfrac{1}{2}\frac{p^2r^2}{b^2}\right)+O(p^3)\right\}f(t)\\ &= -\frac{\cos\theta}{8\pi\rho}\left\{\left(\frac{1}{b^2}-\frac{1}{a^2}\right)+\text{positive powers of } p\right\}f(t)\\ &= -\frac{\cos\theta}{8\pi\rho}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)Z.\end{aligned}$$

And S_2 reduces to $Z/r\rho$, from which Love's formulae (29) and (30) follow at once.

This example may serve to bring out the fact that the symbolic formulae are quite easy to interpret, and lead to results which can be verified by the older methods. That they are more compact than the older forms is at once evident; but there may be a feeling that there is

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* There is a difference in the sign of ϕ which is inherent in the two definitions of ϕ.

† We could obtain the same result by subtracting the terms in $(t-r/b-\tau)$; but the above calculation is shorter.

a risk of error in their use. On the contrary, in actual work it usually turns out that the symbolic forms lead to fewer slips than other methods; no doubt this is simply due to the fact that greater clearness in work follows from any abbreviation of detail.

The results for a similar problem, that of *a steady force acting through the volume of a sphere of finite size*, have been worked out in a similar way by Heaviside*; it will be seen that it is necessary to consider several more stages in the motion; the chief cause of difference is the existence of two pulses of thickness equal to the diameter of the sphere (in each of which the disturbance varies rapidly†).

It may be worth while to point out that from (19′) or (24) we can pass at once to Lord Rayleigh's solution of the problem of secondary waves (*Scientific papers*, 3, 169).

In this solution the distribution of force is assumed to be in a fixed direction in the wave-front. Thus, to obtain a comparison with the solutions found in § 3 above, we take the force now to be parallel to x, and the plane $z = 0$ as the wave-front. That is, we interchange x and z in (19′); and we reject the terms in S_1, for reasons given before. Then (19′) becomes

$$(19'') \qquad \xi = \frac{1}{4\pi p^2}\left(\frac{p^2 S_2}{b^2} - \frac{\partial^2 S_2}{\partial x^2}\right), \quad \eta = -\frac{1}{4\pi p^2}\frac{\partial^2 S_2}{\partial x\,\partial y}, \quad \zeta = -\frac{1}{4\pi p^2}\frac{\partial^2 S_2}{\partial x\,\partial z},$$

and it will be seen that these are of the same type as the terms in the first column of (16).

By choosing a suitable value for F, we can deduce a formula for S_2 which will account for any assigned plane wave; but the formulae so found suffer from the drawback that the secondary waves must be symmetrical about the line of action chosen for the force F. Thus the secondary waves in this solution must travel *backwards*, as well as *forwards*, from the wave-front; in the same way‡ as (8′) of § 3.

Using (24), we see that the leading part of the disturbance is perpendicular to the radius, and varies as $\sin\theta'/r$, where θ' is the angle between x and r: θ' is the same as the angle called ϕ by Stokes and Rayleigh, or $\theta' = Px$ in the diagram of § 3.

For comparison with (7) we may note that in Rayleigh's solution the spherical polar components are

$$\Theta_R = \frac{\cos\theta\cos\phi}{2\pi r} f'(bt-r), \qquad \Phi_R = -\frac{\sin\phi}{2\pi r} f'(bt-r),$$

where ϕ denotes the same angle as on p. 449 in § 3.

Since the formulae (24) are in complete agreement with those given by Stokes for the same problem§, it would seem that the discrepancy between Stokes's formulae for secondary waves and those found above in (16) and in (7) of § 3, can arise only from some difference in taking account of the effect of a thin pulse. It appears that the part of Stokes's solution which is obtained from the *velocities* in the primary wave

* *Electromagnetic theory*, 3, 388–403.

† Compare the simpler problems carried out for sound-waves in § 5 below.

‡ The waves given by (8) also travel backwards, but they have the sign reversed behind the wave-front.

§ It will be proved in § 5 directly that Poisson's solution (in the form of an integral) may be derived from the symbolic method.

(*l.c.* 284) is exactly the same as the result found in the first column of (16) above, and thus the root of the discrepancy must lie in Stokes's calculation of the contributions due to the *displacements* in the primary wave. This calculation is made by differentiating* the corresponding contribution from the velocities (with respect to t); although the details are rather more elaborate (*l.c.* 268 and 285), yet the result amounts to applying the operator $-b(\partial/\partial z)$ (in our notation) to the first part of the solution, at the same time using the given values of the *displacements* in place of the *velocities* in the primary wave.

This process can be verified at once, when (as in a sound-wave) the motion can be expressed by means of *one* potential-function ψ, satisfying the wave-equation. Then, as in (8) and (8′) of §3 above, we can write

$$4\pi\psi = \int\left\{\frac{\partial}{\partial z_1}\left(\frac{e^{-qr}}{r}\right)\psi_1 - \frac{e^{-qr}}{r}\left(\frac{\partial\psi_1}{\partial z_1}\right)\right\}dS \quad (q = p/c).$$

Suppose that ψ is a displacement-potential, and let the primary wave be given by

$$\psi = f(ct-z),$$

then
$$\psi_1 = f(ct), \quad \frac{\partial\psi_1}{\partial z_1} = -f'(ct).$$

Thus the part of the contribution from dS, due to the original *velocities*, is given by

$$-\frac{dS}{4\pi}\frac{e^{-qr}}{r}\frac{\partial\psi_1}{\partial z_1} = +\frac{dS}{4\pi r}e^{-qr}f'(ct) = \frac{dS}{4\pi r}f'(ct-r).$$

The part corresponding to the *displacements* is

$$\frac{dS}{4\pi}\frac{\partial}{\partial z_1}\left(\frac{e^{-qr}}{r}\right)\psi_1 = -\frac{dS}{4\pi}\frac{\partial}{\partial z}\left(\frac{e^{-qr}}{r}\right)f(ct) = -\frac{dS}{4\pi}\frac{\partial}{\partial z}\left\{\frac{1}{r}f(ct-r)\right\}.$$

This second part can be derived from the first by using the operator $-c(\partial/\partial z)$, at the same time replacing $f'(ct)$ by its time-integral $f(ct)/c$.

Difficulties in regard to Stokes's method arise as soon as we try to understand what is involved in the assumption that the disturbance transmitted from a particular point Q is due merely to the displacement and velocity of the point Q itself. Thus, if we select a small volume T round Q, and assign an initial disturbance to the points inside T, the result will be a thin *pulse*, of thickness equal to the greatest diameter of T, and in the pulse the disturbance will vary rapidly as the wave passes over any chosen point of observation.

Some examples are worked out in detail at the end of § 5, which will indicate the kind of variation arising in simple cases. These examples

* Compare (26) and (27) in § 5 below: the hypotheses made there imply naturally that the functions ϕ_0, ϕ_1 are independent of t. But, as remarked elsewhere, Stokes applies a similar group of solutions to construct the secondary waves from the displacement in the primary wave: and the latter certainly depends on t.

refer to *sound-waves*; they suffice to show the general character of the waves sent out, without introducing the complications of an electromagnetic field of force (or of the displacements of an elastic solid). That these examples correspond to similar problems of electromagnetic waves is fairly evident; a reference is given in § 5 to a problem worked out by Heaviside (arising from a spherical electron jerked into slow motion), and it will be found that in one of our examples the function ϕ is almost identical with the magnetic force of that electrical problem. Thus the information to be gained from such calculations is enough to indicate the general nature of the pulses arising from similar disturbances set up in an electromagnetic field (or in an elastic solid).

5. *Comparison of the symbolic method with Poisson's integral solution.*

On account of their very different appearance, it may not be superfluous to establish the identity between these two methods of solution. As is well known, the Poisson integral gives a solution of the equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \Delta_2 \phi \tag{25}$$

in the form

$$\phi_P = \frac{t}{4\pi} \int \phi_1 \, d\omega + \frac{\partial}{\partial t} \left\{ \frac{t}{4\pi} \int \phi_0 \, d\omega \right\}, \tag{26}$$

where ϕ_0, ϕ_1 are the initial values of ϕ, $(\partial\phi/\partial t)$, and the integrals are taken on a sphere of radius ct, with P as centre, the integration being with respect to the solid angle at P as origin.

To treat the same problem symbolically, we replace (25) by the form*

$$p^2 \phi = p^2 \phi_0 + p\phi_1 + c^2 \Delta_2 \phi. \tag{25'}$$

Then the solution is expressed by the integral

$$\phi = \frac{1}{4\pi c^2} \int \frac{e^{-pr/c}}{r} (p^2 \phi_0' + p\phi_1') \, d\tau_1 \tag{27}$$

extended throughout the space T (to which the initial disturbance ϕ_0, ϕ_1 is confined).

A comparison of (26) and (27) shows at once that we need only examine the effect of the initial value of ϕ_1; for (as remarked by Stokes),

* *Proc. London Math. Soc.* (2), 15 (1916), 401, see §§ 1, 5. Note that an extra factor p is introduced in $(p^2\phi_0 + p\phi_1)$ since here we have supposed that the standard integrals of § 1 contain a factor λ in the denominator; this agrees with the results of Heaviside's method and simplifies the treatment of applied steady forces.

the effect of ϕ_0 can be calculated at once by a simple differentiation of the former integral with respect to t. From the integral (27) this is obvious, since we can take the factor p outside the integral, writing the equation as

$$\int p^2 \frac{e^{-pr/c}}{r} \phi_0' . d\tau_1 = \frac{\partial}{\partial t} \left(\int p \frac{e^{-pr/c}}{r} \phi_0' d\tau_1 \right).$$

To transform the ϕ_1-part of (27) into an integral of the type (26) we use r and ω as variables of integration, where ω denotes the solid angle subtended at the point P; and then

$$(27') \qquad \frac{1}{4\pi c^2} \int p\phi_1' \frac{e^{-pr/c}}{r} d\tau_1 = \frac{1}{4\pi c^2} \int d\omega \int_{r_1}^{r_2} pr\phi_1' e^{-pr/c} dr,$$

where r_1, r_2 are the values of r, corresponding to the two points in which the radius cuts the boundary of T. If we compare this form (27′) with (26), we are led to conjecture the symbolic formula

$$ct\phi_1(ct) = \int_{r_1}^{r_2} \frac{p}{c} r\phi_1(r) e^{-pr/c} dr, \quad \text{if } r_1 < ct < r_2,$$

or, more simply,

$$(28) \qquad \psi(ct) = \int_{r_1}^{r_2} \frac{p}{c} \psi(r) e^{-pr/c} dr, \quad \text{if } r_1 < ct < r_2;$$

the limitations of t being introduced to ensure that the sphere cuts T.

The simplest proof of (28) is to observe that

$$e^{-pr/c} \text{ means } 1, \text{ if } ct > r; \text{ or } 0, \text{ if } ct < r,$$

and consequently

$$\int_{r_1}^{r_2} e^{-pr/c} \psi(r) dr = \int_{r_1}^{ct} \psi(r) dr, \quad \text{if } r_1 < ct < r_2.$$

Then, differentiating with respect to t, we see that

$$\int_{r_1}^{r_2} pe^{-pr/c} \psi(r) dr = c\psi(ct), \quad \text{if } r_1 < ct < r_2,$$

which is equivalent to (28).

Another instructive way of dealing with (28) is to examine first the simple case in which $\psi(r)$ is simply a polynomial in r; the general idea of the proof will be sufficiently clear if we limit $\psi(r)$ to be a quadratic, so that $\psi(r) = A + Br + Cr^2$.

Then, on carrying out the integration, we have

$$\int_{r_1}^{r_2} \frac{p}{c} (A + Br + Cr^2) e^{-pr/c} dr = \left[-e^{-pr/c} \left\{ (A + Br + Cr^2) + \frac{c}{p} (B + 2Cr) + \frac{c^2}{p^2} (2C) \right\} \right]_{r_1}^{r_2}.$$

Now, since $r_2 > ct$, all terms from the upper limit will be zero; as regards the lower limit we have

$$e^{-pr_1/c} = 1, \quad \frac{c}{p} e^{-pr_1/c} = ct - r_1, \quad \frac{c^2}{p^2} e^{-pr_1/c} = \tfrac{1}{2} (ct - r_1)^2.$$

Thus, when interpreted, the integral in (28) becomes

$$(A+Br_1+Cr_1^2)+(ct-r_1)(B+2Cr_1)+\tfrac{1}{2}(ct-r_1)^2(2C)=A+B(ct)+C(ct)^2=\psi(ct).$$

It is easy to extend this method, assuming that $\psi(r)$ can be expressed as a power-series in $(r-r_1)$; the interpretation of (28) is found in the form

$$\psi(r_1)+(ct-r_1)\psi'(r_1)+\tfrac{1}{2}(ct-r_1)^2\psi''(r_1)+\ldots=\psi(ct).$$

An alternative methed, involving fewer restrictions on $\psi(r)$, can be constructed on the lines followed in pp. 441–443 of my paper just quoted. But in *this* calculation the path of integration remains parallel to the imaginary axis; and we need only the result

$$\int_{r_1}^{r_2}\frac{p}{c}e^{-pr/c}\,dr=e^{-pr_1/c}-e^{-pr_2/c}=1,\quad \text{if } r_1<ct<r_2.$$

It will be noticed that when r_2 is also less than ct, the contribution from the upper limit is $-\psi(ct)$; so that the total is *zero*. This agrees with the Poisson integral, since the sphere $r=ct$ now does not cut T, at least in the direction of the radius under consideration.

Having proved (28), we see that the two solutions (26) and (27) are equivalent in all respects. *As a simple example, we may evaluate* (27) *for the case when T is a sphere of radius a, with its centre at the origin*; taking $\phi_1=u_0$ (a constant) and $\phi_0=0$, the integral (27) becomes

$$(29)\qquad \phi=\frac{u_0}{cq^2}(qa\cosh qa-\sinh qa)\frac{e^{-qr}}{r},\quad \text{where } q=p/c, \text{ and } r>a,$$

r being measured from the centre. To evaluate (29), we need take only values of t such that $r-a<ct<r+a$, since the result is easily proved to be zero at other times; then (29) becomes

$$\phi=\frac{u_0}{2cr}\left\{\frac{a}{q}e^{-q(r-a)}-\frac{1}{q^2}e^{-q(r-a)}\right\}$$

or
$$\phi=\frac{u_0}{2cr}\{a(ct-r+a)-\tfrac{1}{2}(ct-r+a)^2\}=\frac{u_0}{4cr}\{a^2-(ct-r)^2\}.$$

The corresponding formula when ϕ_0 is constant and $\phi_1=0$ is

$$\phi=-\frac{\phi_0}{2r}(ct-r).$$

These forms are easily verified from Poisson's integral; the results will be found in most text-books on the theory of sound, and certainly *this* calculation is simpler by means of Poisson's integral than by the present method. But (as we shall see shortly) there are many other examples in which the reverse holds good. For the moment, however, it seems to be desirable to refer to a difficulty originally noticed by Prof. A. E. H. Love*; when ϕ is interpreted as a velocity-potential (the problem being taken as referring to *sound-waves*) the pressure is propor-

* *Proc. London Math. Soc.* (2), 1 (1902), 336.

tional to $\partial\phi/\partial t$ or $p\phi$, and this function is found to tend to infinity* near $r = 0$, just as the inward wave passes over the point considered.

Prof. Love shows that we can remove this difficulty provided that the initial values ϕ_0 and ϕ_1 (which is not to be confused with Prof. Love's initial velocity u_0) are connected by the equation

$$c\frac{\partial\phi_0}{\partial r} + \phi_1 = 0, \quad \text{at } r = a;$$

and we must have also $$\frac{\partial\phi_0}{\partial r} = 0, \quad \text{at } r = 0.$$

The simplest functions which will satisfy these two conditions are

$$\phi_0 = \frac{u_0}{2ca}(a^2 - r^2), \quad \phi_1 = \text{const.} = u_0,$$

choosing the additive constant in ϕ_0, so as to make $\phi_0 = 0$ at $r = a$.

To obtain the corresponding formula for ϕ, we begin by taking the initial value of $\partial\phi/\partial t$ as $A(a^2 - r^2)$; then using the potential formulae for a surface-distribution on the sphere $r = s$, namely,

$$\frac{\sinh qs}{qs}\frac{e^{-qr}}{r} \quad \text{or} \quad \frac{\sinh qr}{qr}\frac{e^{-qs}}{s}, \quad \text{according as } r > s \text{ or } r < s,$$

we obtain the formula $$r\phi = \int_0^a \frac{As}{2c}(a^2 - s^2)\left\{e^{-q|r-s|} - e^{-q(r+s)}\right\} ds.$$

This can be integrated just as it stands,† but it is probably easier to interpret the functions at once. We have, in fact,

$$e^{-q|r-s|} = 1, \text{ if } ct > |r-s|, \quad \text{and} \quad e^{-q(r+s)} = 1, \text{ if } ct > r+s,$$

and both functions are zero for other values of t. Thus the integrand is zero except when ct falls between $|r-s|$ and $(r+s)$; or (what is the same thing) except when

$$|r - ct| < s < r + ct.$$

That is, the lower limit for s is $|r-ct|$ and the upper limit is the smaller of a and $(r+ct)$.

Thus, on carrying out the integrations, we have

$$r\phi = \frac{A}{2c}\left\{\tfrac{1}{2}a^2(4rct) - \tfrac{1}{4}(4rct)(2r^2 + c^2t^2)\right\}, \qquad \text{if } r + ct < a,$$

or $$= \frac{A}{2c}\left[\tfrac{1}{2}a^2\left\{a^2 - (r-ct)^2\right\} - \tfrac{1}{4}\left\{a^4 - (r-ct)^4\right\}\right], \quad \text{if } r + ct > a.$$

The result is *zero* if $|r-ct| > a$.

These formulae reduce to $$\phi = A\left\{a^2t - t(r^2 + c^2t^2)\right\}, \quad \text{if } r + ct < a,$$

or $$= \frac{A}{8cr}\left\{a^2 - (r-ct)^2\right\}^2, \qquad \text{if } |r-a| < ct < r+a.$$

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* Such a result really means that the approximations, on which the fundamental equations depend, then break down.

† The results are $$\frac{6A}{cq^4}\left\{(1 + \tfrac{1}{3}q^2a^2)\sinh qa - qa\cosh qa\right\}e^{-qr} \quad (r > a),$$

or $$\frac{Ar}{cq}\left\{(a^2 - r^2) - \frac{6}{q^2}\right\} + \frac{6A}{cq^4}(1 + qa + \tfrac{1}{3}q^2a^2)\,e^{-qa}\sinh qr \quad (r < a).$$

To obtain the result for the actual problem, we now follow Stokes's rule and differentiate with respect to t, obtaining

$$\phi = A(a^2-r^2-3c^2t^2), \qquad \text{if } r+ct<a,$$

or

$$= \frac{A}{2r}\{a^2-(r-ct)^2\}(r-ct), \quad \text{if } |r-a|<ct<r+a,$$

and after the wave has passed (so that $ct>r+a$) ϕ is *zero*. When r is small and $ct>a-r$, the second formula gives

$$\frac{\partial\phi}{\partial t} = \frac{Ac}{2r}\{3(ct-r)^2-a^2\},$$

which (since ct is between $a+r$ and $a-r$) is of order Aca^2/r. From the analysis given for (29) above we see that the corresponding contribution to $\partial\phi/\partial t$, when $\phi_1 = u_0$, a constant, is equal to $-\frac{1}{2}u_0(ct-r)/r$, which is of order $-\frac{1}{2}au_0/r$. The two terms will therefore cancel if $A = \frac{1}{2}u_0/ac$, which is equivalent to Prof. Love's condition.

Thus the *simplest problem of spherical symmetry which is free from complications at the centre* will be given by

$$\phi_0 = \tfrac{1}{2}\frac{u_0}{ac}(a^2-r^2), \quad \left(\frac{\partial\phi}{\partial t}\right)_0 = u_0,$$

and then, during the passage of the wave*,

(30)
$$r\phi = \frac{u_0}{4ac}\left[\{(ct-r)^3-a^2(ct-r)\}+a\{a^2-(ct-r)^2\}\right]$$

$$= \frac{u_0}{4ac}(r+a-ct)^2(ct-r+a) \qquad (|r-a|<ct<r+a).$$

This value of ϕ remains of the same sign throughout the wave; its greatest value is $\frac{8}{27}u_0a^2/cr$, corresponding to $ct-r=-\frac{1}{3}a$. The value of $\partial\phi/\partial t$ has both signs, and its greatest value (at points for which $r>a$) is au_0/r, corresponding to $ct-r=-a$ (the wave-front).

We shall now examine some examples in which there is not a symmetrical solution, so that the Poisson-integral solution is troublesome to evaluate. If we differentiate (29) with respect to z, we obtain a new solution of the wave-equation in the form

(31)
$$\phi = \frac{A\cos\theta}{cq^2r^2}(qa\cosh qa-\sinh qa)(1+qr)\,e^{-qr}, \quad (r>a),$$

or

$$= \frac{A\cos\theta}{cq^2r^2}(qr\cosh qr-\sinh qr)(1+qa)\,e^{-qa}, \quad (r<a).$$

* As a verification of the symbolic methods, I have checked this result by calculating Prof. Love's general functions f, F for this problem; ϕ can also be obtained, although less easily, from Poisson's integral.

It may be conjectured that this solution will correspond to a surface-density proportional to $\cos\theta$ [since the formula (29) refers to a distribution which is constant inside the sphere $r = a$]; and this is confirmed by calculating, from (31), the discontinuity in $\partial\phi/\partial r$ at $r = a$, which is $qA\cos\theta/c$ [indicating, on reference to (27), that the surface-density which corresponds to ϕ_1 is equal to $A\cos\theta$]. During the passage of the wave the formula (31) gives the solution

$$\phi = \frac{A\cos\theta}{4cr^2}\{(r^2+a^2)-c^2t^2\} \quad (|r-a|<ct<r+a). \tag{31'}$$

It is difficult to imagine any way of actually producing these initial conditions, and so this type of solution has not much physical interest for sound-waves; but, by differentiating (29) with respect to x and y, we can obtain Heaviside's solution* for the electromagnetic pulse sent out when a spherical charge is jerked into (comparatively) slow motion. The magnetic field is then in circles about the axis, and is given by (31') with $\sin\theta$ in place of $\cos\theta$.

I have checked the formula (31') from Poisson's integral for points on the axis $\theta = 0$; but I had not gone very far in the general case before it became clear that the labour of integration was so considerable as to make the solution very difficult on these lines, although there may be some way of reducing the integrals which has not occurred to me.

The formula corresponding to (31') for the initial value $\phi_0 = B\cos\theta$ is found, by Stokes's rule, to be†

$$\phi = -\frac{B\cos\theta}{2r^2}ct. \tag{31''}$$

From these formulae it is not difficult to pass to others in which the distribution is through the *volume* of the sphere $r = a$, and in which ϕ_0 and ϕ_1 are proportional to $\cos\theta$; one simple method is to integrate the formula (31).

For example, if $\phi_1 = Cz$, we need only write $A = Cs$ (writing s for a) in (31) and integrate from $s = 0$ to a; of course, at internal points, two integrals will have to be used.

In order to have a solution free from difficulties at the centre, it will be necessary to connect the values chosen for ϕ_0 and ϕ_1; thus, guided by

* *Electrical papers*, 2, 410; or *Electromagnetic theory*, 3, 96. The actual solution can be derived from § 2 by putting $Z_0 = \text{const.}$ inside $r = a$, and all other discontinuities zero.

† It does not seem worth while to examine here the condition under which $\partial\phi/\partial t$ remains finite near $r = 0$; in the electrical problem the corresponding condition is easily satisfied.

our last example, we select the forms

$$\phi_0 = C(a^2-r^2)\,z, \quad \phi_1 = Dz.$$

It is then, perhaps, easier to return to the fundamental equation for ϕ, and to solve directly instead of integrating (31).

Thus we have to solve the equations

$$\Delta_2\phi - q^2\phi = -q^2\phi_0 - \frac{q}{c}\phi_1 = -q^2\left\{(a^2-r^2)\,C + \frac{D}{qc}\right\} r\cos\theta \quad (r<a), \tag{32}$$

and

$$\Delta_2\phi - q^2\phi = 0 \quad (r>a).$$

A particular integral of (32) is readily found by trial in the form $(\alpha+\beta r^2)\,r\cos\theta$; the correct complementary functions can be taken from (31). In this way we find

$$\phi = (\alpha+\beta r^2)\,r\cos\theta + K(qr\cosh qr - \sinh qr)\frac{\cos\theta}{r^2}, \quad \text{if } r<a, \tag{33}$$

and

$$\phi = L(1+qr)\,e^{-qr}\frac{\cos\theta}{r^2}, \quad \text{if } r>a,$$

where

$$\alpha = C\left(a^2 - \frac{10}{q^2}\right) + \frac{D}{qc}, \quad \beta = -C.$$

The conditions of continuity at $r=a$ are then found from ϕ and $\partial\phi/\partial r$; they give the value*

$$Ke^{qa} = \frac{1}{q^3}(1+qa)\left(\frac{30C}{q^2} + 2a^2C - \frac{3D}{qc}\right) + \frac{a^2}{q}\left(\frac{10C}{q^2} - \frac{D}{qc}\right). \tag{34}$$

From (33) we can determine the general character of the function ϕ near $r=0$: the leading terms in ϕ are derived from

$$K(qr\cosh qr - \sinh qr)\frac{\cos\theta}{r^2}, \tag{33'}$$

where, from (34), K is of the type

$$\left\{\frac{K_0}{(qa)^5} + \frac{K_1}{(qa)^4} + \frac{K_2}{(qa)^3} + \frac{K_3}{(qa)^2}\right\} e^{-qa}.$$

Thus, after the first wave has arrived, so that $a-r<ct<a+r$, we have the solution

$$\phi = \left(K_0\frac{x^5}{5!} + K_1\frac{x^4}{4!} + K_2\frac{x^3}{3!} + K_3\frac{x^2}{2!}\right)\frac{\cos\theta}{r^2}, \tag{33''}$$

where

$$x = ct + r - a > 0.$$

Now, during the passage of the first wave, x varies from 0 to $2r$, and accordingly ϕ itself remains finite near $r=0$. But since $\partial\phi/\partial t$ contains the term $cK_3x\cos\theta/r^2$, this can only remain finite if $K_3=0$.

Returning to (34), we see that the necessary condition is

$$2Cac - D = 0, \tag{34'}$$

* The corresponding value of L will not be needed until after we have found the relation between C, D; the formula is then found to simplify considerably.

or, written in terms of ϕ_0, ϕ_1, we must have*

$$c\left(\frac{\partial\phi_0}{\partial r}\right)+\phi_1=0, \quad \text{at } r=a.$$

Making use of the relation (34′) in (33) we see that

$$K=\frac{C}{q^3}(30+24qa+6q^2a^2)\,e^{-qa},$$

from which we deduce the value of L, namely,†

$$L=-\frac{C}{q^3}(15-18qa+9q^2a^2-2q^3a^3)\,e^{+qa}+\tfrac{1}{2}K.$$

These can now be expressed as functions of t, as follows.

$$K=C\left\{\tfrac{1}{4}(ct)^3+a(ct-a)^4+(ct-a)^3\right\} \qquad \text{if } ct>a,$$
$$=\tfrac{1}{4}C(ct-a)^3(ct+a)^2,$$

and
$$L=-C\left\{\tfrac{1}{8}(ct+a)^3-\tfrac{3}{4}a(ct+a)^4+\tfrac{3}{2}a^2(ct+a)^3-a^3(ct+a)^2\right\}, \quad \text{if } ct<a,$$
$$=-\tfrac{1}{8}C(ct+a)^2(ct-a)^3;$$

when $ct>a$, the value of L is to be increased by $\frac{1}{2}K$: thus $L=0$ when $ct>a$.

To obtain the value of ϕ is now easy; we shall content ourselves with considering a point at some distance outside the sphere. The general formula is

$$\phi=\frac{\cos\theta}{r^2}(1+qr)\,e^{-qr}L,$$

and so the most important terms are given by

$$\phi=\frac{\cos\theta}{r}qe^{-qr}L=-\frac{C\cos\theta}{8r}\frac{d}{dx}\left\{x^3(2a-x)^2\right\} \quad (r-a<ct<r+a),$$

where now $x=ct-r+a$, so that x is positive. The result is‡

$$\phi=-\frac{C\cos\theta}{8r}x^2(2a-x)(6a-5x). \tag{35}$$

This function of x changes sign as the wave reaches the point given by $x=\frac{6}{5}a$, and its maximum is rather less than $2a^4$, while its minimum§ is rather greater numerically than $2a^4$. Thus ϕ varies rapidly in crossing the wave, and its magnitude is of the order

$$\frac{Ca^4\cos\theta}{4r}=\tfrac{1}{2}\frac{Da^3\cos\theta}{cr}.$$

* Compare A. E. H. Love, *Proc. London Math. Soc.* (2), 1 (1903), 54; in addition to the reference given in the former case of symmetry (on p. 463 above).

† We could write L in terms of $\sinh qa$ and $\cosh qa$, but there is no advantage in doing so; while the above form is simpler for the wave-calculations.

‡ The presence of the factor x^2 in ϕ agrees with the calculation previously given for the character of $\partial\phi/\partial t$ near $r=0$; this arises from the factor $(ct-a)^3$ in L.

§ The minimum is negative; the two values are roughly $(1\cdot59)\,a^4$ and $-(2\cdot12)\,a^4$.

In the present problem the value of ϕ_1 varies between 0 and $Da\cos\theta$; thus, on comparing our formula with (30), it is evident that the change in the character of the wave has not greatly affected the order of magnitude of the transmitted wave.

On considering the formulae (30) and (34) it is evident that the disturbance sent out from an initial set of displacements and velocities is by no means of a simple character, even when the initial disturbance is concentrated in a (comparatively) small sphere. It is, moreover, of a very different type from the disturbances found (in §§ 3, 4) to represent the secondary waves equivalent to a thin pulse. It would seem clear, therefore, that the method used by Stokes is subject to serious objections in principle. We shall see (in the next section) that the method of symbolic solutions does give the same results for an elastic solid as those found by Prof. Love, so that we have no ground for distrusting the present calculations.

It is evident that solutions can also be constructed corresponding to any initial disturbances of the type

$$\phi_0 = A(a^2-r^2)\,r^m S_n, \quad \phi_1 = Br^m S_n,$$

where S_n is a surface-harmonic of order n, m is arbitrary, and A, B will be connected by the relation corresponding to (34′), so as to satisfy

$$c\left(\frac{\partial\phi_0}{\partial r}\right)+\phi_1 = 0 \quad (\text{at } r=a),$$

and thus we secure the proper character of the solution for $\partial\phi/\partial t$ near $r=0$. But the nature of the solutions so found is not likely to differ much from the solutions already worked out, except that the variations in the wave-pulses will be more complicated. It does not seem practicable to evaluate any such solutions by means of Poisson's integral.

6. *Integration of the elastic-solid equations with given initial conditions.*

It may be worth while to give the solution of this problem, as it is obtained very quickly by our present methods, and the results are very compactly arranged.

According to the rules already explained*, the equations of motion such as

$$(9\ bis) \qquad p^2\bar{\xi} = X+(a^2-b^2)\frac{\partial\Theta}{\partial x}+b^2\Delta_2\bar{\xi},$$

* See § 1 above and *Proc. London Math. Soc.* (2), 15 (1916), 401, §§ 1, 5, in particular pp. 404, 421. The rules given there would lead to using $(\lambda\xi_0+\xi_1)$ in place of X. In the present paper we have inserted an extra factor λ in the denominators of the complex integrals; this makes the notation agree with Heaviside's conventions.

are modified by taking $(p^2\xi_0+p\xi_1)$ in place of X, where ξ_0, ξ_1 are the initial values of ξ and $\partial\xi/\partial t$; similar changes will be made in the equations for η, ζ.

The calculations given in § 4 enable us to write out the solution on the same lines as (19) and (19′); the result for the dilatation δ is

$$4\pi a^2\delta = \frac{\partial R_1}{\partial x}+\frac{\partial R_2}{\partial y}+\frac{\partial R_3}{\partial z}, \tag{36}$$

where
$$R_1 = \int \frac{e^{-pr/b}}{r}(p^2\xi_0'+p\xi_1')\,d\tau_1,$$

with similar formulae for R_2, R_3, in terms of (η_0, η_1), (ζ_0, ζ_1). Then, if we write also

$$S_1 = \int \frac{e^{-pr/b}}{r}(p^2\xi_0'+p\xi_1')\,d\tau_1, \tag{37}$$

with similar formulae for S_2, S_3, in terms of (η_0, η_1) and (ζ_0, ζ_1), the solution is

$$\xi = \frac{\partial\phi}{\partial x}+\frac{S_1}{4\pi b^2}, \quad \eta = \frac{\partial\phi}{\partial y}+\frac{S_2}{4\pi b^2}, \quad \zeta = \frac{\partial\phi}{\partial z}+\frac{S_3}{4\pi b^2}, \tag{38}$$

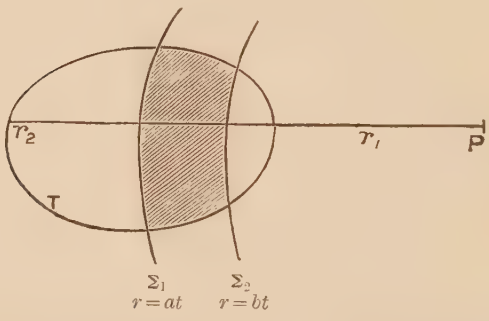
where*
$$\phi = \frac{1}{4\pi p^2}\left\{\frac{\partial}{\partial x}(R_1-S_1)+\frac{\partial}{\partial y}(R_2-S_2)+\frac{\partial}{\partial z}(R_3-S_3)\right\}.$$

It is easy to obtain the transformation of (38) to forms more familiar in appearance, by using methods similar to those of § 5. Thus, as in (28), we see that (so far as ξ_1 is concerned)

$$S_1 = b^2 t\int d\omega\xi_1(bt),$$

or

$$\frac{S_1}{4\pi b^2} = \frac{t}{4\pi}\int d\omega\xi_1(bt), \tag{39}$$



* Note that in (19) above the vector here denoted by (R_1, R_2, R_3) is parallel to z, and is there called S_1; similarly here (S_1, S_2, S_3) corresponds to the former S_2.

the integral being extended over the solid angles of that part of the sphere Σ_2 (that is, $r = bt$) which falls inside the originally disturbed region T. Similarly, the contribution from ξ_0 is given by

$$\frac{S_1}{4\pi b^2} = \frac{\partial}{\partial t}\left\{\frac{t}{4\pi}\int d\omega \xi_0(bt)\right\}; \tag{40}$$

now in carrying out the differentiation indicated in (40), the change in Σ_2 must be taken into account; this change leads to a line-integral* round the curve of intersection of Σ_2 and the boundary of T, in addition to the surface-integral. But actually we must suppose that ξ_0, η_0, ζ_0 are zero at the boundary of T, to avoid an initial rupture of the material, so that this line-integral is zero.

In order to evaluate similarly $\partial\phi/\partial x$, $\partial\phi/\partial y$, $\partial\phi/\partial z$ it is convenient to carry out first the differentiations of R_1, R_2, R_3 with respect to x, y, z. Thus we shall need the formulae

$$r^2\frac{\partial^2}{\partial x^2}\left(\frac{e^{-pr/a}}{r}\right) = (3l^2-1)\left(\frac{1}{r}+\frac{p}{a}\right)e^{-pr/a} + l^2\frac{rp^2}{a^2}e^{-pr/a},$$

$$r^2\frac{\partial^2}{\partial x\,\partial y}\left(\frac{e^{-pr/a}}{r}\right) = 3lm\left(\frac{1}{r}+\frac{p}{a}\right)e^{-pr/a} \qquad + lm\frac{rp^2}{a^2}e^{-pr/a},$$

where l, m, n denote the direction-cosines of the radius r. It will be noticed that we have now to multiply by $(p^2\xi_0+p\xi_1)/p^2$, etc., and to integrate with respect to r and ω; at the same time we have to subtract similar contributions from S_1, S_2, S_3.

It follows that the complete formula for $\partial\phi/\partial x$ (so far as it depends on ξ_1, η_1, ζ_1) is given by

$$\begin{aligned}\frac{1}{4\pi}\int d\omega\int_{r_1}^{r_2}\Bigg[&\left\{\left(\frac{1}{p}+\frac{r}{a}\right)e^{-pr/a}-\left(\frac{1}{p}+\frac{r}{b}\right)e^{-pr/b}\right\}\\ &\times\left\{\left(\frac{3l^2-1}{r}\right)\xi_1+\frac{3lm}{r}\eta_1+\frac{3ln}{r}\zeta_1\right\}\\ &+\left(\frac{rp}{a^2}e^{-pr/a}-\frac{rp}{b^2}e^{-pr/b}\right)l(l\xi_1+m\eta_1+n\zeta_1)\Bigg]dr.\end{aligned} \tag{41}$$

If we write $l\xi_1+m\eta_1+n\zeta_1 = \rho_1$, the last line in (41) can be evaluated by the same method as (39), which gives

$$\frac{t}{4\pi}\int_{\Sigma_1} d\omega\, l\rho_1(at) - \frac{t}{4\pi}\int_{\Sigma_2} d\omega\, l\rho_1(bt). \tag{42}$$

With regard to the first line of (41), we have

$$\{(1/p)+(r/a)\}e^{-pr/a} = \{t-(r/a)+(r/a)\} = t, \quad \text{if } t > r/a,$$

* For the necessary formulae reference may be made to Love's paper [*Proc. London Math. Soc.* (2), 1 (1903), 318].

and the expression is zero when $t < r/a$. Thus

$$\left(\frac{1}{p}+\frac{r}{a}\right)e^{-pr/a}-\left(\frac{1}{p}+\frac{r}{b}\right)e^{-pr/b}=t, \quad \text{if } bt<r<at,$$

and is zero for other values of r. Consequently the first integral in (41) is to be taken through the part of T which falls between the spheres Σ_1, Σ_2. The result can be written

$$\frac{t}{4\pi}\int_{\Sigma_2}^{\Sigma_1} d\tau_1 \left\{\left(\frac{3l^2-1}{r^3}\right)\xi_1'+\frac{3lm}{r^3}\eta_1'+\frac{3ln}{r^3}\zeta_1'\right\}. \tag{43}$$

Finally, to obtain the contributions from ξ_0, η_0, ζ_0 we differentiate (42) and (43) with respect to t: the result is given by Love's formula (64).

When the region T is small compared with its distance from P, the parts played by the waves are separate from each other; thus the first wave to arrive at P is dilatational, and its leading part is represented by the first surface-integral in (42). This gives a radial displacement

$$\dot{\xi}=\frac{l}{4\pi r}\left(\frac{\rho_1}{a}+\frac{\partial\rho_0}{\partial r}\right)\sigma, \quad (t \text{ of order } r/a), \tag{42'}$$

where σ is the section of T perpendicular to the radius r and ρ_1, ρ_0 are averages taken over σ. After this wave has passed, the terms in (43) remain, giving

$$\dot{\xi}=\frac{T}{4\pi r^3}\{3l(\rho_0+t\rho_1)-(\dot{\xi}_0+t\dot{\xi}_1)\} \quad (r/a<t<r/b), \tag{43'}$$

averages again being understood (through the volume T). It is evident that (43') is small compared with (42'); it is a disturbance of an irrotational type in which the dilatation is also zero. Finally, when the rotational wave arrives, its effect is given by combining (39), (40), and the last term in (42), namely

$$\dot{\xi}=\frac{1}{4\pi r}\left\{\frac{\dot{\xi}_1-l\rho_1}{b}+\left(\frac{\partial\dot{\xi}_0}{\partial r}-l\frac{\partial\rho_0}{\partial r}\right)\right\}\sigma \quad (t \text{ of order } r/b). \tag{40'}$$

This is a transverse displacement, and is of the same order as (42').

It is interesting to compare these formulae with the more exact results (30), (34) worked out in § 5, and it will be seen that they agree in general form. Of course, similar calculations could be carried out here; but the results would hardly be worth the labour involved.

It may be worth while to notice that the dilatation and rotations are easily calculated by the same methods; in fact, (36) gives for the part

depending on ξ_1, η_1, ζ_1,

$$\delta = -\frac{1}{4\pi a^2}\int d\omega \int_{r_1}^{r_2} p\left(1+\frac{pr}{a}\right) e^{-pr/a}\rho_1\, dr.$$

To evaluate the integral, we have

$$\int_{r_1}^{r_2} p\left(1+\frac{pr}{a}\right) e^{-pr/a}\rho_1\, dr = p^2\int_{r_1}^{r_2}\left(\frac{1}{p}+\frac{r}{a}\right) e^{-pr/a}\rho_1\, dr = p^2\int_{r_1}^{at} t\rho_1\, dr \quad (r_1 < at < r_2),$$

the transformation being the same as for (43).

The differentiation is now easy, and the result is*

$$\delta = -\frac{1}{4\pi}\int_{\Sigma_1} d\omega\left\{\frac{2}{a}\rho_1(at)+t\frac{\partial\rho_1}{\partial r}\right\}+\frac{1}{4\pi at}\int\rho_1(at)\, ds_1\cot\chi, \tag{44}$$

where the line-integral is taken round the curve of section of Σ_1 (with the boundary of T) and χ is the angle between the radius and the normal. The corresponding part depending on ρ_0 is obtained by differentiating again, but will not be written out here.

We can easily verify Love's conclusion (from a more elaborate calculation) that the dilatation does not depend *solely* on δ_0, δ_1.

In fact we have

$$\frac{\partial P_1}{\partial x} = \int\frac{\partial}{\partial x}\left(\frac{e^{-pr/a}}{r}\right)(p^2\xi_0'+p\xi_1')\, d\tau_1 = -\int\frac{\partial}{\partial x_1}\left(\frac{e^{-pr/a}}{r}\right)(p^2\xi_0'+p\xi_1')\, d\tau_1$$

or

$$\frac{\partial P_1}{\partial x} = -\int\frac{e^{-pr/a}}{r}\lambda\,(p^2\xi_0'+p\xi_1')\, dS+\int\frac{e^{-pr/a}}{r}\left(p^2\frac{\partial\xi_0'}{\partial x}+p\frac{\partial\xi_1'}{\partial x}\right) d\tau_1,$$

where the last transformation follows from Green's theorem and λ, μ, ν denote the direction-cosines of the normal. Hence (36) gives

$$4\pi a^2\delta = -\int\frac{e^{-pr/a}}{r}(pN_0'+pN_1')\, dS+\int\frac{e^{-pr/a}}{r}(p^2\delta_0'+p\delta_1')\, d\tau_1, \tag{45}$$

where
$$N_0 = \lambda\xi_0+\mu\eta_0+\nu\zeta_0 = 0,$$

and
$$N_1 = \lambda\xi_1+\mu\eta_1+\nu\zeta_1.$$

The volume-integral in (45) is transformed in the same way as before; thus the part depending on δ_1 is equal to

$$\frac{t}{4\pi}\int_{\Sigma_1} d\omega\,\delta_1(at), \tag{46}$$

and the part depending on δ_0 is found by differentiation (including a line-integral, since δ_0 need not be zero at S).

* Love, *loc. cit.*, p. 325 (66): the complete formula is given in (67), afterwards transformed to (75), to show that δ is *not* expressible simply in terms of δ_0 and δ_1, when ξ_1, η_1, ζ_1 do not vanish at the surface of τ.

With regard to the surface-integral in (45), we use the fact that

$$e^{-pr/a} = 1 \text{ so long as } r < at;$$

thus
$$p\int \frac{e^{-pr/a}}{r} N_1' \, dS = p\int \frac{N_1'}{r} \, dS,$$

the integral being extended to the part of S which is *inside* Σ_1. On evaluating the differential coefficient, we obtain† (as part of δ)

$$-\frac{1}{4\pi a^2 t}\int N_1' \frac{ds_1}{\sin\chi}. \tag{47}$$

The corresponding calculation for the rotations should now be sufficiently plain without giving all the details; thus in $\partial\xi/\partial y - \partial\eta/\partial z$ we find the extra terms

$$-\int \frac{e^{-pr/b}}{r} \{p(\mu\xi_1' - \nu\eta_1')\} \, dS$$

leading, as before, to the additional term*

$$-\frac{1}{4\pi b^2 t}\int (\mu\xi_1' - \nu\eta_1')\frac{ds_2}{\sin\chi}, \tag{48}$$

which agrees with Love's formula (76).

In the same way we can transform ϕ to show its dependence on δ_0, δ_1; and we obtain

$$4\pi\phi = -\int \frac{1}{pr}(e^{-pr/a} - e^{-pr/b})N_1' \, dS + \int \frac{1}{pr}(e^{-pr/a} - e^{-pr/b})(b\delta_0' + b\delta_1') \, d\tau_1. \tag{49}$$

In formula (49) we see that

$$\frac{\partial}{\partial x}\left\{\frac{1}{pr}(e^{-pr/a} - e^{-pr/b})\right\} = -\frac{l}{r^2}\left\{\left(\frac{1}{p} + \frac{r}{a}\right)e^{-pr/a} - \left(\frac{1}{p} + \frac{r}{b}\right)e^{-pr/b}\right\} = -\frac{lt}{r^2}, \quad \text{if } bt < r < at,$$

and this expression is zero for other values of r.

Hence
$$\frac{\partial\phi}{\partial x} = \int_{\Sigma_1}^{\Sigma_2} \frac{lt}{4\pi r^2} N_1' \, dS - \int_{\Sigma_1}^{\Sigma_2} \frac{lt\delta_1'}{4\pi r^2} \, d\tau_1,$$

where the formula is to be supplemented by the usual differentiation with respect to t (so as to show the effect of the initial values of δ_0); and, on adding (39), the corresponding terms in ξ are

$$\xi = \int_{\Sigma_1}^{\Sigma_2} \frac{lt}{4\pi r^2} N_1' \, dS - \int_{\Sigma_1}^{\Sigma_2} \frac{lt\delta_1}{4\pi r^2} \, d\tau_1 + \frac{t}{4\pi}\int_{\Sigma_2} d\omega \, \xi_1(bt), \tag{50}$$

which corresponds to Love's formula (80). It is important as showing

* Love, *loc. cit.*, p. 327 (75).

that, when the wave transmitted is *purely rotational, we must suppose that initially the normal velocity* $N_1 = 0$, *as well as* $\delta_0 = 0$, $\delta_1 = 0$.

We can make another transformation by observing that (outside the region T) we have

$$\Delta_2 R_1 = \frac{p^2}{a^2} R_1, \quad \Delta_2 S_1 = \frac{p^2}{b^2} S_1.$$

Thus
$$\frac{1}{4\pi p^2} \Delta_2 (R_1 - S_1) = \frac{R_1}{4\pi a^2} - \frac{S_1}{4\pi b^2}.$$

Hence from (38) we get three formulae such as

(51)
$$\xi = \frac{R_1}{4\pi a^2} + \frac{1}{4\pi p^2} \left(\frac{\partial H}{\partial y} - \frac{\partial G}{\partial z} \right),$$

where
$$F = \frac{\partial}{\partial y}(R_3 - S_3) - \frac{\partial}{\partial z}(R_2 - S_2),$$

$$G = \frac{\partial}{\partial z}(R_1 - S_1) - \frac{\partial}{\partial x}(R_3 - S_3),$$

$$H = \frac{\partial}{\partial x}(R_2 - S_2) - \frac{\partial}{\partial y}(R_1 - S_1).$$

These formulae for F, G, H can be manipulated in the same way as (49) above, and they lead to the same results as Love's formula (84). The most interesting conclusion is *that to obtain a purely dilatational wave* the tangential velocity must be zero at the boundary of* T *(as well as the rotations and their time-rates of change at internal points).*

It can then be proved (but the analysis is in no way different from Prof. Love's) that, to avoid infinities in the inward waves, it is necessary that $N_1 = a\delta_0$; and similar relations hold between the tangential velocity and the initial rotations on the boundary of S (the velocity now being b)†

* Represented by the first term in (51).

† These correspond to Heaviside's result (quoted in § 3) for the continuity of $(E - cH)$ in an electric wave; and the relation $N_1 = a\delta_0$ corresponds to the result used in § 5.

THE APPROXIMATE FUNCTIONAL EQUATION FOR THE MULTIPLE THETA-FUNCTION AND THE TRIGONOMETRIC SUMS ASSOCIATED THEREWITH

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1. In a recent note* Hardy and Littlewood have given a new proof of their approximate functional equation for the theta-function. This formula states that

$$(1.1)\qquad \sum_{0\leqslant n^2\leqslant \omega} e^{-n^2\pi ix}\cos 2n\pi\theta - \frac{e^{-\frac{1}{4}\pi i}}{\sqrt{x}}\, e^{\pi i\theta^2/x} \sum_{0\leqslant n^2\leqslant x^2\omega} e^{n^2\pi i/x}\cos\frac{2n\pi\theta}{x} = O\left(\frac{1}{\sqrt{x}}\right),$$

where $0<x<1$, $0\leqslant\theta\leqslant 1$ and $\omega>1$, uniformly in ω and θ.

They observe also that this new method, based on the principle of steepest descents, is applicable, in principle at any rate, to powers of a theta-function such as $\sum r(n)e^{-n\pi ix}$, where $r(n)$ is the number of representations of n as the sum of two squares. While attacking the problem by their method, I discovered, much to my surprise, that it is possible to obtain not merely the approximate functional formula for the powers of a theta-function but also the corresponding formula for the general multiple theta-function, by a process of induction from a slight extension of (1.1). The extension in question is

$$(1.2)\qquad \sum_{0\leqslant(n+\alpha)^2\leqslant\omega} e^{-(n+\alpha)^2\pi ix+2n\pi i\theta} = \frac{e^{-2\pi i\theta\alpha}}{\sqrt{(ix)}} \sum_{0\leqslant(n+\theta)^2\leqslant x^2\omega} e^{(n+\theta)^2\pi i/x-2n\pi i\alpha} + O\left(\frac{1}{\sqrt{x}}\right),$$

(where $0<x<1$, $\omega>1$, α and θ are real) uniformly in ω, α and θ; and

* G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation", *Acta Math.*, 47 (1925), 189–198. For their original proof see *ibid.*, 37 (1914), 193–221. Further references are given by L. J. Mordell, "The approximate functional formula for the theta-function", *Journal London Math. Soc.*, 1 (1926), 68–72. Reference should also be made to a recent paper with the same title by J. R. Wilton, *ibid.*, 2 (1927), 177–180. Wilton gives a determination of the absolute constant occurring in the error term of (1.1).

(1.2) may be proved by very trivial changes in the argument of Hardy and Littlewood which proves (1.1).

The proof by induction of the general formula for the multiple theta-function is in every important respect precisely similar to the ordinary proof by induction of the precise functional equation* for the multiple theta-function from the precise equation for the simple theta-function. I content myself, therefore, with a statement of the general result, giving a proof only for a certain special case which does not need the theorems on positive quadratic forms required in the general case.

I write

$$a_0 = 1, \quad a_j = |a_{kl}| \quad (k, l = 1, 2, \dots j;\ j \geqslant 1). \tag{1.3}$$

Then the general result is contained in the following

(1.4) THEOREM. *Let*

$$Q(y) = Q(y_1, y_2, \dots, y_p) = \sum_{i,k=1}^{p}\sum a_{ik} y_i y_k \quad (a_{ik} = a_{ki}) \tag{1.41}$$

be a positive definite quadratic form so that the numbers $a_1, a_2, \dots, a_p$ are all positive, and let

$$\bar{Q}(y) = \bar{Q}(y_1, \dots, y_p) \tag{1.42}$$

be the quadratic form conjugate to Q.

Then if

$$0 < x < a_{i-1}/a_i \quad (i = 1, 2, \dots, p) \tag{1.43}$$

we have

$$\sum_{0 \leq Q(n+a) \leq \omega} \exp\left\{-\pi i x Q(n+a) + 2\pi i \sum_{1}^{p} n_j \theta_j\right\}$$

$$= a_p^{-\frac{1}{2}} (ix)^{-\frac{1}{2}p} \sum_{0 \leq \bar{Q}(n+\theta) \leq x^2\omega} \exp\left\{\frac{\pi i}{x}\bar{Q}(n+\theta) - 2\pi i \sum_{1}^{p}(n_j+\theta_j)a_j\right\}$$

$$+ O\left\{\phi_p(x, \omega)\right\}, \tag{1.44}$$

(the summations ranging over integral values of n subject to the restrictions stated) uniformly in $\omega > 1$ and in the real numbers a and θ, where

$$\phi_p(x, \omega) = \frac{\omega^{\frac{1}{2}p-\frac{1}{2}}}{x^{\frac{1}{2}}} + \frac{\omega^{\frac{1}{2}p-1}}{x} + \dots + \frac{1}{x^{\frac{1}{2}p}}. \tag{1.45}$$

* For this proof reference may be made to E. Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale* (Leipzig und Berlin, 1918), 58–64.

We may observe that, if $x\omega > A$, we may take

$$\phi_p(x, \omega) = \omega^{\frac{1}{2}p-\frac{1}{2}}x^{-\frac{1}{2}}$$

if $p \geqslant 2$. Further

$$(ix)^{-\frac{1}{2}p} = e^{-\frac{1}{4}p\pi i}x^{-\frac{1}{2}p};$$

and the theorem holds also for the conjugate sums.

2. The special case referred to is given by the form

$$Q(y) = b_1y_1^2+\ldots+b_py_p^2, \tag{2.1}$$

where the numbers $b_1, \ldots b_p$ are all positive. By (1.2) the theorem holds for $p = 1$. We assume the truth of the theorem for the form in $p-1$ variables,

$$R(y) = b_2y_2^2+\ldots+b_py_p^2 \quad (p \geqslant 2), \tag{2.2}$$

and we shall prove that the theorem then holds for the form $Q(y)$.

We may write the sum on the left of (1.44) in the form

$$\Sigma \exp\{-\pi ixb_1(n_1+\alpha_1)^2+2\pi in_1\theta_1\} \tag{2.3}$$
$$\Sigma \exp\left\{-\pi ixR(n+\alpha)+2\pi i\sum_2^p n_j\theta_j\right\}$$

where the inner summation extends over the range

$$0 \leqslant R(n+\alpha) \leqslant \omega-b_1(n_1+\alpha_1)^2 \tag{2.31}$$

and the outer summation over the range

$$0 \leqslant b_1(n_1+\alpha_1)^2 \leqslant \omega. \tag{2.32}$$

Since the theorem holds for the form R, which has the conjugate

$$\bar{R}(y) = b_2^{-1}y_2^2+\ldots+b_p^{-1}y_p^2, \tag{2.33}$$

we may replace the inner sum in (2.3) by

$$(b_2\ldots b_p)^{-\frac{1}{2}}(ix)^{-\frac{1}{2}p+\frac{1}{2}}\Sigma \exp\left\{\frac{\pi i}{x}\bar{R}(n+\theta)-2\pi i\sum_2^p (n_j+\theta_j)\alpha_j\right\} \tag{2.4}$$
$$O\{\phi_{p-1}[x, \omega-b_1(n_1+\alpha_1)^2]\},$$

where the summation ranges over

$$0 \leqslant \bar{R}(n+\theta) \leqslant x^2\omega-x^2b_1(n_1+\alpha_1)^2 \tag{2.41}$$

provided that

$$0 < x < b^{-1} \quad (j = 2, \ldots, p). \tag{2.42}$$

The contribution of the error term in (2.4) to the total sum in (2.3) is plainly

$$O\Big(\sum_{0\leqslant b_1(n_1+a_1)^2\leqslant\omega}\phi_{p-1}\{x,\ \omega-b_1(n_1+a_1)^2\}\Big), \tag{2.43}$$

while if we change the order of summation the exact term contributes

$$(b_2\dots b_p)^{-\frac{1}{2}}(ix)^{-\frac{1}{2}p+\frac{1}{2}}\sum\exp\Big\{\frac{\pi i}{x}\bar{R}(n+\theta)-2\pi i\sum_2^p(n_j+\theta_j)a_j\Big\} \tag{2.5}$$
$$\times\sum\exp\{-\pi i x b_1(n_1+a_1)^2+2\pi i n_1\theta_1\},$$

where, after (2.32) and (2.41), the inner summation has the range

$$0\leqslant x^2b_1(n_1+a_1)^2\leqslant x^2\omega-\bar{R}(n+\theta) \tag{2.51}$$

and the outer summation has the range

$$0\leqslant\bar{R}(n+\theta)\leqslant x^2\omega. \tag{2.52}$$

But by (1.2) the inner sum in (2.5) is equal to

$$(ib_1x)^{-\frac{1}{2}}\sum\exp\Big\{\frac{\pi i}{b_1x}(n_1+\theta_1)^2-2\pi i(n_1+\theta_1)a_1\Big\} \tag{2.6}$$
$$+O\Big\{\phi_1\Big(xb_1,\ \omega-\frac{\bar{R}(n+\theta)}{x^2b_1}\Big)\Big\},$$

the summation ranging over

$$0\leqslant b_1^{-1}(n_1+\theta_1)^2\leqslant x^2\omega-\bar{R}(n+\theta), \tag{2.61}$$

provided that

$$0<xb_1<1. \tag{2.62}$$

Now $a_p=b_1b_2\dots b_p$ and $Q(y)=b_1^{-1}y_1^2+\bar{R}(y)$, so that the exact term in (2.6) contributes in (2.5) the exact term on the right of (1.44). Also the conditions (2.42) and (2.62) imposed on x are precisely the conditions (1.43) of the theorem. It remains, therefore, to show that the contribution of the error term in (2.6) to (2.5) and the error term (2.43) are each of the form $\phi_p(x,\omega)$ prescribed by the theorem.

The first contribution is plainly

$$O\Big(x^{\frac{1}{2}p-\frac{1}{2}}\sum_{0\leqslant\bar{R}(n+\theta)\leqslant x^2\omega}x^{-\frac{1}{2}}\Big),$$

which is $O(x^{-\frac{1}{2}p})$ or $O(x^{\frac{1}{2}p-1}\omega^{\frac{1}{2}p-\frac{1}{2}})$ according as $x^2\omega\leqslant A$ or $x^2\omega>A$, where A is a positive constant. On the other hand, (2.43) is

$$O\{\omega^{\frac{1}{2}}\phi_{p-1}(x,\omega)\},$$

so that we may take

$$\phi_p(x, \omega) = \omega^{\frac{1}{2}}\phi_{p-1}(x, \omega) + O(x^{-\frac{1}{2}p})$$

if $x^2\omega \leqslant A$ and

$$\phi_\rho(x, \omega) = \omega^{\frac{1}{2}}\phi_{\rho-1}(x, \omega) + O(x^{\frac{1}{2}\rho-1}\omega^{\frac{1}{2}p-\frac{1}{2}})$$

if $x^2\omega > A$. Since, by hypothesis, $\phi_{p-1}(x, \omega)$ has the form (1.45), it follows immediately that $\phi_p(x, \omega)$ has the same form.

3. We now apply Theorem (1.4) in the manner used by Hardy and Littlewood* to determine the behaviour for large ω and irrational x $(0 < x < 1)$ of the sum

(3.1) $s(x, \omega, \theta, \alpha)$

$$= \sum_{0 \leqslant (n_1+\alpha_1)^2+\ldots+(n_p+\alpha_p)^2 \leqslant \omega} \exp\left\{-\pi i x \sum_1^p (n_j+\alpha_j)^2 + 2\pi i \sum_1^p n_j\theta_j\right\},$$

where θ and α are real and $p \geqslant 2$.

I give the results for the case $p = 2$ in detail because, although not quite the best possible, they are nevertheless very close to what are apparently the best results. I content myself with stating the results for $p > 3$, because they appear to be far from the real truth. They are, however, the best obtainable by this method from Theorem (1.4).

Taking therefore $p = 2$ and altering the notation slightly, we have

(3.11) $$s(x, \omega, \theta, \alpha) \sum_{0 \leqslant (m+\alpha)^2+(n+\beta)^2 \leqslant \omega} e^{-\pi i x\{(m+\alpha)^2+(n+\beta)^2\}+2\pi i(m\theta+n\phi)},$$

where θ, ϕ, α, β are real and $0 < x < 1$.

We write

(3.2) $$x_0 = x, \quad x_n^{-1} = a_{n+1} + x_{n+1},$$

where $a_n (n = 1, 2, \ldots)$ is a positive integer and $0 < x_n < 1$. The number ρ denotes a number of modulus unity which is not necessarily the same in each equation.

Observing that $$e^{\pi i a n^2} = e^{\pi i a n}$$

for any integers a and n, we see that, by (1.44),

$$s(x, \omega, \theta, \alpha) = \rho x^{-1} s(x_1, x^2\omega, \theta', \alpha',) + O(\omega^{\frac{1}{2}}x^{-\frac{1}{2}}) + O(x^{-1}),$$

* Hardy and Littlewood, "Some problems of Diophantine approximation", *Acta Math.*, 37 (1914), 193–221 (210–216).

where $\theta' = \alpha - \frac{1}{2}a_1\theta$, $\alpha' = \theta$, and $\bar{s}$ denotes the conjugate of s. Continued application of this equation shows that, for any integer $\nu \geqslant 1$,

$$(3.3)\quad s(x, \omega, \theta, \alpha) = \rho(xx_1 \dots x_{\nu-1})^{-1} S(x_\nu, x^2x_1^2 \dots x_{\nu-1}^2\omega, \theta^{(\nu)}, \alpha^{(\nu)}) + O(\Delta_\nu) + O(\Delta'_\nu),$$

where S denotes s or $\bar{s}$, and

$$(3.31)\qquad \Delta_\nu = O\{\omega^{\frac{1}{2}}(x^{-\frac{1}{2}} + x_1^{-\frac{1}{2}} + \dots + x_{\nu-1}^{-\frac{1}{2}})\},$$

$$(3.32)\qquad \Delta'_\nu = O\{x^{-1} + (xx_1)^{-1} + \dots + (xx_1 \dots x_{\nu-1})^{-1}\}.$$

Now plainly

$$(3.4)\qquad \frac{1}{x_\nu x_{\nu+1}} = 1 + \frac{a_{\nu+1}}{x_{\nu+1}} > 2,$$

so that the product $xx_1 \dots x_r$ tends to zero when $r \to \infty$. Hence there is an integer $\mu = \mu(\omega) = \mu(\omega, x)$ such that

$$(3.41)\qquad x^2x_1^2 \dots x_{\mu-1}^2\omega \geqslant 1 > x^2x_1^2 \dots x_\mu^2\omega,$$

and μ lies between two constant multiples of $\log \omega$.

Taking $\nu = \mu$ and $\nu = \mu+1$ in (3.3), we obtain

$$(3.5)\qquad s(x, \omega, \theta, \alpha) = O(xx_1 \dots x_{\mu-1}\omega) + O(\Delta_\mu) + O(\Delta'_\mu)$$

and

$$(3.51)\qquad s(x, \omega, \theta, \alpha) = O\{(xx_1 \dots x_\mu)^{-1}\} + O(\Delta_{\mu+1}) + O(\Delta'_{\mu+1}).$$

Now, by (3.41),

$$(3.61)\qquad \Delta_\mu = O\{\omega^{\frac{1}{2}}(\omega^{\frac{1}{2}} + \dots + \omega^{\frac{1}{2}})\} = O\{\mu\omega^{\frac{3}{2}}\} = O(\omega^{\frac{3}{2}}\log\omega),$$

and

$$(3.62)\qquad \Delta'_\nu = O\{(xx_1 \dots x_{\nu-1})^{-1}(1 + x_{\nu-1} + x_{\nu-1}x_{\nu-2} + \dots)\}$$
$$= O\{(xx_1 \dots x_{\nu-1})^{-1}\}$$

by (3.4).

From (3.41), (3.5), (3.61), and (3.62) we deduce that, for *any irrational x between 0 and 1**,

$$(3.7)\qquad s(x, \omega, \theta, \alpha) = o(\omega)$$

uniformly in θ, ϕ, α, β, and, in particular,

$$(3.71)\qquad \sum_{0 \leqslant n \leqslant \omega} r(n)e^{-n\pi ix} = o(\omega).$$

* The restriction $0 < x < 1$ is now plainly unnecessary.

More generally for the sum in (3 . 1) we have

$$s(x, \omega, \theta, \alpha) = o(\omega^{\frac{1}{2}p}) \tag{3 . 7)'}$$

and, in particular,

$$\sum_{0 \leqslant n \leqslant \omega} r_p(n) e^{-n\pi i x} = o(\omega^{\frac{1}{2}p}). \tag{3 . 71)'}$$

For certain classes of irrationals these results may be improved. Thus, if the elements of the simple continued fraction for x are bounded, *i.e.* if $a_n < K$, so that $x_n > 1/(1+K)$, we see that

$$\Delta_{\mu+1} = O(\mu\omega^{\frac{1}{2}}) = O(\omega^{\frac{1}{2}} \log \omega),$$

so that, from (3 . 51), *for all irrational x between 0 and 1* with bounded partial quotients a_n, and, in particular, for all quadratic surds,*

$$s(x, \omega, \theta, \alpha) = O(\omega^{\frac{1}{2}} \log \omega) \tag{3 . 8}$$

uniformly in θ and α. In particular, for any x of this type,

$$\sum_{0 \leqslant n \leqslant \omega} r(n) e^{-n\pi i x} = O(\omega^{\frac{1}{2}} \log \omega). \tag{3 . 81}$$

In precisely the same way we may prove that, for $p > 2$,

$$s(x, \omega, \theta, \alpha) = O(\omega^{\frac{1}{2}p - \frac{1}{2}}) \tag{3 . 8)'}$$

uniformly in θ and α, for any irrational x with bounded partial quotients a_n, and, in particular, for any quadratic surd.

4. It is natural to inquire whether (3 . 7)–(3 . 8)′ are the best possible results. Now, from the work of Hardy and Littlewood†, it may be deduced that, for any positive function $\phi(n)$ which tends to zero when $n \to \infty$, there exist irrationals x such that

$$\sum_{n \leqslant \omega} r_p(n) e^{-n\pi i x} = \Omega\{\omega^{\frac{1}{2}p} \phi(\omega)\}, \tag{4 . 1}$$

and from (4 . 1) it follows immediately that (3 . 7) and (3 . 7)′ are the best results that can be proved for all irrationals.

From the same work‡ we derive also the result

$$\sum_{n \leqslant \omega} r(n) e^{-n\pi i x} = \Omega(\omega^{\frac{1}{2}}) \tag{4 . 2}$$

for all irrational x; and, from (4 . 2), we see that (3 . 8) cannot be improved by more than the omission of log ω.

* See the last foot-note.

† *L.c. supra*, 229.

‡ *Ibid.*, 228.

As for (3 . 8)′ we observe that

$$r_p(n) \neq o(n^{\frac{1}{2}p-1}) \quad (p \geqslant 3),$$

so that

$$\sum_{\leq \omega} r_p(n) e^{-n\pi i x} \neq o(\omega^{\frac{1}{2}p-1}), \tag{4 . 3}$$

and the difference between the orders in (4 . 3) and (3 . 8)′ is $\frac{1}{2}$.

If we make use of the theory of Fourier series*, it is easily seen that, for almost all x,

$$\sum r_p(n) e^{-n\pi i x} = O(\omega^{\frac{1}{2}p-\frac{1}{2}+\delta}) \tag{4 . 4}$$

for any positive δ. For the series

$$\sum r_p^2(n) n^{-(p-1+2\delta)} \quad (\delta > 0)$$

is convergent, since $r_p(n) = O(n^{\frac{1}{2}p-1+\epsilon})$ for any $\epsilon > 0$, so that, by the Riesz-Fischer theorem, the series

$$\sum r_p(n) n^{-\frac{1}{2}p+\frac{1}{2}-\delta} \begin{matrix}\cos \\ \sin\end{matrix} n\pi x \tag{4 . 5}$$

are Fourier series. Consequently, by a theorem due to Young, the series obtained by multiplying by $n^{-\delta'}$ (for any positive δ') are convergent almost everywhere, so that the series (4 . 5) converge almost everywhere, since δ and δ' are arbitrary. And from this result (4 . 4) follows immediately.

It is possible to improve (4 . 4) slightly by the use of the more precise theorem of Young.

* Cf. Hardy and Littlewood, *l.c.*, 196.

THE 8-TANGENT HYPERQUADRICS OF NOETHER'S CANONICAL CURVE FOR $p = 5$

By WILLIAM P. MILNE.

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1. *Introduction.*

Noether has shown that a plane curve of genus p can be represented point by point on a canonical curve of degree $2(p-1)$ in space of $p-1$ dimensions. This canonical curve for plane curves of genus 3 is the plane quartic curve, and it has long been known that all 4-tangent conics of the same system of the quartic curve can be regarded as the polar conics with respect to a cubic curve of points on a fixed conic. It was more recently proved by Roth* and by myself† that the enveloping cones to one of the 255 quadrinodal cubic surfaces containing Noether's canonical curve for $p = 4$ (*i.e.* the curve of intersection of a cubic surface with a quadric), drawn from points on itself, define the points of contact of all the 6-tangent quadrics of the same system. It is the object of the present paper to investigate the corresponding problem in the case of Noether's canonical curve for $p = 5$ (*i.e.* the curve of intersection of three hyperquadrics in space of four dimensions). It should be noted that two 8-tangent hyperquadrics are said to belong to the same system when a hyperquadric not containing the canonical curve passes through their sixteen points of contact.

2. *On a certain quartic hypersurface* Φ *with 5 nodes.*

I have shown‡ that to every system of 8-tangent hyperquadrics of the curve of intersection $\Gamma \equiv UVW$ of three hyperquadrics U, V, W corresponds a quartic hypersurface Φ having five nodal points and con-

* Roth, *Monatshefte Math. Phys.*, 22 (1911), 64–88.

† W. P. Milne, *Proc. London Math. Soc.* (2), 21 (1922), 373–380.

‡ W. P. Milne, *Proc. London Math Soc.* (2), 26 (1927), 119–134.

taining Γ as a double curve. The general equation to all quartic hypersurfaces containing Γ as a double curve is

$$aU^2+bV^2+cW^2+2fVW+2gWU+2hUV=0. \quad (1)$$

Let the point (ξ, η, ζ) in the plane π correspond to the hyperquadric

$$\xi U+\eta V+\zeta W=0. \quad (2)$$

To points of a line in π correspond hyperquadrics of a pencil of the net (2), and to points on a conic in π correspond the contact-hyperquadrics of a quartic hypersurface of the type (1). If (ξ, η, ζ) lies on the discriminant-quintic of the net (2), the corresponding hyperquadric has a node and it is, in fact, a hypercone. If a conic of π touch the discriminant-quintic, the quartic hypersurface enveloped by the hyperquadrics corresponding to points of the conic has a node at the vertex of the hypercone corresponding to the point of contact. The discriminant-quintic is general in character and has 2015 5-tangent conics, 1023 of which correspond to the 5-nodal members of (1) whose nodal points do not lie on a hyperplane, while the points of the other 992 taken in pairs correspond to the points of contact of the hyperquadrics of the net (2) that touch the 496 4-tangent hyperplanes of the curve Γ^*. The five nodes of each of the 992 quartic hypersurfaces that correspond to the latter group of 992 5-tangent conics lie on a hyperplane, viz., that particular 4-tangent hyperplane of Γ which is touched by the hyperquadrics of the system (2) that correspond to the points of the particular conic. Let us take one of the 1023 5-tangent conics and let its equation be taken in the form $\xi=\phi^2$, $\eta=2\phi$, $\zeta=1$. Let the parameters of its five points of contact with the discriminant-quintic be α_1, α_2, α_3, α_4, α_5 respectively, and let the corresponding nodal points of the quartic hypersurface Φ defined by the above conic be the vertices of reference X_1, X_2, X_3, X_4, X_5.

Let

$$U\equiv\Sigma a_{rs}x_rx_s, \quad V\equiv\Sigma b_{rs}x_rx_s, \quad W\equiv\Sigma c_{rs}x_rx_s \quad (r, s=1, 2, 3, 4, 5),$$

where $a_{rs}=a_{sr}$, etc.

The conditions that the hyperquadric $\alpha_1^2U+2\alpha_1V+W=0$ shall have a nodal point at X_1 are

$$a_{1r}\alpha_1^2+2b_{1r}\alpha_1+c_{1r}=0 \quad (r=1, 2, 3, 4, 5). \quad (3)$$

Also the conditions that the quartic hypersurface $V^2-UW=0$ enveloped by $\phi^2U+2\phi V+W=0$ shall have nodes at the vertices of

* W. P. Milne, *Journal London Math. Soc.*, 2 (1927), 79–84.

reference supply the following further relations :

$$a_{rr}c_{rr} = b_{rr}^2 \quad (r = 1, 2, 3, 4, 5). \tag{4}$$

Hence we obtain

$$a_{rs} : b_{rs} : c_{rs} = 1 : -\tfrac{1}{2}(a_r + a_s) : a_r a_s.$$

Thus the system $\phi^2 U + 2\phi V + W = 0$ becomes

$$\Sigma(\phi - a_r)(\phi - a_s) a_{rs} x_r x_s = 0 \quad (r, s = 1, 2, 3, 4, 5). \tag{6}$$

This is plainly the reciprocal of $\Sigma a_r a_s l_r l_s = 0$ with respect to $\Sigma(a_r - \phi)x_r^2 = 0$ $(r = 1, 2, 3, 4, 5)$. The following result is at once deduced :

The reciprocals of the hyperquadric $\Sigma \equiv \Sigma a_{rs} l_r l_s = 0$ $(r, s = 1, 2, 3, 4, 5)$ *with respect to the members of the pencil*

$$S \equiv a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2 + a_5 x_5^2 = 0,$$

$$S' \equiv x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0,$$

envelop a quartic hypersurface Φ *with nodes at the five vertices of the five hypercones of the pencil* SS', *and containing, as a double curve* Γ, *the locus of points whose polar hyperplanes with respect to every member of the pencil* SS' *touch* Σ.

We shall refer to the plane of intersection of the polar hyperplanes of the point P with respect to the pencil SS' as "the polar plane of P with respect to the pencil SS'".

Φ *is the locus of points whose polar planes with respect to* SS' *each intersect* Σ *in a pair of straight lines.*

To prove this, let the reciprocals of Σ with respect to S, S' be U, V respectively. Let P be a point on the surface U, V, and let the polar hyperplanes of P with respect to S, S' touch Σ at Q, Q' respectively. Plainly, the polar plane of the line QQ' with respect to Σ is the polar plane of the point P with respect to the pencil SS'. Proceeding to the limit when S and S', and therefore U and V, become coincident, we see that P lies on Φ, and that the polar plane of P with respect to the pencil SS' becomes the polar plane with respect to Σ of a tangent line to Σ, and hence intersects Σ in two straight lines meeting at the point of contact Q.

If the line lies wholly on Σ, the tangent hyperplanes of points upon the line have a common axial plane which intersects Σ in two lines

coincident with the given line. We deduce the following result :

The curve Γ *is the locus of points whose polar planes with respect to* SS' *each intersect* Σ *in a pair of coincident straight lines.*

3. *Tabulation of known properties.*

We shall tabulate for convenience of reference certain known properties of a pencil of hyperquadrics which will be required in the sequel*.

(1) The locus of a point P whose polar hyperplane with respect to a member S of a pencil SS' of hyperquadrics contains a given line l is a hypercone L, the polar hypercone of l. One system of generating planes of L consists of the polar planes of points on l with respect to the pencil SS'. The other system of generating planes consists of the polar planes of l with respect to individual members of the pencil. The vertex L_0 of the hypercone is that unique point whose polar plane with respect to SS' contains l. Let the polar plane of P with respect to SS' cut l in Q. The polar plane of Q with respect to SS' will meet the polar plane of l with respect to S in the line PL_0. The polar hyperplane of Q with respect to S contains the polar plane of Q with respect to SS' as well as the polar plane of l with respect to S, and hence touches L along the line PL_0. If the polar plane of P with respect to SS' be π and if the polar hyperplane of P with respect to S be πl, the polar hyperplane of Q with respect to S will be the tangent hyperplane at P to L.

(2) The locus of a point P whose polar hyperplane with respect to a member of the pencil SS' contains a given plane ξ is a cubic scroll Ξ. The polar planes of points of ξ with respect to SS' constitute the conic-planes of Ξ. The polar lines of ξ with respect to individual hyperquadrics of the pencil are the generators of Ξ.

(3) The locus of a point P having a given polar hyperplane ρ with respect to a member of the pencil SS' is a rational normal quartic curve R. The polar planes of points on ρ with respect to SS' are the trisecant planes of R. The curve R may also be regarded as the locus of points whose polar planes with respect to SS' lie in ρ.

* For the general case, see Segre, *Mem. Acc. Torino* (2), 36 (1884), 3–86 (50) or Bertini, *Geometria projettiva degli iperspazi* (1923), 178–180. A summary is given in Segre's report, "Mehrdimensionale Räume", *Encykl. Math. Wiss.*, III C7, 864.

4. *The polar hypercones of lines on* Σ.

We shall now establish the following theorem :

The vertices L_0 *of the polar hypercones* L *with respect to* SS' *of the lines* l *on* Σ *lie on* Φ.

Since the polar plane of L_0 with respect to SS' contains l, it must intersect Σ in a supplementary line l'. Hence the polar plane of L_0 intersects Σ in two straight lines and L_0 must therefore lie on Φ.

We shall next prove the following result :

The polar hypercones with respect to SS' *of all lines on* Σ *are* 8*-tangent hyperquadrics of* Γ.

Let P be a point common to Γ and L, the polar hypercone of l which lies on Σ. Let the polar plane π of P with respect to SS' meet l in Q. Then, since P lies on Γ, π must intersect Σ in two coincident straight lines m, where m passes through Q. Any hyperplane through π must therefore intersect Σ in a quadric T having two coincident intersecting generators m. T must therefore be a cone whose vertex lies on m. In particular, the hyperplane πl must intersect Σ in a cone whose vertex is lm, *i.e.* Q, and must therefore touch Σ at Q. Let S be the hyperquadric of the pencil SS', such that the polar hyperplane of P with respect to S is πl, and let U be the reciprocal of Σ with respect to S. Since the polar hyperplane of P with respect to S touches Σ at Q, the polar hyperplane of Q with respect to S will touch U at P. But we have already seen in (1) of § 3 that the polar hyperplane of Q with respect to S is the tangent hyperplane to L at P. Thus U and L have a common tangent hyperplane at P, and Γ, which lies on U and passes through P, must therefore be touched by L at P. Thus the required result is established. Since Γ is of degree 8 and L is of degree 2, we see that there are 8 points of contact.

Since l lies on Σ, the tangent hyperplanes to Σ at the points of l have a common axial plane π', which must therefore lie in πl, the tangent plane at Q. The polar line of π' with respect to S must therefore pass through P, since the polar hyperplane of P with respect to S is πl, and therefore contains π'. Hence *each of the* 8 *points of contact of* L *with* Γ *lies on the polar cubic scroll of the plane* π' *which touches* Σ *along* l.

Any hyperplane through π' touches Σ at a point of l, and hence the polar line of π' with respect to S lies on U. Hence the polar scroll of π' and U have a generator in common, and similarly with regard to all the other reciprocals of Σ with respect to the members of the pencil

SS'. Let W be the reciprocal of Σ with respect to S'. It is easy to show that the scroll meets U in a generator and further in a quintic curve, of which every generator of the scroll is a bisecant. This quintic curve meets W in ten points, of which two are on the generator common to W and the scroll. The remaining eight points are the eight points of contact with Γ of the polar hypercone of l.

We shall next show that *the points of contact with* Γ *of the polar hypercones with respect to* SS' *of any two non-intersecting lines* l, l' *on* Σ *lie on a hyperquadric not containing* Γ, *i.e. the polar hypercones of* l *and* l' *belong to the same system of* 8*-tangent hyperquadrics of* Γ.

Let the hyperplane ϖ containing the two non-intersecting lines l, l' intersect Σ in the quadric σ. It is plain that the locus of points whose polar planes with respect to SS' cut ϖ in the lines of a linear complex is a hyperquadric M. Consider the linear complex Λ composed of the lines that meet each pair of polar lines of σ intersecting both l and l'. The polar plane of any point Q of ϖ with respect to Λ is the plane containing Q, and the pair of points in which the polar plane of Q with respect to σ cut both l and l'. In particular, if Q lies on l or l', the polar plane of Q with respect to Λ is the tangent plane at Q to σ. Consider now the point of contact P of L with Γ. The polar plane of P with respect to SS' is π meeting l in Q, and we have already seen that the hyperplane πl touches Σ at Q. Now the tangent plane at Q to σ lies in the tangent hyperplane at Q to Σ. Hence the tangent plane at Q to σ and π both lie in the same hyperplane, and hence have a line in common, which belongs to Λ, since it lies in the tangent plane at Q to σ. P must therefore lie on the hyperquadric M, and similarly with regard to the other points of contact of L and L' with Γ. Plainly, M *contains the rational normal quartic curve which is the locus of the poles of* ϖ *with respect to the pencil* SS' *and, in particular, the* 5 *nodes of* Φ *which lie upon this curve.*

5. *The* 8*-tangent hypercones of lines on a hyperplane.*

We proceed to consider still further the polar hypercones of those lines on Σ that lie on its intersection with a hyperplane ϖ. We shall first of all deal with the singly infinite family of polar hypercones of one system of generators l on σ. We note that there are two such generators whose polar hypercones pass through any arbitrary point P, since if the polar hypercone of l passes through P, the polar plane of P with respect to SS' must intersect l. But the polar plane of P intersects ϖ in a straight line which meets σ in two points, and hence meets two generators

l and l' of the same system. We therefore obtain a quadratic system of hypercones $\lambda^2 L+2\lambda M+L'=0$, corresponding to the generators l of the same system of σ, and these hypercones must envelop a quartic hypersurface $\Phi_1 \equiv M^2-LL'=0$. If, in the above reasoning, we let l and l' coincide, we see that Φ_1 *is the locus of a point whose polar plane with respect to* SS' *cuts* ϖ *in a line touching* Σ.

Otherwise, Φ_1 *is the locus of a point whose polar plane with respect to* SS' *cuts* Σ *in a conic touching* ϖ.

Hence, *the polar hypercones of the system of generators* l *on* σ *and the polar hypercones of the opposite system of generators* m *on* σ *both envelop the same quartic hypersurface* Φ_1.

In the case of points lying on Γ these conics degenerate into two coincident straight lines, and therefore must be regarded as meeting ϖ in two coincident points. Hence,

Γ *lies on* Φ_1.

Since the polar hypercones of all the generators l of the same system of σ are contact hyperquadrics of Φ_1 (*i.e.* touch Φ_1 wherever they meet it, except at points on its double curve), and since Γ lies on Φ_1 we see from a different standpoint that *the polar hypercones* L *with respect to* SS' *of the lines* l *on* Σ *are* 8*-tangent hyperquadrics of* Γ.

Also, it is plain from the expression

$$\Phi_1 \equiv M^2-LL'=0,$$

that M passes through all the points of contact of Φ_1 with L, and hence of Γ with L. Thus again we see from another point of view that *a hyperquadric not containing* Γ *passes through the points of contact with* Γ *of the hypercones* L *and* L' *and hence that* L *and* L' *belong to the same system of* 8*-tangent hyperquadrics of* Γ.

6. *The double curve of* Φ_1.

The double curve of Φ_1, regarded as enveloped by the polar hypercones of the generators l of σ, is the curve common to all the polar hypercones of the generators l of the same system of σ. Let m, m' be two given generators of σ of the system opposite to l. Let the variable generator l meet m, m' in Q, Q' respectively. Let S, S' be two given hyperquadrics of the pencil. The polar hyperplanes of Q and Q' with respect to S and S' constitute four projective pencils of hyperplanes meeting in the vertex L_0 of the polar hypercone L of l. The axes of the pencils

of the hyperplanes are the polar planes of m and m' with respect to S and S'. The locus of L_0 is therefore a rational normal quartic curve C, the trisecant planes of which are the polar planes of the system of generators m with respect to the individual members of the pencil SS'. Also the polar plane of L_0 with respect to SS' contains l and hence meets m. Thus L_0 lies on the polar hypercone of m with respect to SS'.

The locus of the vertices of polar hypercones with respect to SS' of the generators l of one system of the quadric σ in which a hyperplane ϖ intersects Σ is a rational normal quartic curve C, lying wholly on Φ, of which the polar planes with respect to individual members of the pencil SS' of generators m on σ of the opposite system to l are trisecant planes. The curve C lies on all the polar hypercones of the generators m. The corresponding result holds on interchanging the generators l and m.

We now proceed to find the double curve of Φ_1, which is the common curve of intersection of all the polar hypercones of the generators l of σ and is of the eighth degree. This double curve is therefore the locus of a point P whose polar plane with respect to SS' cuts Σ in a conic intersecting all the generators l. This conic must therefore be (i) an undegenerate conic lying wholly on σ, or (ii) it must degenerate into a generator m and another line. P must therefore lie on one or other of two rational normal quartic curves, viz. (i) the rational normal quartic curve, which is the locus of the poles of ϖ with respect to the pencil SS'; or (ii) the rational normal quartic curve, which is the locus of the vertices of the polar hypercones of the generators m of σ.

The polar hypercones with respect to SS' of the generators l of σ each contain (i) *the rational normal quartic curve, which is the locus of the poles of ϖ with respect to the pencil SS'; and* (ii) *the rational normal quartic curve, which is the locus of the vertices of the polar hypercones of the generators m of σ. These two rational quartic curves together constitute the curve of the eighth degree, which is the double curve of Φ_1 regarded as enveloped by the polar hypercones of the generators l of σ.* Reasoning similarly, we see that *the rational normal quartic curve which is the locus of the vertices of the polar hypercones of the generators l of σ is also a double curve of Φ_1.*

Consider now the case when l and l' intersect. The points of contact of the polar hypercones of l and l' with Γ lie on the polar scrolls of π and π' respectively, as described in § 4. But both π and π' lie in the tangent hyperplane to Σ at the point ll', and hence have a line n in common, which touches Σ at ll'. The two scrolls thus both lie on the polar hypercone of n, which is thus the hyperquadric containing the points of contact of L and L'.

If l and l' be two intersecting lines lying wholly on Σ, and if π and π' be respectively the polar planes of l and l' with respect to Σ, the polar hypercone with respect to SS' of the line $\pi\pi'$ passes through the points of contact with Γ of the polar hypercones L and L' with respect to SS' of the lines l and l' respectively.

7. *General equation to the polar hypercones of lines on Σ.*

Take the equation to the hyperquadric Σ in the form

$$x^2+yt+zu=0,$$

and consider any hyperplane $\lambda x+\mu y+\nu z=0$ containing the line $x=y=z=0$. Let the polar hyperplanes of the point (1, 0, 0, 0, 0) with respect to S, S' be X, X' respectively, and similarly with respect to the other vertices of reference. The polar hypercone of $x=y=z=0$ will then be

$$TU'-T'U\equiv(TU)=0. \tag{6}$$

The general equation to lines on Σ which do not intersect the line $x=y=z=0$ are

$$\left.\begin{aligned}\lambda x+\mu y+\nu z&=0\\ \nu x+\rho y-\lambda u&=0\\ -\mu x+\rho z+\lambda t&=0\end{aligned}\right\}. \tag{7}$$

The polar hypercone of the line (7) is

$$\lambda^2(YZ)+\mu^2(ZT)+\nu^2(UY)+\rho^2(TU)+\mu\nu\{(ZU)+(TY)\}+\nu\lambda(XY)$$
$$+\lambda\mu(ZX)+\lambda\rho\{(TY)+(UZ)\}+\mu\rho(XT)+\nu\rho(XU)=0. \tag{8}$$

Changing ρ into $\rho+\rho'$ in (8) and taking the coefficient of ρ' in the equation thus obtained, we see that a hyperquadric through the points of contact of (6) and (8) with Γ is

$$\lambda\{(TY)+(UZ)\}+\mu(XT)+\nu(XU)+2\rho(TU)=0. \tag{9}$$

Thus (9), together with the net of hyperquadrics containing Γ, gives the most general linear system of hyperquadrics that can be made to pass through the 8 points of contact of (TU) with Γ, and therefore the points of contact of (8) with Γ include the most general family of points of contact of 8-tangent hyperquadrics with Γ to which (TU) belongs.

THE STRONG AND WEAK CONVERGENCE OF FUNCTIONS OF GENERAL TYPE

By J. C. BURKILL.

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1. *Introduction.* In his presidential address to the Society in 1924, Prof. W. H. Young referred to the concept of *super-summability* and its application in analysis. Prof. Young has developed this theme in a series of papers. In the latest of the series* he proves the main theorem of weak convergence for a sequence of functions $f_n(x)$ for which some function $Q(|f_n|)$ is integrable. In this paper I prove other theorems concerning strong and weak convergence of sequences of functions subject to the condition of a general type of super-summability†.

The enunciations may be compared with those given originally by F. Riesz‡ for the special case in which the function $Q(u)$ expressing the type of super-summability is a power of u. The proofs in the general case are, however, necessarily different. The simplest way of proving a result in the special case is usually to make a sufficient number of applications of Hölder's inequality. It is known that when we proceed from powers of u to a general function $Q(u)$ there is no direct extension of Hölder's inequality in the product form§; the corresponding inequality is, in fact, one of a different type due to Young.

For brevity I restrict the discussion to convergence in a finite interval.

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* "On successions with subsequences converging to an integral", *Proc. London Math. Soc.* (2), 24 (1925), 1-20.

† [*Added* 4 *October*, 1928.] It appears from a paper of Kaczmarz and Nikliborc, *Fundamenta Math.*, 11 (1928), 151–168, that the convergence of such sequences has been studied by Noaillon in an unpublished piece of work, which includes the result of the second part of Theorem 2 of this paper.

‡ "Untersuchungen über Systeme integrierbarer Funktionen", *Math. Annalen*, 69 (1910), 449-497. A convenient account is given by Hobson, *The theory of functions of a real variable*, 2 (1926), 239-254.

§ R. Cooper, *Proc. London Math. Soc.* (2), 26 (1927), 415-432 (431).

2. *Types of super-summability.* We write, for $u \geqslant 0$,

$$Q(u) = \int_0^u q(u)\,du \quad \text{where} \quad q(u) > 0,$$

so that $Q(u)$ is an increasing function.

If the restriction is made that $q(u)$ increases and tends to infinity with u, we say that Q is a Q_1. Thus, for sufficiently large u, a Q_1 increases more rapidly than any multiple of u.

LEMMA 1. *If Q is a Q_1, and $a \geqslant 1$, then $Q(au) \geqslant aQ(u)$.*

For
$$Q(u) = \int_0^u q(u)\,du \leqslant \frac{1}{a}\int_0^{au} q(u)\,du.$$

We shall find that some results hold only if $Q(u)$ is of order not greater than a power of u. A simple way of securing this property is to say that Q is a Q^1 (the upper suffix denoting an upper bound to the order of magnitude) if, for all u,

$$Q(2u) \leqslant kQ(u),$$

where k is independent of u.

It follows at once that

$$Q(u+v) \leqslant k\{Q(u)+Q(v)\}.$$

LEMMA 2. *If Q is a Q^1, then, for all u, a (>1), $Q(au) \leqslant Ca^pQ(u)$, where C, p are constants.*

Choose m so that
$$2^m \leqslant a < 2^{m+1}.$$

Then
$$\begin{aligned} Q(au) &\leqslant Q(2^{m+1}u) \leqslant k^{m+1}Q(u) \\ &\leqslant k \,.\, k^{\log a/\log 2}Q(u) \\ &= ka^{\log k/\log 2}Q(u). \end{aligned}$$

We shall say that $f(x)$—supposed measurable—is of class Q in (a, b) if

$$\int_a^b Q(|f|)\,dx$$

is finite. This integral will generally be written shortly as $\int Q|f|$.

The type of super-summability $R(v)$ conjugate to a given type $Q(u)$ was defined by Young*. He assumes that $q(u)$ and $r(v)$ are continuous

* "On classes of summable functions and their Fourier series", *Proc. Royal Soc.* (A), 87 (1912), 225-229.

increasing functions defined for $u \geqslant 0$, $v \geqslant 0$, which are inverse to one another and defines

$$Q(u) = \int_0^u q(u)\,du, \qquad R(v) = \int_0^v r(v)\,dv$$

(so that Q, R are of types Q_1, R_1 in our notation). Young then proves that

$$uv \leqslant Q(u) + R(v),$$

and hence that, if f, g are of classes Q, R, then fg is integrable and

$$\int fg \leqslant \int Q|f| + \int R|g|.$$

3. *Approximation by simple functions.*

THEOREM 1. *Let f be of class Q. Then, given ϵ, a function ψ, constant in stretches, can be found such that*

$$\int Q|f-\psi| < \epsilon.$$

From absolute continuity, given ϵ, we can find δ such that

$$\int_F Q|f| < \tfrac{1}{2}\epsilon \quad \text{if} \quad mF < \delta.$$

Let n be so large that the set e in which $|f| > n$ has measure less than $\frac{1}{2}\delta$.

Divide the interval $(-n, n)$ into $2n^2$ equal parts, and let e_r $(r = -n^2, \ldots, n^2-1)$ be the set in which

$$\frac{r}{n} \leqslant f < \frac{r+1}{n}.$$

For any η we can decompose the sets e, e_r, so that

$$\mathrm{e} = \mathcal{E} + \mathrm{e}' - \mathrm{e}'',$$

$$\mathrm{e}_r = \mathcal{E}_r + \mathrm{e}'_r - \mathrm{e}''_r,$$

where $\mathcal{E}$, $\mathcal{E}_r$ are sums of a finite number of intervals and $m\mathrm{e}'$, $m\mathrm{e}''$, $m\mathrm{e}'_r$, $m\mathrm{e}''_r$ are all less than η.

Let $\mathcal{E}'_r$ be the part of $\mathcal{E}_r$ which remains after the points common with $\mathcal{E}$ and other $\mathcal{E}_r$ have been removed (so that $\mathcal{E}'_r$ consists of a finite number of intervals).

Then $$m\mathcal{E} + \Sigma m\mathcal{E}'_r \geqslant b - a - (2n^2+1)^2\eta.$$

Define a function ψ equal to r/n in $\mathcal{E}'_r$ and equal to 0 elsewhere.

Then, in $\Sigma\mathcal{E}'_r$, $|f-\psi|<1/n$, except in a set of measure at most $2n^2\eta$, and, in this excepted set,

$$|f-\psi| \leqslant \max(|f|, n).$$

Therefore

$$\int_a^b Q|f-\psi| \leqslant (b-a)\,Q\left(\frac{1}{n}\right)+2n^2\eta Q(n)+\int_F Q|f|, \tag{3.1}$$

where

$$mF \leqslant \tfrac{1}{2}\delta+O(n^4\eta).$$

The right-hand side of (3.1) can be made less than ϵ by choice of n and then of η.

4. *Strong convergence.*

THEOREM 2. *If* $\int Q|f-f_n| \to 0$, *then*

$$\underline{\lim} \int Q|f_n| \geqslant \int Q|f|. \tag{1}$$

If, in addition, Q *is a* Q^1, *then*

$$\lim \int Q|f_n| = \int Q|f|. \tag{2}$$

Here $\int Q|f|$ may be infinite.

To prove (1) first suppose $\int Q|f|$ finite.

Given ϵ, we can find n_0 such that the set E_n in which $Q|f-f_n| \geqslant \epsilon$ has measure less than ϵ for $n \geqslant n_0$. Let F_n be the set in which $Q|f-f_n|<\epsilon$.

We can choose $k(\epsilon)$ and $\delta(\epsilon)$, where $k(\epsilon)\to\infty$ and $\delta(\epsilon)\to 0$ as $\epsilon\to 0$, such that, if

$$|h|<k, \quad |l|<k, \quad \text{and} \quad |h-l|<\epsilon,$$

then

$$|Q(h)-Q(l)|<\delta.$$

Let G_n be the part of F_n in which $|f| \geqslant k-\epsilon$. Then $mG_n \to 0$ as $\epsilon\to 0$.

We have now

$$\int_a^b (Q|f|-Q|f_n|) = \int_{F_n-G_n} (Q|f|-Q|f_n|)+\int_{E_n+G_n} Q|f|-\int_{E_n+G_n} Q|f_n|. \tag{4.1}$$

The first integral on the right-hand side is less than $\delta(b-a)$ in absolute value, and this tends to 0 with ϵ.

The second integral tends to 0 in virtue of absolute continuity.

Hence, making $\epsilon \to 0$, we have

$$\int_a^b Q|f| \leqslant \underline{\lim} \int_a^b Q|f_n|.$$

Suppose now that $\int Q|f|$ is infinite.

Given M, choose N so large that

$$\int_{E_N} Q|f| > M,$$

where E_N is the set in which $|f| < N$. The theorem for the finite case then gives

$$M \leqslant \underline{\lim} \int_{E_N} Q|f_n| \leqslant \underline{\lim} \int_a^b Q|f_n|.$$

Let $M \to \infty$ and we have the result.

Going on to (2), we have nothing to prove unless $\int Q|f|$ is finite. Assuming finiteness, in the second and third integrals on the right-hand side of (4 . 1) use the inequality

$$\big|Q|f|-Q|f_n|\big| \leqslant K(Q|f|+Q|f-f_n|).$$

Then the right-hand side of (4 . 1) tends to 0 with ϵ, and the theorem is proved.

If Q is not a Q^1, a sequence f_n can be found for which $\int Q|f-f_n| \to 0$, and

$$\lim \int Q|f_n| > \int Q|f|,$$

so that the hypothesis of (2) is necessary.

For if Q is not a Q^1, to any sequence l_n tending to infinity corresponds a sequence k_n tending to infinity such that

$$Q(2k_n) > l_n Q(k_n).$$

Let the interval (a, b) consist of intervals of lengths h_r $(r = 1, 2, \ldots)$ placed end to end.

Define $f = k_r$ in the interval h_r,

$f_n = f$ except in h_n,

and $f_n = 2f$ in h_n.

Then $$\int Qf = \Sigma h_r Q(k_r),$$

$$\int Q|f-f_n| = h_n Q(k_n),$$

and $$\int Qf_n = \int Qf - h_n Q(k_n) + h_n Q(2k_n)$$

$$> l_n h_n Q(k_n).$$

Choose (say) $l_n = n^3$ and $h_n = 1/n^2 Q(k_n)$, and we have a sequence having the desired property.

THEOREM 3. *If* $$\int_a^b Q|f_m - f_n| \to 0,$$

as $m \to \infty$, $n \to \infty$, *then there exists a subsequence* f_ν *of the* f_n *converging almost everywhere to a function* f *such that*

$$\int_a^b Q|f-f_n| \to 0.$$

Two such functions f *can differ only in a set of measure zero.*

Given ϵ, we can find n_0 such that, for $m \geqslant n_0$, $n \geqslant n_0$,

$$\int_a^b Q|f_m - f_n| < \epsilon Q(\epsilon).$$

Hence the set of points in which $|f_m - f_n| \geqslant \epsilon$ has measure less than ϵ. That is to say, the sequence f_n is convergent on the average.

Applying a known theorem for such sequences* we see that there is a subsequence f_ν which converges almost everywhere to a function f, and that two limit functions f can differ only in a set of measure zero.

If $n \geqslant n_0$ we have, from Fatou's lemma on the integrals of a sequence of positive functions,

$$\int Q|f-f_n| = \int \lim_\nu Q|f_\nu - f_n|$$

$$\leqslant \underline{\lim} \int Q|f_\nu - f_n|$$

$$\leqslant \epsilon Q(\epsilon).$$

Therefore $$\int Q|f-f_n| \to 0 \quad \text{as} \quad n \to \infty.$$

* Hobson, *The theory of functions of a real variable*, 2 (1926), 242.

5. *Weak convergence.*

THEOREM 4. *If* (1) $\int Q|f_n| < K$ *for all* n, *where* Q *is a* Q_1; (2) $\int R|g_n| \to 0$ *as* $n \to \infty$, *where* R *(the conjugate of* Q*) is an* R^1, *then*

$$\int f_n g_n \to 0.$$

We have
$$\int |f_n g_n| \leqslant \int Q|f_n/a| + \int R|g_n a|$$
$$\leqslant \frac{1}{a}\int Q|f_n| + Ca^p \int R|g_n|,$$

where $a > 1$, from Lemmas 1 and 2.

Given ϵ, choose n_0 so that $\int R|g_n| < \epsilon$ for $n \geqslant n_0$.

Choosing a to make the expression $(K/a) + Ca^p\epsilon$ a minimum, we have

$$\int |f_n g_n| \leqslant C_1 \epsilon^{1/p+1},$$

and hence the result.

THEOREM 5. *If*

(1) $\int Q|f_n| \leqslant K$, *where* Q *is a* Q_1;

(2) $\int_a^x f_n \to \int_a^x f$ *as* $n \to \infty$, *for every* x *in* (a, b) [*so that* f_n *converges weakly* (Q) *to* f];

(3) $\int R|g|$ *is finite, where* R *(the conjugate of* Q*) is an* R^1;

then
$$\int_a^b f_n g \to \int_a^b fg.$$

From (1) and (2) it follows, by Young's theorem*, that $\int Q|f|$ is finite. Hence $\int fg$ is finite.

By Theorem 1 we can find ψ, constant in stretches, such that

$$\int R|g - \psi| < \epsilon.$$

Now
$$\left|\int f_n g - \int fg\right| \leqslant \left|\int (f_n - f)\psi\right| + \left|\int f_n(g-\psi)\right| + \left|\int f(g-\psi)\right|.$$

From Theorem 4, the last two integrals can be made arbitrarily small by choice of ψ (independent of n).

* "On successions with subsequences converging to an integral", *Proc. London Math. Soc.* (2), 24 (1925), 1–20 (3).

Since ψ is constant in stretches, the hypothesis (2) shows that the first integral on the right-hand side tends to 0 as n tends to infinity, and the theorem is proved.

We add an example illustrating the failure of the conclusion of the theorem if the condition that R is an R^1 is omitted.

Take the inverse functions to be

$$q(u) = \log(u+1), \quad r(v) = e^v - 1,$$

so that $$Q(u) = (u+1)\log(u+1) - u, \quad R(v) = e^v - v - 1.$$

Take (a, b) to consist of intervals of lengths h_r $(r = 1, 2, \ldots)$ placed end to end.

Define $$f_n = k_n \text{ in } h_n \text{ and } = 0 \text{ elsewhere},$$

$$g = l_r \text{ in } h_r \quad (r = 1, 2, \ldots).$$

If we now take

$$h_r = 1/r^3 \log(r+1), \quad k_r = r^3, \quad l_r = \log(r+1),$$

it is easy to see that $f_n \to 0$ weakly (Q) and $\int R(g)$ is finite, but $\int f_n g \to 1$.

THE DERIVED DIRECTIONS OF A VECTOR DEFINED ALONG A CURVE IN RIEMANNIAN SPACE

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1. *Introduction.*

1.1. *Summary.* The object of this paper is to construct (by a limiting process) the derived directions of a vector defined along a curve in Riemannian space. A simple constructive process for the $N-1$ principal normals to such a vector is also given, from which the Frenet formulae follow immediately. Expressions for the curvatures in terms of the rates of turning of the vector and normals are also given.

1.2. *Line-element; indicator; magnitude of a vector.* We consider a manifold of N dimensions, whose metric is given by

$$ds^2 = e g_{mn}\, dx^m\, dx^n, \tag{1.21}$$

where e is plus or minus one so that the right-hand side may be positive, unless it be zero.

If X^r is a vector, the *magnitude* of X^r is X, where

$$X^2 = e_X g_{mn} X^m X^n, \quad X \geqslant 0, \quad e_X^2 = 1. \tag{1.22}$$

e_X has been called by Synge* the *indicator* of the vector X^r. When X^r is a *unit* vector, we have

$$g_{mn} X^m X^n = e_X. \tag{1.23}$$

If $X = 0$, X^r is called a *null-vector*†, and we see that all null-vectors lie on the elementary *null-cone*,

$$g_{mn}\, dx^m\, dx^n = 0. \tag{1.24}$$

* Synge, "Normals and curvatures of a curve in the Riemannian manifold", *Proc. Int. Math. Congress* (Toronto), 1924.

† Provided that at least one of the components X^r is not zero.

Also the null-cone separates vectors of different indicators. If the line-element is positive definite, the null-cone is imaginary and all indicators are plus one.

1.3. *Perpendicularity; angle between two vectors.* Two vectors are said to be *perpendicular* when

$$g_{mn}X^m Y^n = 0. \tag{1.31}$$

There is some difficulty in defining the angle between any two vectors in the case of a non-definite line-element, and we confine our attention to vectors X^r, Y^r, which are such that one can be transformed continuously into the other without its magnitude vanishing, and without leaving the original linear two-space (X^r, Y^r). Two such vectors may be called *reconcilable*, and the condition for reconcilability may be expressed by the condition that the vector AX^r+BY^r is not null for any positive values of A and B. Two reconcilable vectors have the same indicator.

We have to distinguish two cases:

(*a*) The two-space defined by X^r, Y^r cuts the null-cone in imaginary vectors. It is easily seen that all vectors in the two-space have the same indicator.

(*b*) The two-space cuts the null-cone in real vectors.

The intersections of the linear two-space (X^r, Y^r), with the null-cone are AX^r+BY^r, where A, B satisfy

$$g_{mn}(AX^m+BY^m)(AX^n+BY^n) = 0, \tag{1.32}$$

or

$$eA^2X^2+2ABg_{mn}X^m Y^n+eB^2Y^2 = 0, \tag{1.33}$$

e being the common indicator of X^r and Y^r. In case (*a*) the roots are imaginary; in case (*b*) they are real. Thus we have

$$(g_{mn}X^m Y^n)^2 < X^2 Y^2, \tag{1.34a}$$

$$(g_{mn}X^m Y^n)^2 \geqslant X^2 Y^2, \tag{1.34b}$$

according to the case, the sign of equality occurring when (X^r, Y^r) touches the null-cone. Thus in case (*a*) a real angle θ is defined by

$$\cos\theta = \frac{eg_{mn}X^m Y^n}{XY}. \tag{1.35a}$$

In case (b), it is evident from (1.33) that $g_{mn}X^mY^n$ cannot vanish for any pair of reconcilable vectors. Thus $eg_{mn}X^mY^n$ retains the same sign as Y^r is transformed into X^r. But, when $Y^r = X^r$,

$$eg_{mn}X^mY^n = X^2 > 0;$$

that is, $eg_{mn}X^mY^n$ is positive, and therefore, by (1.34 b), a real angle θ is defined by*

$$\cosh\theta = \frac{eg_{mn}X^mY^n}{XY}. \tag{1.35 b}$$

We observe that, if a linear two-space touches the null-cone, the angle between two reconcilable vectors in the two-space is zero.

Let λ^r, μ^r be two reconcilable unit vectors at a point O, whose coordinates are x^r and let θ be the angle between them. Along any two curves touching λ^r, μ^r at O let us take small equal distances u, thus giving the points P and Q. We shall denote the small vector $x^r_P - x^r_Q$ by $\overline{QP}$, and its magnitude by QP.

We have
$$x^r_P = x^r + \lambda^r u + O(u^2),$$
$$x^r_Q = x^r + \mu^r u + O(u^2).$$

Therefore

$$\lim_{u\to 0}\frac{\overline{QP}}{u} = \lim_{u\to 0}\frac{(x^r_P - x^r_Q)}{u} = \lambda^r - \mu^r, \tag{1.36}$$

and
$$\lim_{u\to 0}\left(\frac{QP}{u}\right)^2 = |\,g_{mn}(\lambda^m-\mu^m)(\lambda^n-\mu^n)\,| = 2\,|\,e - g_{mn}\lambda^m\mu^n\,|.$$

This last expression is equal to $2(1-\cos\theta)$ or $2(\cosh\theta - 1)$, according as case (a) or (b) occurs. Therefore

$$\lim_{u\to 0}\frac{QP}{u} = 2\sin\tfrac{1}{2}\theta, \tag{1.37 a}$$

or

$$\lim_{u\to 0}\frac{QP}{u} = 2\sinh\tfrac{1}{2}\theta, \tag{1.37 b}$$

according as the two-space of λ^r, μ^r cuts the null-cone in imaginary or real vectors.

1.4. *The derived directions of a vector defined along a curve; parallel propagation of a vector.* If we are given a contravariant vector λ^r, defined

* Cf. Mair, *Fourfold geometry* (1926), 146, where the angle between two time-like vectors in space time is defined by means of the hyperbolic cosine.

along a curve in terms of the arc s of the curve, we know that the quantities defined by the series of equations

$$(1.41) \qquad \frac{\delta^{a+1}\lambda^r}{\delta s^{a+1}} = \frac{d}{ds}\left(\frac{\delta^a\lambda^r}{\delta s^a}\right) + \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} \frac{\delta^a\lambda^m}{\delta s^a}\frac{dx^n}{ds}, \quad \frac{\delta^0\lambda^r}{\delta s^0} \equiv \lambda^r \quad (a \geqslant 0),$$

are all contravariant vectors*. With Levi-Civita we shall call them the successive *derived vectors* of λ^r.

The vector λ^r is said to be propagated parallelly along the curve when it satisfies the differential equations

$$(1.42) \qquad \frac{\delta\lambda^r}{\delta s} = 0$$

at every point of the curve.

Now it is a well known result that, if X^r, Y^r are any two vectors defined parallelly along a curve, then $g_{mn}X^mY^n$ is constant along the curve. In particular, $g_{mn}X^mX^n$, which equals e_XX^2, is constant along the curve, and consequently the magnitude and indicator of a vector defined parallelly is the same at all points of the curve. If X^r, Y^r are reconcilable at one point of the curve, they are reconcilable at all points, and the angle between them is constant.

2. *Constructions for the derived directions of a vector.*

2.1. *First derived direction.* Let λ^r be a unit vector defined (not parallelly) along a curve C in the neighbourhood of a point O of C. Let us take the point O' at the distance s on the curve from O. We denote by λ'^r the value of the defined vector at O', and by λ''^r the parallel at O' to λ^r at O. λ''^r will also be a unit vector by (1.4).

Now along any two curves touching λ'^r and λ''^r at O' let us take small equal distances u, giving the points R and Q. Denoting the infinitesimal vector $x_R^r - x_Q^r$ by $\overline{QR}$, and its magnitude by QR, we have, from (1.36),

$$(2.11) \qquad \lim_{u\to 0} \frac{\overline{QR}}{u} = \lambda'^r - \lambda''^r.$$

But

$$\lambda'^r = \lambda^r + \frac{d\lambda^r}{ds}\, s + O(s^2),$$

$$\lambda''^r = \lambda^r - \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} \lambda^m \frac{dx^n}{ds}\, s + O(s^2),$$

* Bianchi, "Sul parallelismo vincolato di Levi-Civita nella metrica degli spazi curvi", *Rend. R. Acc. Napoli*, 28 (1922), 150–171; Levi-Civita, *The absolute differential calculus* (1927), 139–140.

and hence, letting s tend to zero,

$$\lim_{s\to 0}\frac{1}{s}\left(\lim_{u\to 0}\frac{\overline{QR}}{u}\right)=\frac{\delta\lambda^r}{\delta s}. \tag{2.12}$$

Thus the infinitesimal vector, obtained by the above construction, tends to the direction of the derived vector of λ^r as u and s tend to zero. This result gives us a construction for the first derived direction.

Denoting the magnitude of the first derived vector by $\delta\lambda/\delta s$, equation (2.12) gives us

$$\lim_{s\to 0}\frac{1}{s}\left(\lim_{u\to 0}\frac{QR}{u}\right)=\frac{\delta\lambda}{\delta s}. \tag{2.13}$$

Since λ'^r, λ''^r only differ infinitesimally, they are reconcilable, and we have

$$\lim_{u\to 0}\frac{QR}{u}=2\sin\tfrac{1}{2}\theta, \tag{2.14a}$$

or

$$\lim_{u\to 0}\frac{QR}{u}=2\sinh\tfrac{1}{2}\theta. \tag{2.14b}$$

Hence, by (2.13),

$$\lim_{s\to 0}\left(\frac{\theta}{s}\right)=\frac{\delta\lambda}{\delta s}, \tag{2.15}$$

which gives a value for the magnitude of the first derived vector of λ^r.

2.2. *Case of failure of the construction.* The above construction will fail if $\delta\lambda^r/\delta s=0$ at the point O, for then $\lim_{u\to 0}\overline{QR}/u$ will be of the order s^2 at least.

We shall now prove that, if all the quantities

$$\frac{\delta^a\lambda'}{\delta s^a}\quad(a=1,2,\ldots,P-1),$$

vanish at the point O, then $\lim_{u\to 0}\overline{QR}/u$ will be of order s^P, and

$$\lim_{s\to 0}\frac{1}{s^P}\left(\lim_{u\to 0}\frac{\overline{QR}}{u}\right)=\frac{1}{P!}\frac{\delta^P\lambda^r}{\delta s^P}. \tag{2.21}$$

In general, let us suppose that we have two unit vectors λ^r, $\bar\lambda^r$ defined along the curve C. Let λ'^r and $\bar\lambda'^r$ be their values at O', which is distant s from O. We shall take points R and Q, as before, at small distances

u along curves touching these vectors at O. From (1 . 36),

$$\lim_{u\to 0}\frac{\overline{QR}}{u}=\lambda'^{r}-\bar{\lambda}'^{r}. \tag{2 . 22}$$

Therefore, if at O

$$\lambda^{r}=\bar{\lambda}^{r},\quad \frac{d^{\alpha}\lambda^{r}}{ds^{\alpha}}=\frac{d^{\alpha}\bar{\lambda}^{r}}{ds^{\alpha}}\quad(\alpha=1,\,2,\,\ldots,\,P-1), \tag{2 . 23}$$

we easily see that

$$\lim_{s\to 0}\frac{1}{s^{P}}\left(\lim_{u\to 0}\frac{\overline{QR}}{u}\right)=\frac{1}{P!}\left(\frac{d^{P}\lambda^{r}}{ds^{P}}-\frac{d^{P}\bar{\lambda}^{r}}{ds^{P}}\right). \tag{2 . 24}$$

Now

$$\frac{\delta\lambda^{r}}{\delta s}=\frac{d\lambda^{r}}{ds}+\left\{\begin{matrix} r \\ mn\end{matrix}\right\}\lambda^{m}\frac{dx^{n}}{ds}=\frac{d\lambda^{r}}{ds}+F_{1}^{r}\left(\lambda,\ x,\ \frac{dx}{ds}\right),$$

where, for example, x means that F_1^r is a function of all the quantities x^r. Differentiating these equations successively, we can prove by induction that

$$\frac{\delta^{\alpha}\lambda^{r}}{\delta s^{\alpha}}=\frac{d^{\alpha}\lambda^{r}}{ds^{\alpha}}+F_{\alpha}^{r}\left(\lambda,\ \frac{d\lambda}{ds},\ \ldots,\ \frac{d^{\alpha-1}\lambda}{ds^{\alpha-1}},\ x,\ \frac{dx}{ds},\ \ldots,\ \frac{d^{\alpha}x}{ds^{\alpha}}\right), \tag{2 . 25}$$

where α is any positive integer.

If equations (2 . 23) hold at O, we deduce from (2 . 25) that

$$\lambda^{r}=\bar{\lambda}^{r},\quad \frac{\delta^{\alpha}\lambda^{r}}{\delta s^{\alpha}}=\frac{\delta^{\alpha}\bar{\lambda}^{r}}{\delta s^{\alpha}}\quad(\alpha=1,\,2,\,\ldots,\,P-1), \tag{2 . 26}$$

at the point O, and

$$\frac{d^{P}\lambda^{r}}{ds^{P}}-\frac{d^{P}\bar{\lambda}^{r}}{ds^{P}}=\frac{\delta^{P}\lambda^{r}}{\delta s^{P}}-\frac{\delta^{P}\bar{\lambda}^{r}}{\delta s^{P}}. \tag{2 . 27}$$

Conversely, if equations (2 . 26) hold at O, so also do (2 . 23) and (2 . 27), and (2 . 24) gives

$$\lim_{s\to 0}\frac{1}{s^{P}}\left(\lim_{u\to 0}\frac{\overline{QR}}{u}\right)=\frac{1}{P!}\left(\frac{\delta^{P}\lambda^{r}}{\delta s^{P}}-\frac{\delta^{P}\bar{\lambda}^{r}}{\delta s^{P}}\right). \tag{2 . 28}$$

Let us suppose that the first $P-1$ derived vectors of λ^r all vanish at O, and let us define $\bar{\lambda}^r$ as the parallel propagation of λ^r at O along the curve. Then, obviously, (2 . 26) hold at O, and

$$\lim_{s\to 0}\frac{1}{s^{P}}\left(\lim_{u\to 0}\frac{\overline{QR}}{u}\right)=\frac{1}{P!}\frac{\delta^{P}\lambda^{r}}{\delta s^{P}},$$

which proves our theorem.

Also, if θ be the angle between λ'^r and $\bar{\lambda}'^r$, and $\delta^P\lambda/\delta s^P$ be the magnitude of the P-th derived vector of λ^r, we have

$$\lim_{s\to 0}\frac{\theta}{s^P}=\frac{1}{P!}\frac{\delta^P\lambda}{\delta s^P}. \tag{2.29}$$

2.3. *Constructions for the higher derived vectors.* We can easily modify the construction in (2.2) in order to obtain the derived vectors of any order. Let us suppose that the two vectors λ^r, $\bar{\lambda}^r$ defined along C are no longer unit vectors, and let their magnitudes at O' be λ', $\bar{\lambda}'$. If we take small distances, $\lambda' u$ and $\bar{\lambda}' u$, along any two curves touching λ'^r and $\bar{\lambda}'^r$ at O, thus giving the points R and Q, we find that the equations (2.22) still hold, that is,

$$\lim_{u\to 0}\frac{\overline{QR}}{u}=\lambda'^r-\bar{\lambda}'^r. \tag{2.31}$$

Also, by the same reasoning as before, if (2.26) hold at O, then equations (2.28) will still be true.

Let us now subject the vector $\bar{\lambda}^r$ to the conditions that (2.26) is to be true at the point O, and that all along C the vector satisfies the equations

$$\frac{\delta^P\bar{\lambda}^r}{\delta s'^P}=0. \tag{2.32}$$

It follows from the theory of differential equations that these conditions determine $\bar{\lambda}^r$ completely. When we apply the construction of this section to obtain the points R and Q, we have, from (2.32) and (2.28),

$$\lim_{s\to 0}\frac{1}{s^P}\left(\lim_{u\to 0}\frac{\overline{QR}}{u}\right)=\frac{1}{P!}\frac{\delta^P\lambda^r}{\delta s^P}, \tag{2.33}$$

which shows that the infinitesimal vector $\overline{QR}$ tends to the direction of the P-th derived vector of λ^r as s and u tend to zero.

3. *Constructions for the principal normals of a vector along a curve.*

3.1. *The principal normals and curvatures of a vector; the Frenet formulae of a curve.* We shall now give a simple method by which the principal normals and curvatures of a vector defined along a curve may be obtained. The argument is less formal than those previously given*, and shows how we are naturally led from one normal to the next.

* Eisenhart, *Riemannian geometry* (1926), 103–107 ; Blaschke, "Frenet's Formeln für den Raum von Riemann", *Math. Zeitschrift*, 6 (1920), 94-99 ; Synge, *loc. cit.* The last two papers treat only the particular case where the defined vector is the tangent of the curve, Blaschke for a positive definite line-element and Synge for a non-definite line-element.

Let the unit vector, defined arbitrarily along the curve, be λ_0^r with indicator e_0. Our object is to define $N-1$ new unit vectors λ_α^r with indicator e_α $(\alpha = 1, 2, \ldots, N-1)$, which are orthogonal to λ_0^r and to each other; also to define $N-1$ curvatures κ_α $(\alpha = 1, 2, \ldots, N-1)$ associated with these vectors.

Now let us suppose that we have in some way obtained these vectors up to λ_α^r. Section (2.1) gives us a construction for $\delta\lambda_\alpha^r/\delta s$, the first derived vector of λ_α^r, and the set of vectors $\delta\lambda_\alpha^r/\delta s$, λ_α^r, $\lambda_{\alpha-1}^r$, ..., λ_0^r define a linear $(\alpha+2)$-space. Let us define $\lambda_{\alpha+1}^r$ as the unit vector in this space which is normal to λ_α^r, $\lambda_{\alpha-1}^r$, ..., λ_0^r and lies on the same side of the $(\alpha+1)$-space defined by these vectors as $\delta\lambda_\alpha^r/\delta s$. This process determines the required set of vectors (and, of course, their indicators), their mutual orthogonality being assured by the process.

To obtain the formulae connecting them, we have

$$\kappa_{\alpha+1}\lambda_{\alpha+1}^r = \frac{\delta\lambda_\alpha^r}{\delta s} + A_{\alpha,\alpha}\lambda_\alpha^r + A_{\alpha,\alpha-1}\lambda_{\alpha-1}^r + \ldots + A_{\alpha,0}\lambda_0^r \quad (\alpha = 0, 1, \ldots, N-2), \tag{3.11}$$

where the κ's are positive, and are defined to be the *curvatures*. We may also add the value $\alpha = N-1$, with $\kappa_N = 0$, since $\delta\lambda_{N-1}^r/\delta s$ must be linearly dependent on λ_0^r, λ_1^r, ..., λ_{N-1}^r. To find the coefficients A we multiply across by $g_{rs}\lambda_\beta^s$ $(\beta = 0, 1, \ldots, \alpha)$, and obtain

$$0 = g_{rs}\frac{\delta\lambda_\alpha^r}{\delta s}\lambda_\beta^s + A_{\alpha,\beta}e_\beta, \tag{3.12}$$

or

$$\begin{aligned} e_\beta A_{\alpha,\beta} &= g_{rs}\frac{\delta\lambda_\beta^r}{\delta s}\lambda_\alpha^s \\ &= g_{rs}\lambda_\alpha^s(\kappa_{\beta+1}\lambda_{\beta+1}^r - A_{\beta,\beta}\lambda_\beta^r - A_{\beta,\beta-1}\lambda_{\beta-1}^r - \ldots - A_{\beta,0}\lambda_0^r), \end{aligned} \tag{3.13}$$

Thus

$$\begin{cases} A_{\alpha,\alpha}e_\alpha = -A_{\alpha,\alpha}e_\alpha = 0 \\ A_{\alpha,\alpha-1}e_{\alpha-1} = e_\alpha\kappa_\alpha \quad (\alpha = 1, 2, \ldots, N-1), \\ A_{\alpha,\beta}e_\beta = 0 \quad (\beta = 0, 1, \ldots, \alpha-2). \end{cases} \tag{3.14}$$

Equations (3.11) may therefore be written

$$\frac{\delta\lambda_\alpha^r}{\delta s} = \kappa_{\alpha+1}\lambda_{\alpha+1}^r - e_{\alpha-1}e_\alpha\kappa_\alpha\lambda_{\alpha-1}^r, \quad \kappa_0 = \kappa_N = 0 \quad (\alpha = 0, 1, \ldots, N-1). \tag{3.15}$$

When λ_0^r is the tangent to the curve, these are the Frenet formulae.

3.2. *Values for the curvatures.* The curvatures have been defined in (3.1) by an analytical process. We shall now show how they may be expressed in terms of "rates of turning" of the normals.

Let $d\theta_P$ denote the infinitesimal angle between λ_P^r and its parallel, corresponding to a change ds along the curve. Then, from equation (2.15),

$$\frac{d\theta_P}{ds} = \frac{\delta\lambda_P}{\delta s}, \tag{3.21}$$

where $\delta\lambda_P/\delta s$ is the magnitude of the first derived vector of λ_P^r. Hence

$$\left(\frac{d\theta_P}{ds}\right)^2 = e(e_{P+1}\kappa_{P+1}^2 + e_{P-1}\kappa_P^2), \tag{3.22}$$

e being chosen to make the right-hand side positive. Knowing κ_P we have a value for κ_{P+1}, so that we obtain successive values for the curvatures.

For $P=0$ and $P=N-1$, the formula (3.22) becomes

$$\frac{d\theta_0}{ds} = \kappa_1, \quad \frac{d\theta_{N-1}}{ds} = \kappa_{N-1}. \tag{3.23}$$

In Euclidean three-space these are the well known formulae for the curvature and torsion of a curve in terms of the variations of the angles which the tangent and binormal make with fixed directions. The formula (3.23) for κ_1 has been given by J. Lipka*.

For the case of a positive definite line-element, all indicators are plus one, and (3.22) becomes†

$$\left(\frac{d\theta_P}{ds}\right)^2 = \kappa_{P+1}^2 + \kappa_P^2. \tag{3.24}$$

The general value for κ is therefore

$$\kappa_P^2 = (-1)^{P-1}\sum_{\alpha=0}^{P-1}(-1)^\alpha\left(\frac{d\theta_\alpha}{ds}\right)^2. \tag{3.25}$$

Combining this formula for $P=N-1$ with the second one of (3.23), we have the identical relation

$$\sum_{\alpha=0}^{N-1}(-1)^\alpha\left(\frac{d\theta_\alpha}{ds}\right)^2 = 0. \tag{3.26}$$

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* J. Lipka, "Sulla curvatura geodetica delle linee appartenenti ad una varietà qualunque", *Rend. R. Acc. Lincei*, 31 (1922), 353–356.

† Cf. Cesàro, *Geometrica intrinseca*, p. 230, for the particular case of an E_N.

THE CONTACT OF CURVES IN RIEMANNIAN SPACE

By A. J. McCONNELL.

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1. *Introduction.*

1.1. *Summary and notation.* In this paper we first consider the intrinsic equations of a curve in a Riemannian manifold of N dimensions. Then we define contact of any order of two curves at a point and obtain necessary and sufficient conditions that two curves should have contact of any order. Finally certain curves associated with a given curve are defined and, by means of them, constructions are given for the principal normals of a curve.

The notation to be used is that of the previous paper on *The derived directions of a vector defined along a curve.*

2. *Intrinsic equations of a curve.*

2.1. We first prove the following theorem :

THEOREM I. *There is one and only one curve having assigned tangent and principal normals at a given point and having its curvatures given functions of the arc.*

The equations connecting the coordinates, curvatures, tangent, and principal normals of a curve are*

$$\frac{dx^r}{ds} = \lambda_0^r,$$

$$\frac{\delta\lambda_a^r}{\delta s} = \kappa_{a+1}\lambda_{a+1}^r - e_{a-1}e_a\kappa_a\lambda_{a-1}^r, \quad \kappa_0 = \kappa_N = 0 \tag{2.11}$$

$$(a = 0, 1, 2, \ldots, N-1).$$

* Cf. Equation (3.15) of the paper cited.

Now we assume that at no point of the curves discussed will any normal be a null-direction, so that by continuity the indicators e_α of these normals will be the same at all points of a curve. Therefore, if the indicators of the normals are given at any one point of a curve, they are also given at every point of the curve.

The equations (2 . 11) form a set of differential equations in the dependent variables

$$x^r,\ \lambda_\alpha^r \quad (\alpha = 0, 1, 2, \ldots, N-1), \tag{2 . 12}$$

the independent variable being s. Let us suppose that we are given the values of the variables (2 . 12) for a given value of s, say $s = 0$, the vectors forming an orthogonal N-uple of unit vectors at the given point. We are therefore also given the indicators e. In these circumstances, the equations (2 . 11) have one and only one set of solutions reducing to the required values for $s = 0$.

Let us take the unique solutions,

$$x^r = x^r(s), \tag{2 . 13}$$

of these equations. We have now to show that the curve represented by (2 . 13) is one having the assigned curvatures.

We shall first prove that the λ vectors, which are the solutions of (2 . 11), form an orthogonal N-uple of unit vectors at every point of the curve. Let us use temporarily the notation

$$(\alpha, \beta) = g_{mn}\lambda_\alpha^m\lambda_\beta^n \quad (\alpha, \beta = 0, 1, 2, \ldots, N-1). \tag{2 . 14}$$

Then, since the vectors form an orthogonal N-uple at the initial point O, these quantities have the following values at this point :

$$(\alpha, \alpha) = e_\alpha, \quad (\alpha, \beta) = 0, \quad (\alpha \neq \beta), \quad (\alpha, \beta = 0, 1, 2, \ldots, N-1). \tag{2 . 15}$$

Also, as the quantities λ_α^r satisfy the equations (2 . 11), we easily see that the variables (2 . 14) satisfy the differential equations

$$\frac{d}{ds}(\alpha, \beta) = \kappa_{\alpha+1}(\alpha+1, \beta) + \kappa_{\beta+1}(\alpha, \beta+1) - e_{\alpha-1}e_\alpha\kappa_\alpha(\alpha-1, \beta) - e_{\beta-1}e_\beta\kappa_\beta(\alpha, \beta-1). \tag{2 . 16}$$

These have a unique set of solutions, reducing to the values (2 . 15) at O, and it is seen immediately that the unique solutions are exactly the values (2 . 15) for all values of s, that is, the λ vectors form an orthogonal N-uple at every point of the curve.

In particular, since λ_0^r is a unit vector at all points of the curve, we conclude that the parameter s of the curve (2 . 13) is the arc of the curve,

and hence (2 . 11) are its Frenet formulae. It has therefore the assigned curvatures, and we have proved our theorem.

We may say that, being given the tangent and principal normals of a curve at a point is equivalent to being given its orientation in space at the point. Hence, when the curvatures of a curve are given functions of the arc, the curve is determined uniquely by its orientation at the initial point.

The same reasoning holds if all along the curve $\kappa_P = 0$ $(P < N)$, and we are given the lower curvatures. In this case the set of differential equations ends with the equations

$$\frac{\delta \lambda^r_{P-1}}{\delta s} = -e_{P-2} e_{P-1} \kappa_{P-1} \lambda^r_{P-2}.$$

These are exactly of the same type as before, but with fewer dependent variables, and the curve exists and is determined uniquely when we are given the tangent and the first $P-1$ principal normals at the initial point, the remaining normals being indeterminate.

3. *The contact of curves.*

3 . 1. *Definition and conditions for contact.* Let us take two curves C, $\bar{C}$ having the point O in common, and let us take points R, Q at equal arc lengths s along them. When s is small, $x^r_R - x^r_Q$ will be the components of an infinitesimal vector. We shall denote this vector by $\overline{QR}$ and its magnitude by QR.

If QR is an infinitesimal of order s^{P+1} as s tends to zero, then the curves are said to have contact of order P.

Let x^r be the coordinates of O, and let bars refer to the curve $\bar{C}$. Then

$$(3 . 11) \qquad \overline{QR} = \left(\frac{dx^r}{ds} - \frac{d\bar{x}^r}{ds}\right) s + \ldots$$

$$\ldots + \frac{1}{(P+1)!}\left(\frac{d^{P+1}x^r}{ds^{P+1}} - \frac{d^{P+1}\bar{x}^r}{ds^{P+1}}\right) s^{P+1} + O(s^{P+2}).$$

Hence a necessary and sufficient condition for the two curves to have contact of order P is that the values of

$$(3 . 12) \qquad \frac{d^\alpha x^r}{ds^\alpha} \quad (\alpha = 1, 2, \ldots, P)$$

should be the same for the two curves at O, the point of contact; and then

$$\lim_{s\to 0}\frac{\overline{QR}}{s^{P+1}}=\frac{1}{(P+1)!}\left\{\frac{d^{P+1}x^r}{ds^{P+1}}-\frac{d^{P+1}\bar{x}^r}{ds^{P+1}}\right\}. \tag{3.13}$$

We proceed to put this condition into tensor form. We have

$$\lambda^r=\frac{dx^r}{ds},$$

$$\frac{\delta\lambda^r}{\delta s}=\frac{d^2x^r}{ds^2}+\left\{\begin{matrix} r \\ mn\end{matrix}\right\}\frac{dx^m}{ds}\frac{dx^n}{ds}=\frac{d^2x^r}{ds^2}+f_1^r\left(x,\frac{dx}{ds}\right),$$

and so on. We can easily prove by induction that

$$\frac{\delta^\alpha\lambda^r}{\delta s^\alpha}=\frac{d^{\alpha+1}x^r}{ds^{\alpha+1}}+f_\alpha^r\left(x,\frac{dx}{ds},\ldots,\frac{d^\alpha x}{ds^\alpha}\right), \tag{3.14}$$

where f_α^r is independent in form of the particular curve at the point. From the equations (3.14) we conclude that, if the quantities (3.12) are the same for the two curves of O, then the quantities

$$\lambda^r,\quad \frac{\delta^\beta\lambda^r}{\delta s^\beta}\quad(\beta=1,2,\ldots,P-1), \tag{3.15}$$

have the same values for the curves at O, and then also

$$\left\{\frac{d^{P+1}x^r}{ds^{P+1}}-\frac{d^{P+1}\bar{x}^r}{ds^{P+1}}\right\}=\left\{\frac{\delta^P\lambda^r}{\delta s^P}-\frac{\delta^P\bar{\lambda}^r}{\delta s^P}\right\}. \tag{3.16}$$

Conversely, if the quantities (3.15) have the same values for the two curves at O, this is also true of the quantities (3.12), and equations (3.16) still hold.

Therefore we have the result:

THEOREM II. *A necessary and sufficient condition for two curves to have contact of order P at a point O is that the quantities*

$$\lambda^r,\quad \frac{\delta^\alpha\lambda^r}{\delta s^\alpha}\quad(\alpha=1,2,\ldots,P-1) \tag{3.17}$$

should have the same values for the two curves at O, where λ^r is the unit tangent vector; and then

$$\lim_{s\to 0}\frac{\overline{QR}}{s^{P+1}}=\frac{1}{(P+1)!}\left\{\frac{\delta^P\lambda^r}{\delta s^P}-\frac{\delta^P\bar{\lambda}^r}{\delta s^P}\right\}. \tag{3.18}$$

3.2. *Introduction of the Frenet formulae.* We shall now introduce the principal normals and curvatures of the curves by means of the

Frenet formulae. We shall use the symbol A_P to denote the set of quantities

$$\lambda_0^r, \quad \lambda_\alpha^r, \quad \frac{d^\beta \kappa_\alpha}{ds^\beta} \tag{3.21}$$

$$(\beta = 0, 1, \ldots, P-\alpha; \ \alpha = 1, 2, \ldots, P),$$

and we shall denote by $F(A_P)$ any function of these quantities. In calculating the symbols A_P, $(P \geqslant M)$ for a curve whose M-th curvature vanishes along the curve, we understand that we put the quantities

$$\lambda_\sigma^r, \quad \frac{d^\rho \kappa_\sigma}{ds^\rho} \quad (\sigma \geqslant M; \ \rho \geqslant 0)$$

equal to zero. We know, of course, that $M \leqslant N$ for all curves in a N-space. We have, then, for $P = 1, 2$,

$$A_1; \quad \lambda_0^r, \ \lambda_1^r, \ \kappa_1.$$

$$A_2; \quad \lambda_0^r, \ \lambda_1^r, \ \lambda_2^r, \ \kappa_1, \ \frac{d\kappa_1}{ds}, \ \kappa_2.$$

We shall put, as before, $\lambda_0^r = dx^r/ds = \lambda^r$. We have successively

$$\frac{\delta \lambda^r}{\delta s} = \kappa_1 \lambda_1^r, \tag{3.22}$$

$$\frac{\delta^2 \lambda^r}{\delta s^2} = \kappa_1(\kappa_2 \lambda_2^r - e_0 e_1 \kappa_1 \lambda_0^r) + \frac{d\kappa_1}{ds}\lambda_1^r = \kappa_1 \kappa_2 \lambda_2^r + \frac{d\kappa_1}{ds}\lambda_1^r + F_2^r(A_1), \tag{3.23}$$

$$\frac{\delta^3 \lambda^r}{\delta s^3} = \kappa_1 \kappa_2 \kappa_3 \lambda_3^r + \kappa_1 \frac{d\kappa_2}{ds}\lambda_2^r + \frac{d^2 \kappa_1}{ds^2}\lambda_1^r + F_3^r(A_2), \tag{3.24}$$

and so on. In general, we prove by induction that

$$\frac{\delta^\alpha \lambda^r}{\delta s^\alpha} = \kappa_1 \kappa_2 \ldots \kappa_\alpha \lambda_\alpha^r + \kappa_1 \ldots \kappa_{\alpha-2} \frac{d\kappa_{\alpha-1}}{ds}\lambda_{\alpha-1}^r + \kappa_1 \ldots \kappa_{\alpha-3} \frac{d^2 \kappa_{\alpha-2}}{ds^2}\lambda_{\alpha-2}^r + \ldots \tag{3.25}$$

$$\ldots + \frac{d^{\alpha-1}\kappa_1}{ds^{\alpha-1}}\lambda_1^r + F_\alpha^r(A_{\alpha-1}).$$

The proof follows immediately by differentiating (3.25).

We now take the equations (3.25) successively for $\alpha = 1, 2, \ldots, P$. For $\alpha = 1$, we conclude that, if λ^r, $\delta\lambda^r/\delta s$ have the same values at O for the two curves, then so will the quantities A_1. For $\alpha = 2$, we conclude that, if λ^r, $\delta\lambda^r/\delta s$, $\delta^2\lambda^r/\delta s^2$ have the same values for the curves at O, so will also the quantities A_2. In general, we have inductively that, if

$$\lambda^r, \quad \frac{\delta^\alpha \lambda^r}{\delta s^\alpha} \quad (\alpha = 1, 2, \ldots, P-1), \tag{3.26}$$

have the same values for the curves at O, so will the quantities A_{P-1}, and then

$$(3.27)\quad \frac{\delta^P \lambda^r}{\delta s^P} - \frac{\delta^P \bar{\lambda}^r}{\delta s^P} = \kappa_1 \kappa_2 \dots \kappa_{P-1}(\kappa_P \lambda_P^r - \bar{\kappa}_P \bar{\lambda}_P^r) + \kappa_1 \kappa_2 \dots \kappa_{P-2}\left(\frac{d\kappa_{P-1}}{ds} - \frac{d\bar{\kappa}_{P-1}}{ds}\right)\lambda_{P-1}^r + \dots + \left(\frac{d^{P-1}\kappa_1}{ds^{P-1}} - \frac{d^{P-1}\bar{\kappa}_1}{ds^{P-1}}\right)\lambda_1^r.$$

Conversely, we can prove that, if the quantities A_{P-1} are the same for the curves at O, so are the quantities (3.26), and equations (3.27) still hold. Therefore, we have the theorem* :

THEOREM III. *A necessary and sufficient condition for two curves to have contact of order P at the point O is that the quantities A_{P-1} should have the same values for the two curves at O. In particular, the first $P-1$ principal normals and curvatures must be the same for the curves at the point of contact. Then also*

$$(3.28)\quad \lim_{s\to 0}\left\{\frac{(P+1)!\,\overline{QR}}{s^{P+1}}\right\} = \kappa_1 \kappa_2 \dots \kappa_{P-1}(\kappa_P \lambda_P^r - \bar{\kappa}_P \bar{\lambda}_P^r) + \kappa_1 \dots \kappa_{P-2}\left(\frac{d\kappa_{P-1}}{ds} - \frac{d\bar{\kappa}_{P-1}}{ds}\right)\lambda_{P-1}^r + \dots + \left(\frac{d^{P-1}\kappa_1}{ds^{P-1}} - \frac{d^{P-1}\bar{\kappa}_1}{ds^{P-1}}\right)\lambda_1^r.$$

This theorem is an immediate consequence of Theorem II.

As an illustration, let us apply this theorem to Euclidean three-space. A straight line can have contact only of order 1 with a curve at a point for which $\kappa_1 \neq 0$, and in this case

$$\frac{\overline{QR}}{s^2} \to \frac{\kappa_1 \lambda_1^r}{2!}.$$

If $\kappa_1 = 0$ at the point, the straight line can have contact of order 2, and now

$$\frac{\overline{QR}}{s^3} \to \frac{d\kappa_1}{ds}\frac{\lambda_1^r}{3!}.$$

The point is an inflexion on the curve.

Now let us suppose that we take $\bar{C}$ to be a circle, that is, $\bar{\kappa}_2 = 0$, $\bar{\kappa}_1 = \text{const.}$ Then the circle will have contact of order 2, provided that

* Cf. Hlavatý, "Contact de deux courbes dans une V_n", *Rend. R. Acc. Lincei*, 4 (1927), 415–420.

the tangents and principal normals of the curves coincide at O and that $\kappa_1 = \bar{\kappa}_1$ at O. In other words, the radius of the circle is equal to $1/\kappa_1$ and C is the osculating circle. Also

$$\frac{3!\,\overline{QR}}{s^3} \to \kappa_1\kappa_2\lambda_2' + \frac{d\kappa_1}{ds}\lambda_1'.$$

We get the interesting result that if $d\kappa_1/ds = 0$ at the point, then $\overline{QR}$ tends to the direction of the binormal of the given curve. In order that the osculating circle may have higher order of contact we must have $d\kappa_1/ds = 0$, $\kappa_2 = 0$ at O.

4. *Associated curves.*

4.1. *Definition of the associated curves; constructions for the principal normals of a curve.* Let us suppose that we are given a curve which has more than P non-zero curvatures (P being any integer less than $N-1$), and a point O on it. Let us define a curve passing through O as follows. It is to have its tangent and first $P-1$ principal normals coincident with those of the given curve at the point O, its first $P-1$ curvatures are to be equal to those of the given curve at equal arc lengths along the two curves from O, and finally its P-th curvatures are to be zero at all its points. It follows from (2.1) that the new curve is defined uniquely. We shall call it the *associated curve of order P at the point O*, and we shall denote it by C_P.

Now let us apply Theorem III to the given curve C and its associate of order P at the point O. We see that they have contact of order P at the point O; and, if we perform the construction of (3.1) to obtain the points Q and R, we have, from (3.28),

$$\lim_{s\to 0}\frac{\overline{QR}}{s^{P+1}} = \frac{\kappa_1\kappa_2\ldots\kappa_P}{(P+1)!}\lambda_P', \tag{4.11}$$

in consequence of the relations

$$\bar{\kappa}_P = 0, \quad \frac{d^\alpha\kappa_{P-\alpha}}{ds^\alpha} = \frac{d^\alpha\bar{\kappa}_{P-\alpha}}{ds^\alpha} \quad (\alpha = 1, 2, \ldots, P-1), \tag{4.12}$$

when $\bar{C}$ becomes the associated curve C_P.

Hence $\overline{QR}/s^{P+1}$ tends to the direction of the P-th principal normal of the given curve as s tends to zero. Also

$$\lim_{s\to 0}\frac{QR}{s^{P+1}} = \frac{\kappa_1\kappa_2\ldots\kappa_P}{(P+1)!}. \tag{4.13}$$

The result (4.11) gives us our construction for the P-th principal normal of the curve C, which may be stated in the form of a theorem:

THEOREM IV. *If we take two points R, Q at equal distances s along a given curve C, and its associated curve C_P, of order P at the point O, then the limiting vector* $\lim_{s\to 0} \overline{QR}/s^{P+1}$ *gives the direction of the P-th principal normal of C at O.*

We get the sense of the P-th normal by going from the associated curve C_P to the given curve C.

We have therefore a limiting construction for all the principal normals of a curve at any point, and successive applications of (4.13) give values for all the curvatures.

The associated curve of order 1 has its κ_1 zero, that is, it is the geodesic tangent to the curve at O. Therefore the construction for the case $P = 1$ is the well known construction for the first normal and equation (4.13) gives the well known value for the first curvature*.

As an example, let us consider the case of Euclidean three-space. The associated curve of order 2 at the point O of a twisted curve is the plane curve whose first curvatures are the same as those of the given curve at equal distances from O. Then if we take equal distances s along these two curves from O, as before, we have

$$\lim_{s\to 0} \frac{\overline{QR}}{s^3} = \frac{\kappa_1 \kappa_2 \chi_2^2}{3!},$$

that is, $\overline{QR}$ tends to the binormal of the given curve at O.

The associated curve of order 3 of a twisted curve in a three-space coincides with the curve itself; and so, for any number of dimensions, the associated curve of order N of a space curve in an N-space coincides with the curve itself. If the P-th curvature of a curve vanishes at all points, P being less than N, its associated curve of order P coincides with it, as do also all the associated curves of order higher than P.

* Bianchi, *Geometria differenziale*, II_2 (1924), 455; Eisenhart, *Riemannian geometry* (1926), 61.

SOME PROBLEMS OF "PARTITIO NUMERORUM" (VIII): THE NUMBER $\Gamma(k)$ IN WARING'S PROBLEM

By G. H. Hardy *and* J. E. Littlewood.

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1. *Introduction.*

1.1. In this memoir we continue our researches concerning the number $\Gamma(k)$, some of whose properties were developed in the fourth and sixth memoirs of the series†.

In § 2 we give a formal proof of what we have stated before‡ concerning the arithmetical interpretation of $\Gamma(k)$, viz. that $\Gamma(k)$ is (except when $k=4$) the least number s such that every arithmetical progression contains an infinity of numbers which are sums of at most s positive k-th powers.

In §§ 3–5 we are concerned with upper bounds for $\Gamma(k)$. Since each of $k+1$ and $\Gamma(k)$ is a lower bound for $G(k)$, it is of interest to determine completely all cases in which $\Gamma(k) > k+1$. Our earlier researches left little doubt which these cases are, but an exact proof that $\Gamma(k) \leqslant k+1$

† G. H. Hardy and J. E. Littlewood, "Some problems of 'Partitio Numerorum': (IV): The singular series in Waring's Problem and the value of the number $G(k)$", *Math. Zeitschrift*, 12 (1922), 161–188: and (VI): "Further researches in Waring's Problem", *ibid.*, 23 (1925), 1–37. We refer to these memoirs as P.N. 4, P.N. 6. Some errors in P.N. 4 are corrected in P.N. 6, but the main results of P.N. 6 will not be required here.

We assume that the reader is acquainted with the ideas and notation of P.N. 4, but, for the sake of clearness, we summarise the principal results, so far as they are relevant to our present purpose, in §§ 1.2–1.3.

We may add that "P.N. 7", which is still unpublished, contains an application of our methods to the problem of the order of magnitude of the difference between successive primes. We prove (subject to our generalized form of the Riemann hypothesis) that

$$\lim_{n\to\infty} \frac{p_{n+1}-p_n}{\log n} \leqslant \tfrac{2}{3}.$$

‡ P.N. 4, 188; P.N. 6, 7.

in all others was lacking; and this we now supply, the key to the proof being contained in our present Theorem 3 (§ 4).

In the cases which were formerly the main obstacle, however, we now find that we can prove that $\Gamma(k) \leqslant k$ as easily as $\Gamma(k) \leqslant k+1$. We are naturally led to attempt to determine all cases in which $\Gamma(k) > k$, and this, we find, involves the solution of a further problem, to which the key is Theorem 2 (§ 3). This theorem, like Theorem 3, is difficult; but the trouble involved in proving it is not wasted, partly because the theorem has a certain intrinsic interest, and partly because the lemmas on which it depends are important in the actual calculation of $\Gamma(k)$ for particular values of k.

In § 5 we prove Theorem 4, which shows that $\Gamma(k) \leqslant k$ except in five standard cases, in each of which $\Gamma(k)$ is determined explicitly. To these five cases we add a sixth, in which we can prove that $\Gamma(k) = k$.

Finally, § 6 is concerned with questions of numerical calculation.

1.2. It is convenient to insert here a brief summary of the notations and conclusions of P.N. 4, so far as these are directly relevant. It is assumed there that $k > 2$, and we shall suppose this here also, except where the contrary is explicitly indicated.

The singular series is

$$\mathfrak{S} = \mathfrak{S}(k, s, n) = \sum_{q=1}^{\infty} A_q,$$

where $$A_1 = 1, \quad A_q = A_q(k, s, n) = \sum_p \left(\frac{S_{p,q}}{q}\right)^s e^{-2n p \pi i/q} \quad (q > 1),$$

$S_{p,q}$ being a generalized Gaussian sum and p running through values less than q and prime to q. Here $A_{qq'} = A_q A_{q'}$ if $(q, q') = 1$, so that

$$\mathfrak{S} = \chi_2 \chi_3 \chi_5 \ldots = \prod_{\varpi} \chi_{\varpi},$$

where ϖ is a prime and

$$\chi_{\varpi} = 1 + A_{\varpi} + A_{\varpi^2} + \ldots = \sum_{\lambda} A_{\varpi}^{\lambda}.$$

Series and product are absolutely convergent for $s \geqslant 4$† (except for $k = 2$). Given k and ϖ we define θ and ϕ by

$$\varpi^{\theta} \| k\S, \quad \phi = \theta + 1 \ (\varpi > 2), \quad \phi = \theta + 2 \ (\varpi = 2),$$

† In P.N. 4 necessities of printing compelled us to write ϖ as π.

‡ P.N. 4, 176 (Theorem 3).

§ $x \mid y$ means "x is a divisor of y"; $x^{\alpha} \| y$ means "x^{α} is the highest power of x which divides y".

and we sometimes write ρ for ϖ^{ϕ}. We have $\phi < k$, except† when $k = 4$, $\varpi = 2$.

We denote by

$$N(\varpi^{\lambda}, \nu) = N(\varpi^{\lambda}, \nu, s)$$

the number of primitive solutions of

$$x_1^k + x_2^k + \ldots + x_s^k \equiv \nu \pmod{\varpi^{\lambda}},$$

i.e. the number of solutions for which $0 \leqslant x_r < \varpi^{\lambda}$ $(r \leqslant s)$ and not every x_r is divisible by ϖ. If $(\varpi^k)^{\beta} \mid n$‡ then§

$$\chi_{\varpi} = B\varpi^{\phi(1-s)} N(\rho, 0) + \varpi^{\beta(k-s)+\phi(1-s)} N\left(\rho, \frac{n}{\varpi^{\beta k}}\right),$$

where $B = 0$ if $\beta = 0$ and $B > 0$ if $\beta > 0$. Thus in all cases $\chi_{\varpi} \geqslant 0$.

We denote by $\gamma'_{\varpi} = \gamma'_{\varpi}(k)$ the least number m such that

$$\chi_{\varpi} > 0$$

for $s \geqslant m$ and all n, and by $\gamma_{\varpi} = \gamma_{\varpi}(k)$ the least number m such that (for some h)

$$\chi_{\varpi} \geqslant h_{\varpi} = h(k, s, \varpi) > 0$$

for $s \geqslant m$ and all n. Then $\gamma'_{\varpi} = \gamma_{\varpi}$ except when $k = 4$, $\varpi = 2$ (when $\gamma'_2 = 15$ and $\gamma_2 = 16$)‖. Our number $\Gamma(k)$ is defined by

$$\Gamma(k) = \operatorname*{Max}_{\varpi} \gamma_{\varpi}.$$

In order that $\chi_{\varpi} \geqslant h_{\varpi} > 0$ for all n, *i.e.* that $s \geqslant \gamma_{\varpi}$, it is sufficient that $\chi_{\varpi} \geqslant h_{\varpi} > 0$ for all sufficiently large n. The necessary and sufficient condition for this is that

$$N(\rho, n) > 0$$

for all n¶, in which case $N(\varpi^{\lambda}, n) > 0$ for $\lambda \geqslant \phi$††.

1.3. Much of our analysis depends on a particular division into classes of the residues to modulus ρ. We suppose that $\varpi > 2$, so that

† P.N. 4, 178.

‡ Not $\varpi^{\beta k} \mid n$.

§ P.N. 4, 166, Theorem 2. We do not need here the actual value of B when $\beta > 0$.

‖ P.N. 4, 178 (Lemma 19).

¶ P.N. 4, 178 (Lemma 19).

†† This is a particular consequence of P.N. 4, Lemma 5.

$\phi = \theta+1$. We write†

$$k = \varpi^\theta k^* = \varpi^\theta \epsilon k_0,$$

where $$\epsilon = (k^*, \varpi-1) = (\varpi^{-\theta} k, \varpi-1);$$

and $$d = \frac{\varpi-1}{\epsilon},$$

so that $(k_0, d) = 1$‡. Finally we write

$$c = c_\varpi = c_\varpi(k) = \frac{\varpi^\phi-1}{\varpi-1}\epsilon+1 = \frac{\varpi^\phi-1}{d}+1,$$

and we divide the residues into c classes $C_0, C_1, \ldots, C_{c-1}$ as follows§.

If G is a primitive root of ρ, and n_0 is prime to ρ, we have

$$n_0 \equiv G^{m_0\psi_0+e} \pmod{\rho},$$

where $$\psi_0 = \varpi^{\phi-1}\epsilon,$$

m_0 has one of the values $0, 1, \ldots, d-1$, and e one of the values $0, 1, \ldots, \psi_0-1$. The d values of n_0 which have a common e we class together in the class

$$C_e^0 \quad (e = 0, 1, \ldots, \psi_0-1),$$

and we denote a typical member of the class by α_e^0. These ψ_0 classes contain all residues prime to ρ.

Next, if n_i is a residue for which $\varpi^i \mid n_i$, where $0 < i < \phi$, we can write

$$n_i \equiv \varpi^i N_i \equiv \varpi^i G^{m_i\psi_i+e} \pmod{\rho},$$

where $$\psi_i = \varpi^{\phi-i-1}\epsilon,$$

m_i runs from 0 to $d-1$, and e from 0 to ψ_i-1. We thus define ψ_i new classes

$$C_e^i \quad (e = 0, 1, \ldots, \psi_i-1)$$

of members α_e^i. Finally, the residue 0 is the sole member of a class C_0^ϕ. The total number of classes is c, and we may denote them, in the

† We did not use the symbol k^* in P.N. 4.

‡ To fix ideas, we give the examples:

$k = 34, \ \varpi = 3, \ \theta = 0, \ \phi = 1, \ \rho = 3, \ k^* = 34, \ \epsilon = 2, \ d = 1, \ k_0 = 17;$

$k = 34, \ \varpi = 17, \ \theta = 1, \ \phi = 2, \ \rho = 289, \ k^* = 2, \ \epsilon = 2, \ d = 8, \ k_0 = 1;$

$k = 34, \ \varpi = 103, \ \theta = 0, \ \phi = 1, \ \rho = 103, \ k^* = 34, \ \epsilon = 34, \ d = 3, \ k_0 = 1$

$$34 = 3^0 . 2 . 17 = 17^1 . 2 . 1 = 103^0 . 34 . 1.$$

§ A misprint, C_c for C_{c-1}, runs through §§ 5.4—5.6 of P.N. 4.

order in which they have been defined, by C_0, C_1, ..., C_{c-1}, and typical members by α_0, α_1, ..., α_{c-1}.

The class C_0, which is particularly important, consists of the residues of k-th powers prime to ϖ. In particular, 1 is an α_0. Any two α_0's are incongruent (mod ρ), and therefore incongruent (mod ϖ). The product of an α_0 and an α_r is an α_r, and $\alpha_0\alpha_r$, where α_r is given, may be identified with any α_r by choice of α_0†.

One additional remark will be useful. Throughout P.N. 4 we supposed that $k \geq 2$. But our analysis of the classes does not depend on this assumption, and is equally valid when $k = 1$ or $k = 2$. Thus when $k = 1$ we have $\theta = 0$, $\phi = 1$, $\epsilon = 1$, $d = \varpi - 1$, $c = 2$. There are two classes, C_0 consisting of all residues save 0, and C_1 consisting of 0 alone. It is, in fact, obvious that all residues save 0 are representable by a single first power prime to ϖ, and 0 by two.

1.4. We shall require, besides the ideas summarized in §§ 1.2—1.3, a number of propositions which we state here as lemmas. These lemmas are either contained substantially in P.N. 4, or are easy developments of propositions proved there.

LEMMA 1. *If $d > 1$ we have*

$$\Sigma\alpha_0 \equiv 0 \pmod{\rho},$$

the summation extending over every residue of C_0.

For
$$\Sigma\alpha_0 \equiv \sum_{m_0=0}^{d-1} G^{m_0\psi_0} = \frac{G^{d\psi_0}-1}{G^{\psi_0}-1} = \frac{G^{\varpi^{\phi-1}(\varpi-1)}}{G^{\psi_0}-1} \equiv 0,$$

since G^{ψ_0} is an α_0 different from 1 and therefore‡ incongruent to 1 (mod ϖ).

---

† For all this see P.N. 4, 180–181. The point of the incongruence of the α_0's is not made explicitly. To establish it we observe that if G is a primitive root (mod ρ) it is a primitive root (mod ϖ). Hence from

$$G^{m_0\psi_0} \equiv G^{m_0'\psi_0} \pmod{\varpi}$$

we deduce successively

$$(m_0 - m_0')\psi_0 \equiv 0 \pmod{\varpi-1}, \quad (m_0 - m_0')\epsilon \equiv 0 \pmod{\varpi-1},$$
$$m_0 - m_0' \equiv 0 \pmod{d},$$
$$m_0\psi_0 \equiv m_0'\psi_0 \quad \{\mathrm{mod}\ \psi_0 d = \varpi^{\phi-1}(\varpi-1)\},$$
$$G^{m_0\psi_0} \equiv G^{m_0'\psi_0} \pmod{\varpi^\phi}.$$

‡ See § 1.3.

LEMMA 2. *If $d>1$ and a is an α_0 other than 1, then*

$$1+a+a^2+\ldots+a^{d-1}\equiv 0 \pmod{\rho}.$$

For $a\equiv x^k\not\equiv 1$, and

$$\sum_{r=0}^{d-1} a^r=\frac{x^{dk}-1}{x^k-1}, \quad x^{dk}=x^{\varpi^s(\varpi-1)k_0}\equiv 1.$$

LEMMA 3. *For $k\geqslant 2$ every residue (mod ρ), except perhaps 0, is representable (mod ρ) as the sum of at most $c-1$ α_0's, and every residue by c.*

When $k>2$ this is contained in P.N. 4, Lemma 21, and the result is valid for $k=2$ (see the end of § 1 . 3).

LEMMA 4. *If $1\leqslant c'<c$ and $\mu(c')$ is the number of classes representable by at most c' α_0's, then*

$$\mu(c'+1)\geqslant \min\{\mu(c')+1,\ c\}.\dagger \tag{1.41}$$

If (1 . 41) is false for c', then

$$\mu(c'+1)=\mu(c')<c.$$

Suppose that C^* is a typical class of the $\mu(c')$ classes, and that C_r is a C^*. If a_r is any member of C_r, a_r+1 must belong to a C^*, since no new classes are representable by $c'+1$ α_0's.

Similarly $(a_r+1)+1=a_r+2$, a_r+3, ..., and so all residues, belong to some C^*, which contradicts $\mu(c')<c$.

LEMMA 5. *If $d_1|d$ and $d_1>1$, then the residue 0 is representable (mod ρ) by d_1 α_0's.*

This is proved in P.N. 4, in the course of the proof of Lemma 21.

LEMMA 6. *If $\varpi>2$ and $\mathbf{k}=k/k_0$, then*

$$\gamma_\varpi(k)\leqslant 3 \qquad (\mathbf{k}\leqslant 2),$$
$$\gamma_\varpi(k)=\gamma_\varpi(\mathbf{k}) \quad (\mathbf{k}>2).$$

(i) Suppose, first, that $\mathbf{k}=\varpi^\theta\epsilon>2$. Then ϕ, ϵ, d, and ψ_0 are the same for k and $\mathbf{k}$, and the classes C_0 for k and $\mathbf{k}$, are identical since each consists of the residues

$$G^{m_0\psi_0} \quad (m_0=0,\ 1,\ \ldots,\ d-1);$$

which proves the lemma.

† Compare P.N. 4, 183, where we prove that

$$\nu(c'+1)\geqslant \min\{\nu(c')+1,\ c-1\},$$

$\nu(c')$ being the number of representable classes *exclusive of* C_{c-1} (the class consisting of 0 only).

(ii) Next suppose that $\mathbf{k} \leqslant 2$. Then plainly $\theta = 0$, $\phi = 1$, and we have to prove that any residue (mod ϖ) is representable by at most three $\mathbf{k}$-th powers. We have pointed out already† that, when $\mathbf{k} = 1$, two are sufficient. When $\mathbf{k} = 2$ we must have $\theta = 0$, $\phi = 1$, $\epsilon = 2$, and $c = 3$, so that our conclusion follows from Lemma 3.

LEMMA 7. *If*

(i) $\varpi = 2$, $\theta = 0$, *then* $\gamma_2 = 2$;

(ii) $\varpi = 2$, $\theta > 0$, *then* $\gamma_2 = 2^{\theta+2}$;

(iii) $\varpi > 2$, $\epsilon = \varpi - 1$, *then* $\gamma_\varpi = \varpi^{\theta+1}$;

(iv) $\varpi > 2$, $\epsilon = \frac{1}{2}(\varpi - 1)$, *then* $\gamma_\varpi = \frac{1}{2}(\varpi^{\theta+1} - 1)$, *except in the case* $\varpi = 3$, $\theta = 0$.

See P.N. 4, Lemma 22. The exceptional case in (iv) is there omitted by inadvertence. It arises from the failure of $2 \leqslant c-1$, implicitly assumed, and true in general.

LEMMA 8. *If*

(i) $k = 2^\theta$, $\theta > 1$, *then* $\Gamma(k) = 2^{\theta+2}$;

(ii) $k = 2^\theta 3$, $\theta > 1$, *then* $\Gamma(k) = 2^{\theta+2}$;

(iii) $k = \varpi^\theta(\varpi - 1)$, $\varpi > 2$, $\theta > 0$, *then* $\Gamma(k) = \varpi^{\theta+1}$;

(iv) $k = \frac{1}{2}\varpi^\theta(\varpi - 1)$, $\varpi > 2$, $\theta > 0$, *then* $\Gamma(k) = \frac{1}{2}(\varpi^{\theta+1} - 1)$.

See P.N. 4, Theorems 6, 7, 9, 10.

LEMMA 9. *If* $\delta = (\varpi - 1, k)$ *and*

$$S_{p,\varpi} = S_{p,\varpi,k} = \sum_{h=0}^{\varpi-1} e^{2ph^k \pi i/\varpi} = \sum_{h=0}^{\varpi-1} e_\varpi(ph^k),$$

then

$$|S_{p,\varpi}| \leqslant (\delta - 1)\sqrt{\varpi}.$$

See P.N. 4, Lemma 13. If $\delta = 1$ the sum vanishes.

LEMMA 10. *If* $\chi_\kappa(p)$ *is a character* (mod ϖ), κ' *a* κ *for which* χ_κ^k *is identical with the principal character* χ_1, *and*

$$z_{\bar{\kappa}} = \sum_{l=0}^{\varpi-1} \bar{\chi}_\kappa(l)\, e_\varpi(l),$$

† In § 1.4.

then (1) *no* τ *exceeds* $\sqrt{\varpi}$ *in absolute value, and* (2)

$$S_{p,\varpi} = \sum_{\kappa} \tau_{\kappa'} \chi_{\kappa'}(p)$$

when $\delta > 1$.

These propositions also are included in the statement or proof of P.N. 4, Lemma 13.

2. *Arithmetical progressions.*

2.1. In this section we prove

THEOREM 1. *The necessary and sufficient condition that every arithmetical progression should contain an infinity of numbers which are sums of s positive k-th powers is*

$$s \geqslant \Gamma(k),$$

except when $k = 4$, *in which case it is* $s \geqslant 15$.

The materials for the proof are contained in P.N. 4. We suppose, first, that $k \neq 4$.

(i) Suppose that $s \geqslant \Gamma(k)$. Since $s \geqslant \gamma_{\varpi}$, the congruence

$$\sum_{1}^{s} x_r^k \equiv n$$

is soluble to any modulus ϖ^{λ} (for $\lambda \geqslant \phi$ and therefore for all λ), and therefore to any modulus q. If $x_1, x_2, \ldots, x_r$ is a solution, then

$$\sum_{1}^{s} (mq + x_r)^k \equiv n \pmod{q}$$

for every m, so that the progression $Nq+n$ contains an infinity of numbers of the form required.

(ii) Suppose that every arithmetical progression contains numbers representable by s k-th powers (and that $k \neq 4$). Then for any ϖ, n there are numbers $\varpi^k m + n$ representable by s k-th powers, and the congruence

$$n \equiv \Sigma x_r^k \pmod{\varpi^k}$$

is soluble. For each of the values $n = 1, 2, \ldots, \varpi^{\phi}$ the solution is necessarily a primitive one, since otherwise n would be divisible by ϖ^k, and so by $\varpi^{\phi+1}$. There exists, *a fortiori*, a primitive solution of the congruence

$$n \equiv \Sigma x_r^k \pmod{\varpi^{\phi}}$$

for each of these values of n, and since now n ranges through all residues

we have $N(\rho, n) > 0$, and so $s \geqslant \gamma'_{\varpi} = \gamma_{\varpi}$. Since this is true for all ϖ we have $s \geqslant \Gamma(k)$.

The argument in (i) fails for $k = 4$ because γ_{ϖ} and γ'_{ϖ} differ when $\varpi = 2$. We have, in fact,

$$\gamma_2 = 16, \quad \gamma'_2 = 15, \quad \text{while} \quad \gamma_{\varpi} = \gamma'_{\varpi} < 15$$

for every other ϖ. It is γ'_{ϖ} which is relevant in the argument (i) here, and every progression is permissible when $s > 15$. On the other hand, consider the progression $16m+15$. No number of this is representable by fewer than 15 fourth powers, and $s \geqslant 15$ is necessary.

3. *A theorem concerning the case* $\theta = 0$.

3.1. Our main theorem in this section is

THEOREM 2. *If* $k > 2$, $\theta = 0$, $d > 1$, *then*

$$\gamma_{\varpi} \leqslant k.$$

The principal difficulty lies in the representation of the residue 0 (mod ϖ), and the essential lemma is

LEMMA 11. *If* $\varpi > 2$, $\theta = 0$, *then the residue* 0 *has a representation by at most* s a_0*'s whenever*

$$s > 2, \quad \varpi \geqslant (\epsilon-1)^{2(s-1)/(s-2)}.$$

If $\epsilon = 1$, $d = \varpi - 1$, $c = 2$, and 0 is representable by 2 a_0's. If $\epsilon = 2$, $d = \frac{1}{2}(\varpi-1)$, $c = 3$, and it is representable by 3. These cases (in which the main inequality is nugatory) are therefore trivial, and we may suppose that $\epsilon > 2$.

Let $N+1$ be the number of solutions of

$$X = \sum_{r=1}^{s} x_r^k \equiv 0 \pmod{\varpi} \quad (0 \leqslant x_r < \varpi).$$

One solution is given by $x_r \equiv 0$ for all r, and so what we have to prove is that $N > 0$. Now $\sum_p e_{\varpi}(pX)$ has the value $\varpi - 1$ if $X \equiv 0$ and the value -1 if $X \not\equiv 0$. Hence

$$\begin{aligned}
\sum_p S^s_{p,\varpi} &= \sum_p \sum_{x_r=0}^{\varpi-1} e_{\varpi}(pX) \\
&= \sum_{x_r} \sum_p e_{\varpi}(pX) = \sum_{X \equiv 0} \sum_p e_{\varpi}(pX) + \sum_{X \not\equiv 0} \sum_p e_{\varpi}(pX) \\
&= (N+1)(\varpi-1) - (\varpi^s - N - 1) = (N+1)\varpi - \varpi^s,
\end{aligned}$$

$$(N+1)\varpi \geqslant \varpi^s - \sum_p |S_{p,\varpi}|^s. \tag{3.11}$$

Now $$\delta = (\varpi - 1, k) = \epsilon,$$

and so, by Lemma 9,

$$|S_{p,\varpi}| \leqslant (\epsilon-1)\sqrt{\varpi}. \tag{3.12}$$

Also
$$\sum_p |S_{p,\varpi}|^2 = \sum_p \sum_{x,y=0}^{\varpi-1} e_\varpi\{p(x^k-y^k)\}$$
$$= \Big(\sum_{x^k \equiv y^k} + \sum_{x^k \not\equiv y^k}\Big) \sum_p e_\varpi\{p(x^k-y^k)\}$$
$$= \sum_{x^k \equiv y^k} (\varpi-1) + \sum_{x^k \not\equiv y^k} (-1).$$

Given $x \not\equiv 0$, there are just ϵ y's for which $y^k \equiv x^k$. Hence

$$\sum |S_{p,\varpi}|^2 = \{1+\epsilon(\varpi-1)\}(\varpi-1) - \{\varpi^2-1-\epsilon(\varpi-1)\} \tag{3.13}$$
$$= (\epsilon-1)\varpi(\varpi-1).$$

From (3.12) and (3.13) we deduce

$$\sum |S_{p,\varpi}|^s \leqslant \{(\epsilon-1)\sqrt{\varpi}\}^{s-2} \sum |S_{p,\varpi}|^2 = (\epsilon-1)^{s-1}\varpi^{\frac{1}{2}s}(\varpi-1). \tag{3.14}$$

It then follows from (3.11) that $N > 0$ whenever

$$\varpi^s - \varpi > (\epsilon-1)^{s-1}\varpi^{\frac{1}{2}s}(\varpi-1),$$

and *a fortiori* when $\varpi^{s-1} \geqslant (\epsilon-1)^{s-1}\varpi^{\frac{1}{2}s}$, the inequality of the lemma.

3.2. There is a companion result for residues other than 0. This is not required in the proof of any of our main theorems, but it is essential in the calculation of special values of $\Gamma(k)$, and has a certain independent interest.

LEMMA 12. *Suppose that $\theta = 0$ and $n \not\equiv 0 \pmod{\varpi}$. Then n has a representation by at most s a_0's whenever*

$$s > 1, \quad \varpi > (\epsilon-1)^{2s/(s-1)}.$$

As in Lemma 11, we may suppose $\epsilon > 2$. We have to prove that $N > 0$, N being now the number of solutions of

$$X \equiv n \pmod{\varpi}, \quad 0 \leqslant x_r < \varpi.$$

Now

$$\sum_p S_{p,\varpi}^s e_\varpi(-np) = \sum_p \sum_{x_r=0}^{\varpi-1} e_\varpi\{p(X-n)\} \tag{3.21}$$
$$= \sum_{X \equiv n} \sum_p e_\varpi\{p(X-n)\} + \sum_{X \not\equiv n} \sum_p e_\varpi\{p(X-n)\}$$
$$= N(\varpi-1) - (\varpi^s - N) = N\varpi - \varpi^s.$$

But, by Lemma 10,

$$S_{p,\varpi} = \Sigma z_{\bar{\kappa}'} X_{\kappa'}(p).$$

Hence

$$N\varpi - \varpi^s = \sum_p \Sigma^* \mathbf{T}\mathbf{X}(p)\, e_\varpi(-np) = \Sigma^* \mathbf{T} \sum_p \mathbf{X}(p)\, e_\varpi(-np),$$

where $\mathbf{T}$ is the product of s τ's, $\mathbf{X}(p)$ the product of s χ's (and so a character), and the number of terms in Σ^* is $(\epsilon-1)^s$. The inner sum is -1 if $\mathbf{X}(p)$ is the principal character, since $n \not\equiv 0$, and it is the product of a χ and a τ otherwise; so that, by Lemma 10, its absolute value does not exceed $\sqrt{\varpi}$. Hence

$$|N\varpi - \varpi^s| \leqslant (\epsilon-1)^s \varpi^{\frac{1}{2}(s+1)},$$

$$N\varpi \geqslant \varpi^s - (\epsilon-1)^s \varpi^{\frac{1}{2}(s+1)},$$

and $N > 0$ if $\varpi^s > (\epsilon-1)^s \varpi^{\frac{1}{2}(s+1)}$, the inequality of the lemma.

Since $(s-1)^2 > s(s-2)$, we have as a corollary

LEMMA 13. *The result of Lemma* 11 *is true also for residues other than* 0.

We shall also require the trivial

LEMMA 14. *If x is integral and not less than* 4, *then*

$$(x-1)^{2(x-1)/(x-2)} < x(2x-1).$$

3.3. *Proof of Theorem* 2. Since $\varpi \geqslant d+1 > 2$, we have $\phi = 1$, $c = \epsilon+1$. We begin by showing that we can make certain simplifying assumptions.

First, we may suppose $\epsilon > 2$. For $k = \epsilon k_0$, $\mathbf{k} = k/k_0 = \epsilon$. Hence, if $\epsilon \leqslant 2$, we have, by Lemma 6,

$$\gamma_\varpi(k) \leqslant 3 \leqslant k.$$

Next, we may suppose $k_0 = 1$, so that $k = \epsilon$. For suppose the theorem proved in this case. Then, since $\mathbf{k} = \epsilon > 2$, we have, by Lemma 6,

$$\gamma_\varpi(k) = \gamma_\varpi(\mathbf{k}) \leqslant \mathbf{k} = \epsilon \leqslant k.$$

Next, it follows from Lemma 3 that every residue (mod ϖ), except perhaps 0, is representable by at most $c-1 = \epsilon = k$ a_0's, and we have only to show that 0 is so representable.

Finally, it follows from Lemma 5 that 0 is so representable whenever d is even, and, in particular, whenever ϵ is odd; since we may then take $d_1 = 2$. We may therefore suppose that ϵ is even and d is odd.

To sum up, if the theorem is false for some k and ϖ, then a k and ϖ (not necessarily the same) exist for which

(A) $\varpi > 2$, $\theta = 0$, $k_0 = 1$, $k = \epsilon$, $c = \epsilon+1$;

(B) *ϵ is even*, $\epsilon \geqslant 4$, *d is odd*;

(C) *the residue* 0 *is not representable* (mod ϖ) *by fewer than* $\epsilon+1$ a_0's.

From these assumptions we deduce successively (D), (E), (F), (G) below. Since (G) contradicts (D), the theorem is then proved.

3.41. (D) $\varpi < \epsilon(2\epsilon-1)$, $d < 2\epsilon-1$†.

Take $s = \epsilon$ in Lemma 11. The result contradicts (C) unless $\varpi < (\epsilon-1)^{2(\epsilon-1)/(\epsilon-2)}$. Since $\epsilon \geqslant 4$, by (B), Lemma 14 gives $\varpi < \epsilon(2\epsilon-1)$; and then

$$d = (\varpi-1)/\epsilon < 2\epsilon-1.$$

3.42. (E) *There is an a_0, say a_0^*, such that a_0^*+1 is also an a_0.*

First, the number of classes representable by two or fewer a_0's is two at most; for otherwise we should have, by Lemma 4,

$$\mu(c-1) \geqslant \min\{\mu(2)+c-3,\ c\} = c,$$

and all residues would be representable by $c-1 = \epsilon$ a_0's, contrary to (C).

Next, if no a_0^* exists, no a_0+1 is an a_0, and, in particular, $1+1 = 2$ is not an a_0, so that every a_0+1 belongs to the same class as 2. It follows that $a_0+1 \equiv 2a_0'$ for every a_0 and some a_0'. Incongruent a_0's correspond to incongruent a_0''s and a_0' runs through the whole of C_0 with a_0. Hence, by Lemma 1,

$$\Sigma(a_0+1) \equiv 2\Sigma a_0, \quad d \equiv \Sigma a_0 \equiv 0;$$

and this is impossible because $d \mid \varpi-1$.

3.43. (F) $a_0^* \not\equiv 1$ (mod ϖ).

If $a_0^* \equiv 1$, 2 is an a_0, by (E); and so 1, 2, 2^2, 2^3, ... are all a_0's. Now any number up to $2^{\epsilon+1}-2$ can be represented in the binary scale by at most ϵ non-zero digits; and, after (D),

$$\varpi < \epsilon(2\epsilon-1) < 2^{\epsilon+1}-2$$

(since $\epsilon \geqslant 4$). Hence

$$\varpi = 2^{n_1}+2^{n_2}+\ldots+2^{n_r} \quad (0 \leqslant n_r < n_{r-1} < \ldots < n_1,\ r \leqslant \epsilon),$$

and $\varpi \equiv 0$ is representable by at most ϵ a_0's, contrary to (C).

† To avoid any misunderstanding, we repeat that (D)–(G) are not true theorems, but deductions from hypotheses which prove untenable.

3 . 43. (G) $d > 2\epsilon - 1$.

Take $a = a_0^*$ in Lemma 2. Then a and $1+a$ are a_0's, so that $a^m(1+a)$ is an a_0 for every m. Since d is odd, Lemma 2 gives

$$0 \equiv 1+a+a^2+\ldots+a^{d-1} = (1+a)+(1+a)\,a^2+\ldots+(1+a)\,a^{d-3}+a^{d-1}.$$

This is a representation of 0 by $\frac{1}{2}(d-3)+2 = \frac{1}{2}(d+1)$ a_0's. Hence, by (C),

$$\tfrac{1}{2}(d+1) > \epsilon, \quad d > 2\epsilon - 1.$$

Finally, (G) contradicts (D). Hence (A), (B), (C) are not simultaneously possible; and this proves the theorem.

4. *A theorem concerning the case* $\theta > 0$.

4 . 1. THEOREM 3. *If* $\theta > 0$, $d > 2$, *then*

$$\gamma_\varpi \leqslant k.$$

The main difficulties here arise with residues (mod ρ) other than 0.

We denote by $N[m]$, where $m > 0$, any number that is a sum of m k-th powers (positive or zero). The symbol is not a one-valued function of m; m (together with k and ϖ) determines a certain class of integers, and the symbol denotes an undetermined member of this class. An equation in N's implies that any number that is of the form on the left of the equation is also of the form on the right.

4 . 2. LEMMA 15. $N[m]N[n] = N[mn]$.

For
$$N[m]N[n] = \sum_{i=1}^{m} x_i^k \sum_{j=1}^{n} y_j^k = \sum_{i,j} (x_i y_j)^k.$$

LEMMA 16. *For any* n

$$n \equiv N[\epsilon] \pmod{\varpi}.$$

This is trivial if $n \equiv 0 \pmod{\varpi}$, since 0 is the sum of ϵ 0^k's. If $n \not\equiv 0 \pmod{\varpi}$, let $k' = \epsilon$, and let c', a_0' denote the number c and a typical a_0 for the index k'. Then by Lemma 3 n can be expressed as a sum of at most $c'-1 = \epsilon$ a_0''s. Now an a_0 taken (mod ϖ) is plainly an a_0'; moreover, since the a_0's are incongruent (mod ϖ) (§ 1 . 3), and since the number d is the same for k and ϵ (for the prime ϖ), it follows that the set of a_0' is identical with the set of a_0 considered to mod ϖ. Hence n is representable (mod ϖ) by not more than ϵ a_0's.

4 . 3. We come now to the lemma containing the crucial step in the proof.

LEMMA 17†. *If $d>2$ then there is a u, not a multiple of ϖ, such that*

$$u\varpi \equiv N[\varpi-1] \pmod{\varpi^2}.$$

Since $d \geqslant 3$ there exist three k-th powers 1^k, α^k, β^k prime to ϖ and incongruent (mod ϖ). We define two positive integers a, b by

$$a \equiv \alpha^k \pmod{\varpi^2} \quad (0 < a < \varpi^2),$$

$$b \equiv \beta^k \pmod{\varpi^2} \quad (0 < b < \varpi^2).$$

Since neither a nor b is congruent to 0 or to 1 (mod ϖ) there exist integers μ, ν such that

$$(\mu-1)\varpi+2 \leqslant a \leqslant \mu\varpi-1 \quad (0 < \mu \leqslant \varpi),$$

$$(\nu-1)\varpi+2 \leqslant b \leqslant \nu\varpi-1 \quad (0 < \nu \leqslant \varpi).$$

Then $$\mu\varpi = a+m \equiv \alpha^k+m \pmod{\varpi^2},$$

where $$1 \leqslant m \leqslant \varpi-2.$$

Thus $\mu\varpi$ is representable (mod ϖ^2) as a sum of $1+m \leqslant \varpi-1$ k-th powers, and similarly for $\nu\varpi$. The result of the lemma therefore holds (with $u=\mu$ or $u=\nu$) unless $\mu=\nu=\varpi$. Now in this case, since a and b cannot both be ϖ^2-1, one at least, say a, must satisfy

$$(\varpi-1)\varpi+2 \leqslant a \leqslant \varpi^2-2,$$

so that $\varpi^2 = a+m$, where $2 \leqslant m \leqslant \varpi-2$. Then, h denoting *any* positive integer,

$$-\varpi = -hm+(hm-\varpi) \equiv h\alpha^k+(hm-\varpi) \pmod{\varpi^2}.$$

In this we take $h=h_m$, the least integer not less than ϖ/m. Then $-\varpi$ is expressible (mod ϖ^2) as a sum of

$$s_m = h_m+(h_m m-\varpi)$$

k-th powers, and the result of the lemma (with $u=-1$) will certainly hold if it is true that $s_m \leqslant \varpi-1$ for every m in $2 \leqslant m \leqslant \varpi-2$. Now

† In our original version the form of this lemma was slightly different. The present form, which leads rather more simply to Theorem 3, was suggested to us by Mr. Ingham. We are indebted to him also for the proof given here of the lemma; this represents a great gain in simplicity over the original. We have finally to thank Mr. Ingham also for detecting a number of minor errors in the paper as a whole.

$m(h_m-1) \leqslant \varpi-1$. Hence

$$s_m \leqslant (m+1)\left(1+\frac{\varpi-1}{m}\right)-\varpi = m+\frac{\varpi-1}{m}.$$

If now $\frac{1}{2}(\varpi-1) < m \leqslant \varpi-2$, then $s_m < \varpi-2+2 = \varpi$. If on the other hand $2 \leqslant m \leqslant \frac{1}{2}(\varpi-1)$, then $s_m \leqslant \frac{1}{2}(\varpi-1)+\frac{1}{2}(\varpi-1) = \varpi-1$. In either case $s_m \leqslant \varpi-1$, and the proof of the lemma is completed.

LEMMA 18. *For any integer* n

$$n\varpi \equiv N[\epsilon(\varpi-1)] \pmod{\varpi^2}.$$

We choose v so that $uv \equiv n \pmod{\varpi}$, u being the u of Lemma 17. Then

$$n\varpi \equiv v\,.\,u\varpi \equiv N[\epsilon]\,.\,u\varpi \pmod{\varpi^2} \quad \text{(Lemma 16)}$$

$$\equiv N[\epsilon]N[\varpi-1] \equiv N[\epsilon(\varpi-1)] \pmod{\varpi^2} \quad \text{(Lemma 15)}.$$

4 . 4. *Proof of Theorem* 3. For any n we have

$$\begin{aligned} n &= N[\epsilon]+n_1\varpi \quad \text{(Lemma 16)} \\ &= N[\epsilon]+N[\epsilon(\varpi-1)]+n_2\varpi^2 \quad \text{(Lemma 18)} \\ &= N[\epsilon]+N[\epsilon(\varpi-1)]+\varpi N[\epsilon(\varpi-1)]+n_3\varpi^3 \quad \text{(Lemma 18)} \\ &= \ldots\ldots\ldots \\ &= N[\epsilon]+N[\epsilon(\varpi-1)]+\varpi N[\epsilon(\varpi-1)]+\ldots+\varpi^{\theta-1}N[\epsilon(\varpi-1)]+n_{\theta+1}\varpi^{\theta+1} \\ &\equiv N[\epsilon+\epsilon(\varpi-1)+\epsilon(\varpi^2-\varpi)+\ldots+\epsilon(\varpi^\theta-\varpi^{\theta-1})] \pmod{\rho} \\ &\equiv N[\epsilon\varpi^\theta] \pmod{\rho}. \end{aligned}$$

If $n \not\equiv 0 \pmod{\rho}$ the $\epsilon\varpi^\theta$ k-th powers by which n is thus represented $(\mathrm{mod}\,\rho)$ cannot all be divisible by ϖ, since $\phi \leqslant k$. Hence any $n \not\equiv 0$ is representable $(\mathrm{mod}\,\rho)$ as a sum of at most $\epsilon\varpi^\theta$ a_0's. The same thing is true of the residue 0, by Lemma 5 (since $1 < d < \varpi^\theta\epsilon$). Hence

$$\gamma_\varpi \leqslant \varpi^\theta\epsilon \leqslant k.$$

5. *Determination of all cases in which* $\Gamma(k) > k$.

5 . 1. We use π, like ϖ, to denote an odd prime; there is no danger of confusion with the common notation†. We distinguish six special

† π, ϑ are given by k; ϖ is a prime which may assume any value when k is given, and θ is determined by k and ϖ.

classes of values of k, viz. :

I. $k = 2^{\vartheta}$ $(\vartheta > 1)$.

II. $k = 2^{\vartheta}.3$ $(\vartheta > 1)$.

III. $k = \pi^{\vartheta}(\pi-1)$ $(\vartheta > 0)$.

IV. $k = \frac{1}{2}\pi^{\vartheta}(\pi-1)$ $(\vartheta > 0)$.

V. $k = \pi-1$, *k not belonging to any preceding class.*

VI. $k = \frac{1}{2}(\pi-1)$, *k not belonging to any preceding class.*

It is easily verified that these classes are mutually exclusive.

THEOREM 4. *The values of $\Gamma(k)$ when k belongs to a special class are given by the formulae :*

I. $\Gamma(k) = \gamma_2 = 2^{\vartheta+2} = 4k > k+1.$

II. $\Gamma(k) = \gamma_2 = 2^{\vartheta+2} = \frac{4}{3}k > k+1.$

III. $\Gamma(k) = \gamma_\pi = \pi^{\vartheta+1} > k+1.$

IV. $\Gamma(k) = \gamma_\pi = \frac{1}{2}(\pi^{\vartheta+1}-1) > k+1$, *except in the particular case* $k = 3 = \frac{1}{2}.3\ (3-1)$, *when* $\Gamma(k) = k+1.$

V. $\Gamma(k) = \gamma_\pi = \pi = k+1.$

VI. $\Gamma(k) = \gamma_\pi = \frac{1}{2}(\pi-1) = k.$

When k does not belong to a special class, then

$$\Gamma(k) \leqslant k.$$

The values of $\Gamma(k)$ in cases I–IV are given in Lemma 8. We suppose then that k does not belong to any of these classes.

Let ϖ be any prime. We shall prove that $\gamma_\varpi \leqslant k$, except when k belongs to class V and $\varpi = \pi$. There are four cases to consider :

(i) $\varpi = 2$, $\theta = 0$. Here $\gamma_\varpi = \gamma_2 = 2 < k$, by Lemma 7.

(ii) $\varpi = 2$, $\theta > 0$. Here $\epsilon = 1$, and k_0 is odd and greater than 3, since otherwise k would belong to class I, class II, or class III. Hence, by Lemma 7,

$$\gamma_\varpi = \gamma_2 = 2^{\theta+2} = 4k/k_0 \leqslant k.$$

(iii) $\varpi > 2$, $\theta = 0$. If $k_0 > 1$, then

$$\gamma_\varpi \leqslant c = \epsilon+1 \leqslant \epsilon k_0 = k,$$

by Lemma 3. If $k_0 = 1$, $d > 1$, then $\gamma_\varpi \leqslant k$ by Theorem 2. If $k_0 = 1$, $d = 1$, then $\epsilon = \varpi - 1$, $k = \varpi - 1$; and this is case V, with $\pi = \varpi$.

(iv) $\varpi > 2$, $\theta > 0$. If $k_0 \geqslant 1$, then, by Lemma 3,

$$\gamma_\varpi \leqslant c = \frac{\varpi^{\theta+1}-1}{\varpi-1}\epsilon+1 < 2\varpi^\theta\epsilon+1 \leqslant \varpi^\theta\epsilon k_0+1 = k+1,$$

and so $\gamma_\varpi \leqslant k$. If $k_0 = 1$, then $d > 2$, since $d = 1$ or 2 makes k belong to class III or class IV. Hence $\gamma_\varpi \leqslant k$, by Theorem 3.

It remains only to observe that $\gamma_\pi = \pi = k+1$ in case V, and $\gamma_\pi = \frac{1}{2}(\pi-1) = k$ in case VI, by Lemma 7. With this the proof is completed.

6. *The numerical calculation of* $\Gamma(k)$.

6.1. Table I gives the values of $\Gamma(k)$ for $k \leqslant 36$, the special class, if any, to which k belongs, and (under the heading ϖ) the least ϖ (in case more than one exists) for which $\gamma_\varpi = \Gamma(k)$.

The value of $\Gamma(k)$ may be found for any k by a finite process of calculation. A routine method may be set out as follows.

We observe first that $\Gamma(k) \geqslant 3$. Since $\Gamma(kk') \geqslant \Gamma(k)$ and $\Gamma(2^\mathfrak{s}) = 2^{\mathfrak{s}+2} > 3$ when $\mathfrak{s} \geqslant 2$, it is necessary to prove this only when k is an odd prime π. In this case there are $\pi-1$ π-th powers (mod π^2), at most $(\pi-1)^2 < \pi^2$ numbers are representable by *one or two* of them, and so $\gamma_\pi(\pi) \geqslant 3$.

We cannot prove that $\Gamma(k) \geqslant 4$, though no case of $\Gamma(k) = 3$ is known.

Lemmas 11 and 13 show that $\gamma_\varpi \leqslant 3$ provided that

$$\varpi > (k-1)^{2(3-1)/(3-2)} = (k-1)^4\dagger.$$

The number γ_2 is given for any k by Lemma 7, and, in particular, we know that for an even $k > 3$ we have $\gamma_2 \geqslant 8$, $\Gamma(k) \geqslant 8$. If now no (odd) ϖ exists for which $\gamma_\varpi > \text{Max}(\gamma_2, 3)$, then $\Gamma(k) = \text{Max}(\gamma_2, 3)$. If, however, such ϖ's exist, they must satisfy $\varpi \leqslant (k-1)^4$, they can be found by a finite search, and $\Gamma(k)$ is equal to the greatest of the γ_ϖ corresponding.

A prime ϖ giving in this way $\Gamma(k) = \gamma_\varpi > \text{Max}(\gamma_2, 3)$ must further satisfy either (i) $\varpi \mid k$, or (ii) $\varpi \equiv 1 \pmod{\delta}$, where $\delta \mid k$ and δ is not unity or a positive power of 2. For if such a ϖ satisfies neither (i) nor (ii) we must have $\theta = 0$ and $\epsilon = 2^\alpha$, where $\alpha \geqslant 0$, in which case

$$\gamma_\varpi \leqslant c = 2^\alpha+1 \leqslant 2^{\alpha+1} \leqslant \gamma_2.$$

In case (ii) ϖ belongs to the progression $m\delta+1$; it is fairly certain in practice that it will be the *least* prime in its progression.

An examination of the prime divisors of k and of the least primes in the progressions $m\delta+1$ generally suggests fairly strongly some numerical

† So that the condition $\theta = 0$ for the applicability of Lemmas 11 and 13 is satisfied.

value s for $\Gamma(k)$. We have then to prove, first, that $\gamma_{\varpi} = s$ for the selected ϖ, and then that $\gamma_{\varpi} \leqslant s$ both for every prime divisor of k and for every ϖ, in each of the progressions, for which

$$\varpi \leqslant (\epsilon - 1)^{2(s-1)/(s-2)}.$$

If k is odd, however, we need consider only, in the progressions, primes satisfying the more favourable inequality

$$\varpi \leqslant (\epsilon - 1)^{2s/(s-1)}.$$

For in this case the α_0's are equal and opposite in pairs, and 0 is representable by two of them, so that we may use Lemma 12 instead of Lemma 13.

It is easy to compute $\Gamma(k)$ with *practical* certainty for values of k considerably larger than 36; but the necessity of examining primes of the order k^4 makes a complete proof impracticable for those k whose $\Gamma(k)$ happens to be small.

6.2. *An example*: $k = 34$. We illustrate the process by proving that $\Gamma(34) = 10$. Here $\gamma_2 = 8$, by Lemma 7. It is improbable that $\Gamma(k)$ is so small as this, and the most likely values for the critical ϖ are 17 and 103, the smallest prime in the progression $17m+1$, the only progression that has to be considered. Since $103 < 17^2 = 289$, we begin by examining $\varpi = 103$.

We denote by ν_r a typical number, other than 0, representable (mod 103) by just r α_0's; ν_1 is an α_0. Since $d = 3$, 0 is representable by three α_0's. Calculation gives

ν_1 1, 46, 56.

ν_2 2, 9, 47, 57, 92, 102.

ν_3 3, 10, 35, 45, 48, 55, 58, 65, 93.

ν_4 4, 8, 11, 18, 36, 49, 59, 66, 81, 91, 94, 101.

ν_5 5, 12, 19, 24, 34, 37, 44, 50, 54, 60, 64, 67, 74, 82, 90, 95.

ν_6 6, 7, 13, 17, 20, 25, 27, 38, 51, 61, 68, 70, 75, 80, 83, 96, 100.

ν_7 14, 21, 23, 26, 28, 33, 39, 43, 52, 53, 62, 63, 69, 71, 73, 76, 84, 97.

ν_8 15, 16, 22, 29, 40, 72, 77, 79, 85, 89, 98, 99.

ν_9 30, 32, 41, 42, 78, 86.

ν_{10} 31, 87, 88.

These numbers exhaust the residues: hence $\gamma_{103} = 10$.

It remains to verify that $\gamma_{\varpi} \leqslant 10$ for the remaining primes $34m+1$, and that $\gamma_{17} \leqslant 10$. Taking $\varpi = 34m+1$, and $s = 10$ in Lemmas 11 and 12, we have to verify that any residue (mod ϖ), other than 0, is

representable by ten α_0's when

$$(6.21) \qquad \varpi \leqslant (34-1)^{2.10/(10-1)} < 2380,$$

and that 0 is similarly representable in the further range

$$(6.22) \qquad 2381 \leqslant \varpi \leqslant (34-1)^{2(10-1)/(10-2)} < 2614.$$

The values of ϖ satisfying (6.22) are 2381, 2551. For these d is 70 and 75, both divisible by $d_1 = 5$. Hence in these cases, by Lemma 5, 0 is representable by five α_0's.

The ϖ's of the range (6.21) are 137, 239, 307, 409, 443, 613, 647, 919, 953, 1021, 1123, 1259, 1327, 1361, 1429, 1531, 1667, 1871, 1973, 2143, 2347.

We have to show that any residue to any of these moduli is representable by at most ten 34-th powers, and calculation shows that this is possible with a good deal to spare. With the higher moduli it is not necessary to calculate more than about ten α_0's.

Finally, to prove that $\gamma_{17} \leqslant 10$, we have to show that all residues (mod 289) are similarly representable; and here again there is something to spare.

6.3. It is natural to expect that $\Gamma(k)$ tends to infinity with k. But so far are we from being able to prove this that (as we stated in § 6.1) we cannot prove even

$$\underline{\lim}\, \Gamma(k) \geqslant 4;$$

and these problems seem to be extremely difficult. Thus to prove $\Gamma(k) \geqslant 4$ for all, or all large, k, or, what is the same thing, to prove $\Gamma(\pi) \geqslant 4$ for all, or all large, π, it is necessary to prove, for each π, that either $\gamma_\pi(\pi) \geqslant 4$ or $\gamma_\varpi(\pi) \geqslant 4$ for some prime $\varpi = 2m\pi+1$. The first alternative is improbable for a given π (since the number of α_0's is $d = \pi-1$). The second would be established if we could show that, for any large π, a prime $\varpi = 2m\pi+1$ exists with $d = 2m < \varpi^{\frac{1}{3}}+1$†, which is roughly the same as $2m < \pi^{\frac{1}{3}}$. Such a theorem, if true, must be very deep, and it is not easy to find other ways of attacking the problem.

6.4. This being so, it is worth while to carry our numerical computations beyond the limits of Table 1, contenting ourselves with lower bounds for $\Gamma(k)$ when the exact calculation becomes impracticable. In what follows we suppose that $\varpi > 2$, and we use the following considerations to assign a lower bound for γ_ϖ (where we cannot appeal to any more precise result).

† For if $d + d^2 + d^3 < \varpi$ there must exist some residue not representable by one, two, or three α_0's.

We use $N_s(d)$ to denote either the number of distinct residues (mod ρ), other than 0, which are representable by s or fewer α_0's, *or any upper bound of this number.* Here d has its usual meaning. If $N_s(d)$ were the exact number of residues, the notation would be inappropriate, since it would depend on k otherwise than through d; but it will appear that, by taking an appropriate upper bound, we can make it depend on d and s only. Similarly we use $\nu_s(d)$ to denote an upper bound for the number of residues representable by s α_0's and not by less†. We may plainly take

$$N_s(d) = \sum_{r=1}^{s} \nu_r(d). \tag{6.41}$$

Let $P_r(s)$ be the number of partitions of s into just r of 1, 2, 3, ..., repetitions being allowed and order counting. Then

$$P_r(s) = \binom{r-1}{s-1} \quad (r \leqslant s), \quad P_r(s) = 0 \quad (r > s)\ddagger. \tag{6.42}$$

Let

$$Q_r(s) = \sum_{t=1}^{s} P_r(t) = \sum_{t=r}^{s} P_r(t) = \binom{s}{r}. \tag{6.43}$$

Finally, when $d > 1$, let $D_r(d)$ be an upper bound for the number of combinations of d α_0's just r at a time, order not counting, repetitions not being allowed, and any combination which contains a set of α_0's whose sum is congruent to 0 (mod ρ) being rejected§. Evidently $D_r(d) = 0$ when $r > k$; and this is true also for $r = d$, since $\Sigma\alpha_0 \equiv 0$, by Lemma 1. And

$$D_r(d) \leqslant \binom{d}{r} \quad (r < d). \tag{6.44}$$

A little consideration shows that the number of residues which are representable by just s α_0's, and in which just r different α_0's appear, is at most $P_r(s)\,D_r(d)$. Hence we may take

$$\nu_s(d) \leqslant \sum_{r=1}^{d-1} P_r(s)\,D_r(d)$$

$$N_s(d) \leqslant \sum_{r=1}^{d-1} Q_r(s)\,D_r(d) = \sum_{r=1}^{d-1} \binom{s}{r} D_r(d) \leqslant \sum_{r=1}^{d-1} \binom{s}{r}\binom{d}{r}. \tag{6.45}$$

It is evident that, if $N_s(d) < \varpi^{\phi} - 1$, then $\gamma_{\varpi} \geqslant s+1$. Hence (6.45) may be used to find a lower bound for γ_{ϖ}. If d is *even*, however, we

† Thus if -1 is an α_0, $\alpha_0 + 1 + (-1)$ represents α_0 by three α_0's, but this representation is to be rejected.

‡ $P_r(s)$ is the coefficient of x^s in $(x + x^2 + x^3 + \ldots)^r$.

§ Without this provision the number of combinations would of course be $\binom{d}{r}$.

can do better than (6 . 45). For, if $d = 2d_1$, the a_0's are of the type

$$\pm a_1, \pm a_2, \ldots, \pm a_{d_1},$$

and, in estimating $D_r(d)$, we can reject any combination in which a_r and $-a_r$ both occur. This leads to the inequalities

$$(6 . 46) \quad D_r(d) \leqslant \frac{d(d-2)\ldots(d-2r+2)}{r!} \quad (0 < r \leqslant d_1), \quad D_r(d) = 0 \ (r > d_1).$$

$$(6 . 47) \quad N_s(d) \leqslant \sum_{r=1}^{d_1} \binom{s}{r} \frac{d(d-2)\ldots(d-2r+2)}{r!}.$$

We use (6 . 45) when $d = 3$, and (6 . 46) when d is even and greater than 3. It so happens that we do not find it necessary to consider odd values of d greater than 3.

The values of $N_s(d)$, given by (6 . 45) and (6 . 46), are set out in Table 2. If one of them is less than ρ, then $\gamma_\varpi \geqslant s+1$.

6 . 5. Table 3 gives (1) the values of $\Gamma(k)$ when k belongs to a special class, (2) lower bounds for $\Gamma(k)$ in other cases, for $36 < k \leqslant 200$. The special values are starred; in all other cases we know, of course, that $\Gamma(k) \leqslant k$.

The lower bounds are found in four ways:

(i) In the first place

$$\Gamma(k) \geqslant \operatorname*{Max}_{\delta \ k} \Gamma(\delta).$$

When the lower bound is found in this way, we write $k > \delta$ (*e.g.* $45 > 9$) in the k-column, δ being the relevant divisor of k.

(ii) If ϖ is the least prime $3k+1$ or $dk+1 = 2d_1k+1$, we can obtain a lower bound for γ_ϖ, and so for $\Gamma(k)$, by the use of Table 2 and the argument of § 6 . 4. In this case the relevant value of d is entered in the second column. It so happens that this d is, in fact, the least d, odd or even, for which $dk+1$ is prime, except in the single case $k = 62$, when $d = 5$ and $d = 6$ both give primes, and $d = 6$ gives the better lower bound. In a larger table, of course, it might be necessary to consider odd d greater than 3.

(iii) In one case, $k = 197$, when $d = 18$, $\varpi = 3547$, we have, in order to show that $\Gamma(197) > 4$, gone beyond the principles of § 6 . 4 and resorted to calculation†. The tables than show that, for $k \leqslant 200$, $\Gamma(k) > 3$, and $\Gamma(k) > 4$ except for 2, 3, 7, and 19.

† For which we are indebted to Mr. F. G. Maunsell. The a_0's occur in equal and opposite pairs, half of them being

1, 291, 447, 1162, 1163, 1177, 1468, 1548, 1552.

The residue 7 requires five of these.

(iv) In two cases, $k = 38$ and $k = 62$, the best result is derived from the fact that $\gamma_2 = 8$ (Lemma 7). Here we have written $38 = 2 \cdot 19$ and $62 = 2 \cdot 31$ in the k-column.

When the actual lower bound is given by (i) or (iv), we have still entered the appropriate d in the second column, so that it can be verified that (ii) does not give a better result. Further refinements of the principles employed in (ii) appear to be practically ineffective.

6.6. We have explored the possibilities $\Gamma(k) = 3$, $\Gamma(k) = 4$ in the range $2 < k \leqslant 3000$. Our results are that $\Gamma(k) > 3$ in all cases; and $\Gamma(k) > 4$, except for $k = 2, 3, 7, 19$, for which it is 4, and possibly (but very improbably) $k = 1163, 1637, 1861, 1997, 2053$.

We need only consider odd k, since $\gamma_2 \geqslant 8$ for even k. In the second it is easy to see that we can confine ourselves to proving that only these exceptions occur among *prime* $k < 3000$. For this result, together with the inequality $\Gamma(kk') \geqslant \Gamma(k)$, shows that $\Gamma(k) > 3$ for odd composite $k \leqslant 3000$ (indeed, $k \leqslant 200^2$), and $\Gamma(k) > 4$, unless k is divisible by *two* of 3, 7, 19. Now all products of two of these numbers, except $19^2 = 361$, occur in Table 3 with a $\Gamma > 4$. In the case $k = 361$ we consider $\gamma_{19}(361)$, when $\varpi^\phi = 19^3$, $d = 18$, and the argument of § 6.4 shows that $\gamma_{19} > 4$. It follows that we need consider only *prime* values of k.

Considering now only primes below 3000, we calculate for each k the least d (necessarily even) for which $dk+1$ is prime. We find

$d \leqslant 12$ for $200 < k < 227$;

$d \leqslant 16$ for $227 \leqslant k < 540$, except for $k = 227$ $(d = 24)$ and $k = 457$ $(d = 30)$;

$d \leqslant 22$ for $540 < k < 1000$;

$d \leqslant 26$ for $1000 < k < 1302$, except for $k = 1163$ $(d = 32)$;

$d \leqslant 30$ for $1302 < k < 2000$, except for $k = 1637$ $(d = 38)$, $k = 1861$ $(d = 40)$, and $k = 1997$ $(d = 44)$;

$d \leqslant 34$ for $2000 < k < 2500$, except for $k = 2053$ $(d = 46)$;

$d \leqslant 36$ for $2500 < k < 3000$.

The sufficient condition for $\Gamma(k) > 3$ is

$$N_3(d) < \varpi - 1 = dk,$$

which is

$$d^2 + 3d + 8 < 6k; \tag{6.61}$$

and that for $\Gamma(k) > 4$ is

$$d^3+4d^2+20d+32 < 24k. \tag{6.62}$$

The inequality (6.61) is satisfied in all cases with a good deal to spare, and the limit 3000 could be extended considerably without excessive labour. The inequality (6.62) is satisfied by the upper bounds of d set out for the various ranges of k above. Thus $\Gamma(k) > 4$, except in the exceptional cases, which require detailed calculation. Mr. Maunsell has carried this out for $k = 227$ and $k = 457$†, but beyond this point the work becomes very heavy.

TABLE 1.

k	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\Gamma(k)$	4	16	5	9	4	32	13	12	11	16	6	14	15	64	6	27	4
Class	IV	I	VI	III	—	I	IV	IV	VI	II	—	VI	VI	I	—	III	—
ϖ	3	2	11	3	7	2	3	5	23	2	53	29	31	2	103	3	229

k	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
$\Gamma(k)$	25	24	23	23	32	10	26	40	29	29	31‡	5	128	33	20	35	36
Class	III	IV	V	VI	II	—	VI	IV	V	VI	V	—	I	VI	—	VI	V
ϖ	5	7	23	47	2	101	53	3	29	59	31	311	2	67	103	71	73

TABLE 2: $N_s(d)$.

s \ d	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
3	30	45	62	84	108	135	165	198	234	273	315	360	408	459	513	570	630	693
4	40	60	82	112	144	180	220	264	312	364	420	480	544	612	684	760	840	
6	120	210	336	504	740	990	1320	2052										
8	320	680	1288															
10	680	1682																
12	1288																	

† For $k = 227$, $\varpi = 3449$, the a_0's are

1, 78, 489, 543, 603, 621, 635, 1238, 1474, 1475, 1518, 2007,

and the same numbers with negative sign. The residue 21 requires 5.

For $k = 457$, $\varpi = 13711$, the a_0's are

1, 702, 792, 1060, 1394, 2186, 2425, 3127, 3148, 3149, 3442, 3726, 5107, 6167, 6543,

and the same numbers with negative sign. The residue 52 requires 5.

‡ There is a misprint, 30 for 31, in the value of $\Gamma(30)$ given in the table of P.N. 4,

TABLE 3.

k	d	Γ	k	d	Γ
37	4	$\geqslant 9$	$*81 = \frac{1}{2}3^4(3-1)$	—	121
$38 = 2.19$	6	$\geqslant 8$	*82	1	83
*39	2	39	*83	2	83
*40	1	41	$84 > 42$	$\geqslant 3$	$\geqslant 43$
*41	2	41	85	$\geqslant 12$	$\geqslant 5$
$*42 = 7(7-1)$	—	49	*86	2	86
*43	2	43	$87 > 29$	$\geqslant 3$	$\geqslant 29$
*44	2	44	*88	1	89
$45 > 9$	4	$\geqslant 13$	*89	2	89
*46	1	47	*90	2	90
47	6	$\geqslant 6$	91	6	$\geqslant 8$
$*48 = 3.2^4$	—	64	$92 > 4$	$\geqslant 3$	$\geqslant 16$
49	4	$\geqslant 10$	93	4	$\geqslant 14$
*50	2	50	94	3	$\geqslant 14$
*51	2	51	*95	2	95
*52	1	53	$96 = 3.2^5$	—	128
*53	2	53	97	4	$\geqslant 14$
$*54 = 3^3(3-1)$	—	81	*98	2	98
$*55 = \frac{1}{2}11(11-1)$	—	60	*99	2	99
*56	2	56	$*100 = 5^2(5-1)$	—	125
57	4	$\geqslant 11$	101	6	$\geqslant 8$
*58	1	59	*102	1	103
*59	2	59	103	6	$\geqslant 8$
*60	1	61	*104	2	104
61	6	$\geqslant 7$	*105	2	105
$62 = 2.31$	6	$\geqslant 8$	*106	1	107
*63	2	63	107	6	$\geqslant 8$
$*64 = 2^6$	—	256	$108 > 54$	$\geqslant 4$	$\geqslant 81$
*65	2	65	109	10	$\geqslant 5$
66	3	$\geqslant 11$	$*110 = 11(11-1)$	—	121
67	4	$\geqslant 12$	*111	2	111
*68	2	68	*112	1	113
*69	2	69	*113	2	113
*70	1	71	*114	2	114
71	8	$\geqslant 5$	$115 > 23$	$\geqslant 4$	$\geqslant 23$
*72	1	73	*116	2	116
73	4	$\geqslant 12$	$117 > 9$	$\geqslant 6$	$\geqslant 13$
*74	2	74	$118 > 59$	$\geqslant 3$	$\geqslant 59$
*75	2	75	*119	2	119
76	3	$\geqslant 12$	*120	2	120
$77 > 11$	6	$\geqslant 11$	$121 > 11$	$\geqslant 6$	$\geqslant 11$
$*78 = \frac{1}{2}13(13-1)$	—	84	122	3	$\geqslant 16$
79	4	$\geqslant 13$	$123 > 41$	$\geqslant 4$	$\geqslant 41$
*80	2	80	124	3	$\geqslant 16$

TABLE 3—(*continued*).

k	d	Γ	k	d	Γ
*125	2	125	163	4	$\geqslant 18$
*126	1	127	$164 > 41$	$\geqslant 3$	$\geqslant 41$
127	4	$\geqslant 16$	*165	2	165
$*128 = 2^7$	—	512	*166	1	167
$129 > 43$	$\geqslant 4$	$\geqslant 43$	167	14	$\geqslant 5$
*130	1	131	*168	2	168
*131	2	131	169	4	$\geqslant 18$
132	3	$\geqslant 16$	170	6	$\geqslant 10$
133	12	$\geqslant 5$	$171 > 9$	$\geqslant 6$	$\geqslant 13$
*134	2	134	*172	1	173
*135	2	135	*173	2	173
$*136 = \frac{1}{2}17(17-1)$	—	144	*174	2	174
137	6	$\geqslant 9$	$175 > 35$	$\geqslant 4$	$\geqslant 35$
*138	1	139	*176	1	177
139	4	$\geqslant 17$	177	4	$\geqslant 19$
*140	2	140	*178	1	179
*141	2	141	*179	2	179
142	4	$\geqslant 17$	*180	1	181
143	6	$\geqslant 9$	181	6	$\geqslant 10$
$144 > 72$	$\geqslant 3$	$\geqslant 72$	182	3	$\geqslant 19$
$145 > 29$	$\geqslant 4$	$\geqslant 29$	*183	2	183
*146	2	146	$184 > 46$	$\geqslant 4$	$\geqslant 46$
$*147 = \frac{1}{2}7^2(7-1)$	—	171	$185 > 37$	$\geqslant 8$	$\geqslant 9$
*148	1	149	*186	2	186
149	8	$\geqslant 6$	$187 > 11$	$\geqslant 6$	$\geqslant 11$
*150	1	151	$188 > 4$	$\geqslant 5$	$\geqslant 16$
151	6	$\geqslant 9$	*189	2	189
$152 > 8$	$\geqslant 3$	$\geqslant 32$	*190	1	191
*153	2	153	*191	2	191
154	3	$\geqslant 18$	$*192 = 3.2^6$	—	256
*155	2	155	193	4	$\geqslant 20$
$*156 = 13(13-1)$	—	169	*194	2	194
157	6	$\geqslant 9$	$195 > 39$	$\geqslant 4$	$\geqslant 39$
*158	2	158	$196 > 28$	$\geqslant 3$	$\geqslant 28$
*159	2	159	197	18	$\geqslant 5$
$160 > 32$	$\geqslant 3$	$\geqslant 128$	*198	1	199
161	6	$\geqslant 9$	199	4	$\geqslant 20$
*162	1	163	*200	2	200

GENERALIZED ASYMPTOTIC SERIES

By H. V. LOWRY.

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1. Heaviside* has expanded several functions in what he called "generalized series". The most important of the results he used is

$$e^x = \sum_{m=-\infty}^{\infty} \frac{x^{r+m}}{\Pi(r+m)},$$

where $\Pi(r)$ is Gauss' function defined by $\Pi(r) = \Gamma(r+1)$.

Jeffreys and Ingham† have recently shown that this result is true if stated in the form

$$\sum_{m=1}^{\infty} \frac{x^{r+m}}{\Pi(r+m)} \sim e^x - \sum_{m=-\infty}^{0} \frac{x^{r+m}}{\Pi(r+m)}. \tag{1}$$

The object of this paper is to explain an elementary method of obtaining asymptotic expansions for some functions of a *real positive* variable similar to the other series used by Heaviside. The method is somewhat easier than the use of contour integrals of Pochhammer's loop type, or Barnes's gamma function type. Some of the results found are cases of formulae for a complex variable found by contour methods. In particular, (1) is a case of the asymptotic expansion of ${}_1F_1(\alpha;\ \rho,\ x)$ found by Barnes‡, and (21) and (24) are cases of the analytic continuation of a hypergeometric series. The method is an extension of the following way of obtaining (1).

If $$\frac{dz}{dx} - z = \frac{x^r}{\Pi(r)} \quad \text{and} \quad \frac{dy}{dx} - y = 0,$$

then $$\frac{d}{dx}(yz) = \frac{yx^r}{\Pi(r)},$$

* *Electromagnetic theory*, 2, Ch. 7.

† See Harold Jeffreys, *Operational methods in mathematical physics* (1927).

‡ *Trans. Camb. Phil. Soc.*, 20 (1908), 253-279.

and therefore, if $z = 0$ when $x = 0$,

$$yz = \int_0^x \frac{yx^r}{\Pi(r)}\, dx.$$

Integrating by parts we get, if $r > -1$,

$$z = \sum_{m=1}^{p} \frac{x^{r+m}}{\Pi(r+m)} - \frac{1}{y}\int_0^x \frac{yx^{r+p}}{\Pi(r+p)}\, dx \tag{3}$$

and

$$z = \int_0^\infty \frac{yx^r}{\Pi(r)}\, dx - \sum_{m=-q}^{0} \frac{x^{r+m}}{\Pi(r+m)} + \frac{1}{y}\int_x^\infty \frac{yx^{r+q-1}}{\Pi(r+q-1)}\, dx. \tag{4}$$

(3) is convergent because the integral in it tends to zero as $n \to \infty$, and (4) is asymptotic to z because the last integral is less than

$$\frac{x^{r+q}}{\Pi(r+q)},$$

since $y \sim e^{-x}$.

Equations (3) and (4) show that (1) is true if $r > -1$, but since we can transfer any number of terms from one side of (1) to the other, (1) is true for all values of r.

2. This method will now be applied to

$$L(z) \equiv \frac{d}{dx}\left(U\frac{dz}{dx}\right) + Vz = \phi(x),$$

where U and V are functions of x.

In general this equation has a formal solution $F(x)$ in ascending powers of x and another $f(x)$ in descending powers of x. When U, V, and $\phi(x)$ are powers of x these formal solutions are

$$F(x) = (\nabla^{-1} - \nabla^{-2} + \nabla^{-3} - \ldots)\phi(x)/V$$

and

$$f(x) = (1 - \nabla + \nabla^2 - \ldots)\phi(x)/V,$$

in which

$$\nabla = \frac{1}{V}\frac{d}{dx}\, U\, \frac{d}{dx}, \qquad \nabla^{-1}z = Q\,\frac{1}{U}\,QVz,$$

where

$$Qz = \int_0^x z\, dx.$$

In general $F(x)$ and $f(x)$ are related by an equation of the form

$$F(x) \sim Ay_1 + By_2 + f(x),$$

where y_1 and y_2 are solutions of $L(y) = 0$, and A and B are constants.

Now $$U\left\{\frac{d}{dx}\left(z\frac{dy}{dx}-y\frac{dz}{dx}\right)\right\} = yL(z)-zL(y)$$

$$= y\phi(x),$$

and therefore $$U\left(y\frac{dz}{dx}-z\frac{dy}{dx}\right) = \int_0^x y\phi(x)dx,$$

provided that $$U\left(y\frac{dz}{dx}-z\frac{dy}{dx}\right) = 0 \quad \text{when} \quad x = 0.$$

Now integrating by parts and using $L(y) = 0$, we have

$$\int_0^x y\phi(x)dx = yQ\phi(x)-\int_0^x \frac{dy}{dx}\,Q\phi(x)dx$$

$$= yQ\phi(x)-U\,\frac{dy}{dx}\,Q\,\frac{1}{U}\,Q\phi(x)-\int_0^x Vy\nabla^{-1}\left(\frac{\phi(x)}{V}\right)dx.$$

Repeating the process, we get

$$\int_0^x y\phi(x)\,dx = U\left(y\,\frac{d}{dx}-\frac{dy}{dx}\right)\sum_{m=1}^{q}(-1)^{m-1}\nabla^{-m}\left(\frac{\phi(x)}{V}\right)-\int_0^x Vy\nabla^{-p}\left(\frac{\phi(x)}{V}\right)dx.$$

In most cases

$$\int_0^x Vy\nabla^{-p}\left(\frac{\phi(x)}{V}\right)dx \to 0 \quad \text{as} \quad x \to \infty$$

and we therefore get a convergent series

$$U\left(y\frac{dz}{dx}-z\frac{dy}{dx}\right) = U\left(yF'(x)-\frac{dy}{dx}F(x)\right). \tag{5}$$

In the same way

$$\int_0^x y\phi(x)\,dx = \int_0^\infty y\phi(x)\,dx-\int_x^\infty y\phi(x)\,dx$$

$$= \int_0^\infty y\phi(x)\,dx+U\left(y\,\frac{d}{dx}-\frac{dy}{dx}\right)\sum_{m=0}^{y-1}(-1)^m\nabla^m\left(\frac{\phi(x)}{V}\right)$$

$$+\int_x^\infty Vy\nabla^q\left(\frac{\phi(x)}{V}\right)dx.$$

This gives an asymptotic series if the last integral is of the same order of magnitude as

$$U\,\frac{dy}{dx}\,\nabla^{q-1}\left(\frac{\phi(x)}{V}\right).$$

Therefore, if this condition is satisfied,

$$U\left(y\frac{dz}{dx}-z\frac{dy}{dx}\right)\sim\int_0^\infty y\phi(x)\,dx-U\left(yf'(x)-\frac{dy}{dx}f(x)\right). \tag{6}$$

Substituting y_1 and y_2 in (5) and (6) and eliminating dz/dx,

$$z=F(x), \tag{7}$$

and

$$z\sim\frac{1}{C}\left\{y_2\int_0^\infty y_1\phi(x)\,dx-y_1\int_0^\infty y_2\phi(x)\,dx\right\}-f(x), \tag{8}$$

where

$$C=U\left(y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}\right),$$

which is a constant.

Although (7) and (8) are restricted as solutions of $L(z)=\phi(x)$ because the integration by parts is not always possible, these restrictions do not generally apply to

$$F(x)\sim\frac{1}{C}\left\{y_2\int_0^\infty y_1\phi(x)dx-y_1\int_0^\infty y_2\phi(x)dx\right\}-f(x). \tag{9}$$

Heaviside would have called

$$C\sum_{m=-\infty}^{\infty}\nabla^m\left(\frac{\phi(x)}{V}\right),$$

a generalized series for

$$y_2\int_0^\infty y_1\phi(x)dx-y_1\int_0^\infty y_2\phi(x)dx.$$

The method of finding (5) and (6) may be true for only one solution of $L(y)=0$, and then we shall have

$$yF'(x)-\frac{dy}{dx}F(x)\sim\frac{1}{U}\int_0^\infty y\phi(x)dx-yf'(x)+\frac{dy}{dx}f(x),$$

instead of (9).

3. *Expansions derived from* $\phi(x)=Ax^r$.

If we take

$$L(z)\equiv\frac{d^2z}{dx^2}+z=\frac{x^r}{\Pi(r)},$$

and

$$y_1=\sin x,\quad y_2=\cos x,$$

(9) becomes

$$-\sum_{m=1}^{\infty}\frac{(-1)^m x^{r+2m}}{\Pi(r+2m)}\sim\cos\left(x-\tfrac{1}{2}r\pi\right)+\sum_{m=-r}^{0}\frac{(-1)^m x^{r+2m}}{\Pi(r+2m)}. \tag{11}$$

In this case the two forms of z are solutions of $L(z) = \phi(x)$ only if $-2 < r < 0$, but it is easy to see that this restriction does not apply to (11).

If we put ix instead of x in (1) we get (11), and this shows that (1) is true for imaginary values of x.

Bessel's equation.

If
$$L(z) \equiv \frac{d}{dx}\left(x\frac{dz}{dx}\right) + xz = \frac{x^{r+1}}{2^r(\Pi\frac{1}{2}r)^2},$$
and
$$y_1 = J_0(x), \quad y_2 = Y_0(x),$$
(9) becomes

$$-\sum_{m=1}^{\infty}\frac{(-1)^m(\frac{1}{2}x)^{r+2m}}{\{\Pi(\frac{1}{2}r+m)\}^2} \sim -\sin\tfrac{1}{2}r\pi\, Y_0(x) - \cos\tfrac{1}{2}r\pi\, J_0(x) + \sum_{m=-\infty}^{0}\frac{(-1)^m(\frac{1}{2}x)^{r+2m}}{\{\Pi(\frac{1}{2}r+m)\}^2}, \qquad (12)$$

where $Y_0(x)$ is Weber's function*.

4. *Expansions derived from* $\phi(x) = Ae^{-x}x^r$.

With this form of $\phi(x)$, $F(x)$ and $f(x)$ cannot be derived directly from
$$\frac{1}{1+\nabla}\left(\frac{\phi(x)}{V}\right),$$
but they come quite easily from the integration by parts.

Thus, if
$$L(z) \equiv \frac{d^2z}{dx^2} + z = \frac{2^{\frac{1}{2}r}e^{-x}x^r}{\Pi(r)},$$
(9) becomes

$$-e^{-x}\sum_{m=1}^{\infty}u_m \sim 2^{-\frac{1}{2}}\sin\{\tfrac{1}{4}(r+1)\pi - x\} - e^{-x}\sum_{m=-\infty}^{0}u_m, \qquad (13)$$

where
$$u_m = (-1)^m 2^{\frac{1}{2}r+2m-2}\left\{\frac{x^{r+4m-2}}{\Pi(r+4m-2)} + \frac{2x^{r+4m-1}}{\Pi(r+4m-1)} + \frac{2x^{r+4m}}{\Pi(r+4m)}\right\}.$$

This result can also be derived from (2) by taking the right-hand side as $A\sin x \,.\, x^r$ or $A\cos x \,.\, x^r$.

Taking
$$L(z) \equiv \frac{d}{dx}\left(x\frac{dz}{dx}\right) + xz = \frac{2^r e^{-x}x^r}{\{\Pi(r)\}^2\Pi(-r-\frac{3}{2})},$$

* The values of $Y_0(x)$ and $K_0(x)$ are those given in Watson's *Theory of Bessel functions*, (1922), 64 and 78.

we get

$$-e^{-x}\sum_{m=1}^{\infty}\frac{(-1)^m(2x)^{r+m}}{\{\Pi(r+m)\}^2\Pi(-r-m-\frac{1}{2})}$$

$$\sim -\pi^{-\frac{1}{2}}\{\cos\pi r I_0(x)-\pi^{-1}\sin\pi r K_0(x)\}$$

$$+e^{-x}\sum_{m=-\infty}^{0}\frac{(-1)^m(2x)^{r+m}}{\{\Pi(r+m)\}^2\Pi(-r-m-\frac{1}{2})}. \qquad (14)$$

This includes the well known asymptotic expansion for $K_0(x)$ as the particular case $r=\frac{1}{2}$, but it does not include the parallel expression for $I_0(x)$. When $r=0$, (14) becomes Kummer's formula for $e^x I_0(x)$.

Equations similar to (14) can be obtained by writing

$$L(z)\equiv\frac{d}{dx}\left(x\frac{dz}{dx}\right)+xz=\frac{2^r x^r}{(\Pi r)^2\Pi(-r-\frac{3}{2})}\begin{matrix}\cos\\ \sin\end{matrix}(\tfrac{1}{2}r\pi-x).$$

These values of $\phi(x)$ give

$$-\cos(\tfrac{1}{2}r\pi-x)\sum_{m=1}^{\infty}u_m+\sin(\tfrac{1}{2}r\pi-x)\sum_{m=1}^{\infty}v_m$$

$$\sim \pi^{-\frac{1}{2}}\{\cos r\pi J_0(x)-\tfrac{1}{2}\sin r\pi Y_0(x)\}$$

$$+\cos(\tfrac{1}{2}r\pi-x)\sum_{m=-\infty}^{0}u_m-\sin(\tfrac{1}{2}r\pi-x)\sum_{m=-\infty}^{0}v_m \qquad (15)$$

and

$$-\sin(\tfrac{1}{2}r\pi-x)\sum_{m=1}^{\infty}u_m-\cos(\tfrac{1}{2}r\pi-x)\sum_{m=1}^{\infty}v_m$$

$$\sim \tfrac{1}{2}\pi^{-\frac{1}{2}}\sin r\pi J_0(x)+\sin(\tfrac{1}{2}r\pi-x)\sum_{m=-\infty}^{0}u_m+\cos(\tfrac{1}{2}r\pi-x)\sum_{m=-\infty}^{0}v_m, \qquad (16)$$

where $$u_m=\frac{(-1)^m(2x)^{r+2m-2}}{\{\Pi(r+2m-2)\}^2\Pi(-r-2m+\frac{3}{2})}$$

and $$v_m=\frac{(-1)^m(2x)^{r+2m-1}}{\{\Pi(r+2m-1)\}^2\Pi(-r-2m+\frac{1}{2})}.$$

From these

$$-\sum_{m=1}^{\infty}u_m\sim\tfrac{1}{2}\pi^{-\frac{1}{2}}[\{2\cos r\pi\cos(\tfrac{1}{2}r\pi-x)+\sin r\pi\sin(\tfrac{1}{2}r\pi-x)\}J_0(x)$$

$$-\sin r\pi\cos(\tfrac{1}{2}r\pi-x)Y_0(x)]+\sum_{m=-\infty}^{0}u_m.$$

The standard asymptotic series for $J_0(x)$ and $Y_0(x)$ are the particular case $r=\frac{1}{2}$.

5. *Asymptotic expansions involving Bessel's functions of any order.*

By taking $$L(y) \equiv \frac{d}{dx}\left(x\frac{dy}{dx}\right) \pm xy - \frac{n^2}{x}y$$

and choosing a suitable form for $\phi(x)$ we can obtain expansions depending on $J_n(x)$, etc.

Thus, with $$\phi(x) = \frac{x^{r+1}}{2^r \Pi(\frac{1}{2}r-\frac{1}{2}n)\Pi(\frac{1}{2}r+\frac{1}{2}n)},$$

we get

$$\sum_{m=1}^{\infty} u_m \sim \frac{1}{\sin n\pi}\left\{\sin\tfrac{1}{2}(n+r)\pi J_n(x)+\sin\tfrac{1}{2}(n-r)\pi J_{-n}(x)\right\} - \sum_{m=-\infty}^{0} u_m, \quad (17)$$

where $$u_m = \frac{(-1)^m x^{r+2m}}{2^{r+2m}\Pi(\frac{1}{2}r+m+\frac{1}{2}n)\Pi(\frac{1}{2}r+m+\frac{1}{2}n)}.$$

Also, with $$\phi(x) = \frac{2^r e^{-x} x^r}{\Pi(-r-\frac{3}{2})\Pi(r-n)\Pi(r+n)},$$

$$e^{-x}\sum_{m=1}^{\infty} u_m \sim \frac{\pi^{-\frac{1}{2}}}{2\sin n\pi}\left\{\sin(n+r)\pi I_n(x)+\sin(n-r)\pi I_{-n}(x)\right\} - e^{-x}\sum_{m=-\infty}^{0} u_m,$$

where $$u_m = \frac{(-1)^m (2x)^{r+m}}{\Pi(-r-\frac{1}{2}-m)\Pi(r+m-n)\Pi(r+m+n)}. \quad (18)$$

The parallel expansions involving $J_n(x)$ are

$$-\cos(\tfrac{1}{2}r\pi-x)\sum_{m=1}^{\infty} u_m + \sin(\tfrac{1}{2}r\pi-x)\sum_{m=1}^{\infty} v_m$$

$$\sim \tfrac{1}{2}\pi^{-\frac{1}{2}}\frac{\cos\frac{1}{2}n\pi}{\sin n\pi}\left\{\sin(n+r)\pi J_n(x)+\sin(n-r)\pi J_{-n}(x)\right\}$$

$$+\cos(\tfrac{1}{2}r\pi-x)\sum_{m=-\infty}^{0} u_m - \sin(\tfrac{1}{2}r\pi-x)\sum_{m=-\infty}^{0} v_m \quad (19)$$

and

$$-\sin(\tfrac{1}{2}r\pi-x)\sum_{m=1}^{\infty} u_m - \cos(\tfrac{1}{2}r\pi-x)\sum_{m=1}^{\infty} v_m$$

$$\sim \tfrac{1}{2}\pi^{-\frac{1}{2}}\frac{\sin\frac{1}{2}n\pi}{\sin n\pi}\left\{\sin(n+r)\pi J_n(x)-\sin(n-r)\pi J_{-n}(x)\right\}$$

$$+\sin(\tfrac{1}{2}r\pi-x)\sum_{m=0}^{\infty} u_m + \cos(\tfrac{1}{2}r\pi-x)\sum_{m=0}^{\infty} v_m. \quad (20)$$

From these

$$-\sum_{m=1}^{\infty} u_m \sim \frac{\pi^{-\frac{1}{2}}}{\sin n\pi}\left[\cos\left\{\tfrac{1}{2}(r-n)\pi - x\right\}\sin(n+r)\pi J_n(x) + \cos\left\{\tfrac{1}{2}(r+n)\pi - x\right\}\sin(n-r)\pi J_{-n}(x)\right] + \sum_{m=-\infty}^{0} u_m,$$

where

$$u_m = \frac{(-1)^m (2x)^{r+2m-2}}{\Pi(-r+\frac{3}{2}-2m)\Pi(r+2m-2-n)\Pi(r+2m-2+n)}$$

and

$$v_m = \frac{(-1)^m (2x)^{r+2m-1}}{\Pi(-r+\frac{1}{2}-2m)\Pi(r+2m-1-n)\Pi(r+2m-1+n)}.$$

(15) and (16) can, of course, be derived from (19) and (20) by making $n \to 0$.

If n is an integer, the right side of (20) takes the form

$$\tfrac{1}{2}\pi^{-\frac{1}{2}} \sin r\pi J_n(x) + \ldots.$$

The standard asymptotic expansion of $J_n(x)$ is obtained from (19) and (20) by putting $r = \frac{1}{2}$ and eliminating $J_{-n}(x)$.

6. *Two expansions connected with the binomial theorem.*

In all the cases considered above the convergent parts on the left-hand side of the equations are convergent for all positive values of x, so that the expansions are properly called asymptotic for all positive values of x. The next example is of a different nature because the convergent part converges only if $x < 1$.

If

$$\frac{dz}{dx} + \frac{n+1}{1+x} z = \frac{x^r}{\Pi(r)\Pi(n-r-1)},$$

$$z = (1+x)^{n+1} \int_0^x \frac{(1+x)^{-n-1} x^r}{\Pi(r)\Pi(n-r-1)}\, dx,$$

and

$$z = \frac{(1+x)^{n+1}}{\Pi(n)} - \int_x^\infty \frac{(1+x)^{-n-1} x^r}{\Pi(r)\Pi(n-r-1)}\, dx,$$

provided that

$$n > -1 \quad \text{and} \quad r > -1.$$

Hence, integrating by parts,

$$z = (1+x) \sum_{m=1}^{p} \frac{x^{r+m}}{\Pi(r+m)\Pi(n-r-m)} + R_p,$$

and $$z=\frac{(1+x)^{n+1}}{\Pi(n)}-1+x\sum_{m=-q}^{0}\frac{x^{r+m}}{\Pi(r+m)\Pi(n-r-m)}+R_q,$$

where $$R_p=(1+x)^{n+1}\int_0^x\frac{(1+x)^{-n-1}x^r\,dx}{\Pi(r+p)\Pi(n-r-p-1)}$$

$$=O(x^{r+p+1}),$$

and $$R_q=(1+x)^{n+1}\int_x^\infty\frac{(1+x)^{-n-1}x^r\,dx}{\Pi(r-q-1)\Pi(n-r+q)}$$

$$=O(x^{r-q}),$$

provided that $p>-r-1$, and $q>r$.

If $x<1$, $R_p\to 0$ as $p\to\infty$, and the first form of z is convergent. Therefore

$$\sum_{m=1}^{\infty}\frac{x^{r+m}}{\Pi(r+m)\Pi(n-r-m)}\sim\frac{(1+x)^n}{\Pi(n)}-\sum_{m=-\infty}^{0}\frac{x^{r+m}}{\Pi(r+m)\Pi(n-r-m}, \tag{21}$$

when $x<1$ and $n>-1$. The restrictions on r can be removed for the same reasons as before.

If $r=0$, (21) becomes the ordinary binomial expansion of $(1+x)^n$ in ascending powers of x, and if $r=n$ it becomes

$$0\sim(1+x)^n-\left\{x^n+nx^{n-1}+\frac{n(n-1)}{2!}x^{n-2}+\ldots\right\}. \tag{22}$$

If $x=1$ and $n\geqslant 0$ both forms of z are convergent, and (21) then becomes the identity

$$\frac{2^n}{\Pi(n)}=\sum_{m=-\infty}^{\infty}\frac{1}{\Pi(r+m)\Pi(n-r-m)}. \tag{23}$$

When $n=0$, (23) becomes

$$\pi\operatorname{cosec}r\pi=\sum_{m=-\infty}^{\infty}\frac{(-1)^m}{r+m},$$

the expansion of $\operatorname{cosec}r\pi$ in partial fractions.

A result similar to (21) can be found involving $(1-x)^{-n}$.

If $$\frac{dz}{dx}-\frac{(n-1)z}{1-x}=\frac{x^r}{\Pi(r)\Pi(-n-r-1)},$$

$$z(1-x)^{n-1}=(1-x)^n\sum_{m=1}^{\infty}\frac{x^{r+m}}{\Pi(r+m)\Pi(-n-r-m)},$$

and

$$z(1-x)^{n-1} \sim \frac{\Pi(n-1)}{\Pi(r+n)\Pi(-n-r-1)} - \sum_{m=-\infty}^{0} \frac{x^{r+m}}{\Pi(r+m)\Pi(-n-r-m)},$$

if $x<1$ and $n>0$.

Therefore

$$\sum_{m=1}^{\infty} \frac{x^{r+m}}{\Pi(r+m)\Pi(-n-r-m)} \sim -\frac{\sin(r+n)\pi\,(1-x)^{-n}}{\sin n\pi\,\Pi(-n)} - \sum_{m=-\infty}^{0} \frac{x^{r+m}}{\Pi(r+m)\Pi(-n-r-m)}. \quad (24)$$

The errors due to stopping at the smallest terms in the asymptotic series in (21), (22), and (24) become smaller the nearer x is to 1.

Heaviside developed most of his generalized series by substituting an operator for x in (1) and (21).

Thus $$J_0(x) = p(1+p^2)^{-\frac{1}{2}},$$

and if we write $1/p^2$ for x and $-\frac{1}{2}$ for n in (21) we get

$$\sum_{m=1}^{\infty} \frac{(-1)^m x^{2r+2m}}{2^{2r+2m}\Pi(r+m)^2} \sim \left(1+\frac{1}{p^2}\right)^{-\frac{1}{2}} - \sum_{m=-\infty}^{0} \frac{(-1)^m x^{2r+2m}}{2^{2r+2m}\Pi(r+m)^2}.$$

This suggested the possibility of a relationship like (12), and Heaviside tried to find the constants multiplying $J_0(x)$ and $Y_0(x)$ by numerical substitution in the series. The method used in this paper finds the constants in several cases. In all of these $y\phi(x)$ vanishes at $x=0$ and at another positive value (1 or ∞). There remain some series given by Heaviside, such as

$$e^{-x}\cos r\pi = \sum_{m=-\infty}^{\infty} \frac{(-1)^m x^{r+m}}{\Pi(r+m)},$$

in which this restriction is not satisfied.

[I have to thank the referees for valuable criticism and many useful suggestions during the preparation of this paper.]

ADDENDA AND CORRIGENDA.

The following notes have been received from the authors of the papers concerned.

THE BEHAVIOUR OF CERTAIN SERIES ASSOCIATED WITH LIMITING CASES OF ELLIPTIC THETA-FUNCTIONS.

I find that the transformation formula which I gave in the third part of my paper [*Proc. London Math. Soc.* (2), 27 (1928), 410-426], §4.1 (422–3), and which I used to determine the analytic character of functions such as $\sum_{1}^{x} e^{\pi i n^2 y} z^n$, had been established in a slightly different form by Prof. L. J. Mordell in a paper "The value of a definite integral", *Quart. J. of Math.*, 48 (1920), 329–342 (341–2).

27 *July*, 1928. R. Cooper.

NOTE ON THE INVERSION OF A REPEATED INTEGRAL.

I am indebted to Mr. E. G. Phillips for pointing out a numerical error in my note, *Proc. London Math. Soc.* (2), 24 (1925), 1-li (*Records* for 14 May, 1925). The correct values of the integrals, which are quoted from my *Course of pure mathematics*, 356 (Ex. 38) where the same error occurs, are

$$\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\, dy = \tfrac{1}{2}, \quad \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\, dx = -\tfrac{1}{2},$$

and not 1 and -1, as there stated.

22 *September*, 1928. G. H. Hardy.

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